

The Yamabe Problem

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These notes are from a five-part set of expository lectures that I gave for a graduate class at Northwestern University in January 2018. Our presentation largely follows that given in the survey [LP87] and the book [Aub82], though for the analytic portion we follow the concentration compactness approach of Lions [Lio84a, Lio84b] instead of the original PDE approach. For brevity we have omitted some proofs, but we try to give references for where the reader can find them.

1 Introduction and preliminaries

The uniformization theorem says that every closed two-dimensional Riemannian manifold is conformally equivalent to one of constant sectional curvature. In higher dimension this is false; the problem of prescribing sectional curvature is highly overdetermined. The Yamabe problem can be seen as one appropriate higher dimensional analogue of this problem.

The Yamabe Problem. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Is there a metric \tilde{g} conformal to g that has constant scalar curvature?*

A metric \tilde{g} is conformal to g if there is a smooth function $\varphi > 0$ such that $\tilde{g} = \varphi^{4/(n-2)}g$. In [Yam60], Yamabe claimed to give an affirmative answer to the question above. In [Tru68], Trudinger found a nontrivial gap in the proof and provided a partial repair. Further contributions of Aubin [Aub76a, Aub76b] and Schoen [Sch84] finally closed the proof nearly a quarter century after Yamabe's initial work.

Given two conformal metrics g and $\tilde{g} = \varphi^{4/(n-2)}g$, and let R and \tilde{R} denote the scalar curvatures of g and \tilde{g} respectively. These quantities are related by the identity

$$\tilde{R} = \varphi^{1-2^*}(-c_n \Delta \varphi + R\varphi). \tag{1.1}$$

Here and in the sequel, $c_n = 4(n-1)/(n-2)$ and our sign convention is $\Delta \varphi = \nabla^a \nabla_a \varphi$. The number $2^* = 2n/(n-2)$ is the exponent in the critical Sobolev embedding, a fact that plays a decisive role in the problem. Rearranging (1.1), we see that solving the Yamabe problem on (M, g) is equivalent to finding a positive smooth solution of the nonlinear eigenvalue problem

$$-c_n \Delta \varphi + R\varphi = \lambda \varphi^{2^*-1} \tag{*}$$

for some $\lambda \in \mathbb{R}$. Stated another way, solving the Yamabe problem is equivalent to finding a positive smooth critical point of the associated energy functional

$$Q(u) = \frac{\int_M c_n |\nabla u|^2 + Ru^2 \, d\text{vol}_g}{\|u\|_{2^*}^2}.$$

With this in mind, we define the *Yamabe constant* of (M, g) by

$$\lambda(M) = \inf \{Q(u) : u \in W^{1,2}(M)\} \quad (1.2)$$

Note that $|\lambda(M)| < \infty$ by the Sobolev inequality, and $\lambda(M)$ is conformally invariant. In fact, by (1.1), it can be rewritten as

$$\lambda(M) = \inf \left\{ \frac{\int_M \tilde{R} d\text{vol}_{\tilde{g}}}{\text{vol}_{\tilde{g}}(M)^{2/2^*}} : \tilde{g} \text{ conformal to } g \right\}$$

We will frequently abuse notation by letting $\lambda(M)$ denote both the infimum value and the variational problem itself. Let us see that (\star) is the Euler Lagrange equation corresponding to this variational problem.

Claim. *Let $u \in W^{1,2}(M)$ be a critical point of $\lambda(M)$. Then u satisfies (\star) for some λ . Furthermore, if u is a minimizer with $\|u\|_{2^*} = 1$, then $\lambda = \lambda(M)$.*

Proof. Let $\varphi \in C_c^\infty(M)$. Then

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \frac{\int_M c_n |\nabla u + \epsilon \nabla \varphi|^2 + R(u + \epsilon \varphi)^2 d\text{vol}_g}{\|u + \epsilon \varphi\|_{2^*}^2} \\ &= \frac{\int 2c_n \langle \nabla u, \nabla \varphi \rangle + 2Ru\varphi d\text{vol}_g}{\|u\|_{2^*}^2} - \underbrace{\frac{2 \int_M c_n |\nabla u|^2 + Ru^2 d\text{vol}_g}{\|u\|_{2^*}^2}}_{=2Q(u)} \frac{\int u^{2^*-1} \varphi d\text{vol}_g}{\|u\|_{2^*}^{2^*}} \\ &= \frac{2}{\|u\|_{2^*}^2} \int \varphi \left(-c_n \Delta u + Ru - Q(u) \frac{u^{2^*-1}}{\|u\|_{2^*}^{2^*-2}} \right) d\text{vol}_g. \end{aligned}$$

As this holds for any variation φ , we see that u satisfies (\star) with $\lambda = Q(u)/\|u\|_{2^*}^{2^*-2}$. If u is a minimizer, then $Q(u) = \lambda(M)$. \square

To summarize thus far, to solve the Yamabe problem, it suffices to show the existence of a smooth positive minimizer of $\lambda(M)$. The problem lacks compactness, which makes establishing the existence of such a minimizer difficult. The solution of the problem comes from combining the following three theorems.

Theorem 1.1 (Yamabe, Trudinger, Aubin). *Suppose $\lambda(M) < \lambda(S^n)$. Then there exists a minimizer of $\lambda(M)$ and hence a solution of the Yamabe problem on M .*

Here $\lambda(S^n)$ be the Yamabe constant of (S^n, g_0) where g_0 is the round metric (which, of course, has constant scalar curvature).

Theorem 1.2 (Aubin). *In M has dimension $n \geq 6$ and is not locally conformally flat at some point $p \in M$, then $\lambda(M) < \lambda(S^n)$.*

A Riemannian manifold (M, g) is said to be locally conformally flat at a point $p \in M$ if there exists a conformal change of metric for which the curvature tensor vanishes in a neighborhood of p .

Theorem 1.3 (Schoen). *If M has dimension 3, 4, or 5 or if M is locally conformally flat at some point $p \in M$, then $\lambda(M) < \lambda(S^n)$ unless M is conformal to S^n .*

Theorem 1.1 represents the analytic portion of the problem. The idea behind Theorem 1.1 is the following: a minimizing sequence for $\lambda(M)$ must either converge in $W^{1,2}(M)$ to a minimizer, or else it must concentrate at a point $p \in M$. A concentration (or “bubble”) contributes $\lambda(S^n)$ to the energy, so if $\lambda(M) < \lambda(S^n)$, this possibility cannot occur.

Theorems 1.2 and 1.3 represent the contributions on the geometry side of the problem. To show that an infimum $\lambda(M)$ is strictly less than a certain number $\lambda(S^n)$, one must construct a test function φ with $Q(\varphi) < \lambda(M)$. For both theorems, the test functions involve suitable modifications of minimizers for $\lambda(S^n)$, though we crucially must first choose the right conformal representative and coordinate system.

At first sight of Theorem 1.1, several natural questions arise:

1. Why can't we just apply the direct method of the calculus of variations?
2. Why does the value of the infimum $\lambda(M)$ affect the existence of minimizers?
3. Why does $\lambda(S^n)$ play a role?

Analytically speaking, the direct method falls short because the embedding of $W^{1,2}(M)$ into $L^{2^*}(M)$ is not compact. The following example shows that the direct method can be used to establish existence of minimizers in a particular case and is instructive for understanding the first two questions.

Example 1.4 (Existence when $\lambda(M) \leq 0$). When $\lambda(M) \leq 0$, we may use the direct method to show the existence of a minimizer of $\lambda(M)$. It is not hard to show that a u is a minimizer of $\lambda(M)$ if and only if it is a minimizer of

$$E(u) = \int c_n |\nabla u|^2 + Ru^2 d\text{vol}_g - \lambda(M) \|u\|_{2^*}^2$$

among all $u \in W^{1,2}(M)$. Now, suppose $\lambda(M) \leq 0$, and let $\{u_k\}$ be a minimizing sequence for $E(u)$. Then $E(u_k) \leq C$, hence $\|u_k\|_{W^{1,2}(M)} \leq C$ (by the Sobolev inequality). So, we have

$$\begin{aligned} u_k &\rightharpoonup u && \text{in } W^{1,2}(M) \text{ and } L^{2^*}(M), \\ u_k &\rightarrow u && \text{in } L^2(M). \end{aligned}$$

by the Banach-Olaglu theorem and the Rellich-Kondrachov theorem respectively. Norms are lower semicontinuous with respect to weak convergence. Therefore, and because the coefficient $\lambda(M)$ has the right sign, we see that $E(u) \leq \liminf E(u_k)$ and so u is a minimizer.

Remark 1.5. In the general case, we will have lower semicontinuity of the energy only if we can show *strong* convergence of a minimizing sequence in $L^{2^*}(M)$.

Along the same lines, the direct method can be used to show existence for a related problem with subcritical scaling.

Exercise 1.6 (Existence for the subcritical power). Suppose we instead wanted to consider the analogue of (\star) with a subcritical exponent, that is,

$$-c_n \Delta u + Ru = \lambda u^{p-1}$$

for $1 \leq p < 2^*$. Use the direct method and the Rellich Kondrachov compact embedding theorem to establish existence of minimizers of the corresponding energy

$$Q_p(u) = \frac{\int_M c_n |\nabla u|^2 + Ru^2 \, d\text{vol}_g}{\|u\|_p^2}.$$

2 The Yamabe constant on the sphere

The statements of Theorems 1.1, 1.2, and 1.3 indicate that the Yamabe constant on the sphere will play an important role in what follows. In this section, we give explicit describe minimizers of the variational problem (1.2) and consequently give establish the value $\lambda(S^n)$. Recall that if we consider the embedded round sphere $S^n \subset \mathbb{R}^{n+1}$ with north pole $N = (0, \dots, 0, 1)$, then stereographic projection $\Psi : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ is a conformal diffeomorphism given by

$$\Psi(s^1, \dots, s^n, \xi) = (x^1, \dots, x^n) \quad \text{where} \quad x^i = s^i / (1 - \xi). \quad (2.1)$$

If we define the conformal factor v_1 by

$$(\Psi^{-1})^* g_0 = 4v_1^{4/(n-2)} g_{\text{euc}},$$

then v_1 is explicitly given by

$$v_1(x) = (1 + |x|^2)^{(2-n)/2} \quad (2.2)$$

where g_{euc} is the Euclidean metric on \mathbb{R}^n . Thanks to the conformal invariance of the Yamabe constant and the fact that Euclidean space has vanishing scalar curvature, we have

$$\lambda(S^n) = \inf\{Q(u) : u \in W^{1,2}(S^n)\} = c_n \sigma_n^2, \quad (2.3)$$

where we let

$$\sigma_n^2 := \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\left(\int_{\mathbb{R}^n} u^{2^*}\right)^{2/2^*}} : u \in W^{1,2}(\mathbb{R}^n) \right\}. \quad (2.4)$$

In other words, *understanding $\lambda(S^n)$ is equivalent to understanding the sharp Sobolev inequality on \mathbb{R}^n* . Aubin [Aub76b] and Talenti [Tal76] independently proved the following:

Theorem 2.1 (Talenti, Aubin). *Fix $n \geq 3$ and let σ_n be as above. Then*

$$\sigma_n^2 = \frac{n(n-1)\omega^{2/n}}{c_n}.$$

Furthermore, minimizers of (2.4) are exactly the function v_1 defined in (2.2) and its invariant scalings: translations, dilations, and constant multiples.

Theorem 2.1 gives the sharp Sobolev inequality on Euclidean space, together with its characterization of extremal functions. That is, for any $u \in W^{1,2}(\mathbb{R}^n)$,

$$\sigma_n \|u\|_{2^*} \leq \|\nabla u\|_2$$

Equality holds if and only if u is a translation, dilation, or constant multiple of v_1 .

Remark 2.2. If we pull back the minimizer v_1 via Ψ^{-1} , we obtain (a multiple of) the round metric. However, pulling back translations and dilations of v_1 , we obtain conformal factors that localize most of the mass near any point on S^n . In particular, these symmetries imply that the set of minimizers $\lambda(S^n)$ is noncompact in $W^{1,2}(S^n)$.

Notice the following immediate corollary of Theorem 2.1 and the dilation invariance of Sobolev extremal functions.

Corollary 2.3. *Let (M, g) be any closed Riemannian manifold of dimension $n \geq 3$. The*

$$\lambda(M) \leq \lambda(S^n).$$

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. That is, when $\lambda(M) < \lambda(S^n)$, we can find a minimizer of (1.2) and therefore a solution to the Yamabe problem.

Remark 3.1 (A historical remark). The original proof of Theorem 1.1 followed this basic idea: as we saw in Exercise 1.6, we can easily show existence of minimizers u_p of the subcritical functional Q_p . In Yamabe's original paper, he claimed to show uniform (in p) $C^{2,\alpha}$ estimates for u_p , and then passed to the limit $p \rightarrow 2^*$ to get a solution on (\star) . Such uniform estimates are false in general; we saw this in Remark 2.2. It turns out that these uniform estimates hold when $\lambda(M) < \lambda(S^n) = c_n \sigma_n^2$. The proof uses the sharp Sobolev inequality and requires the strict inequality to have some room to absorb error terms. The survey [LP87] has a clear account of this approach.

We present another approach here using concentration compactness. The key tool will be Lions' (second) concentration compactness lemma [Lio84a, Lio84b] (see also the book [Str08] and Lemma 3.2 below), which tells us two structural facts about Sobolev functions:

1. If a sequence of uniformly bounded Sobolev functions does not converge strongly in L^{2^*} , the only thing that can go wrong is that it concentrates at countably many points. (Such a sequence cannot, for instance, concentrate on some two dimensional surface.)
2. These atoms where the sequence concentrates satisfy a sort of "Sobolev inequality" for measures with the sharp Sobolev constant σ_n .

Lemma 3.2 (Lions). *Suppose $\{u_k\}$ is uniformly bounded in $W^{1,2}(M)$, so $u_k \rightharpoonup u \in W^{1,2}(M)$. Up to subsequences,*

$$\begin{aligned} \mu_k &:= |\nabla u_k|^2 d\text{vol}_g \xrightarrow{*} \mu, \\ \nu_k &:= |u_k|^{2^*} d\text{vol}_g \xrightarrow{*} \nu. \end{aligned}$$

Then

$$\nu = |u|^{2^*} d\text{vol}_g + \sum_{j \in J} \nu_j \delta_{p_j}, \quad (3.1)$$

$$\mu \geq |\nabla u|^2 d\text{vol}_g + \sigma_n^2 \sum_{j \in J} \nu_j \delta_{p_j} \quad (3.2)$$

where J is an at most countable set.

Before proving Lemma 3.2 let us see how Theorem 1.1 follows.

Proof of Theorem 1.1. Let $\{u_k\}$ be a minimizing sequence for $\lambda(M)$. Without loss of generality, we may assume that $\|u_k\|_{2^*} = 1$. Up to a subsequence, $u_k \rightarrow u$ in $L^2(M)$ and $u_k \rightharpoonup u$ in $W^{1,2}(M)$ and $L^{2^*}(M)$ with $\|u\|_{2^*}^2 = t \in [0, 1]$. Note that if $t = 1$, then $u_k \rightarrow u$ strongly in L^{2^*} and, as noted in Remark 1.5, we are done. So, by Lemma 3.2,

$$\lambda(M) = \lim Q(u_k) \geq \int c_n |\nabla u|^2 + Ru^2 + c_n \sigma_n^2 \sum \nu_j^{2/2^*}$$

Note that $\int c_n |\nabla u|^2 + Ru^2 d\text{vol}_g = t^{2/2^*} Q(u) \geq t^{2/2^*} \lambda(M)$. Together with Jensen's inequality (and recalling that $\lambda(S^n) = c_n \sigma_n^2$), this implies

$$\begin{aligned} \lambda(M) &\geq t^{2/2^*} \lambda(M) + c_n \sigma_n^2 \sum \nu_j^{2/2^*} \\ &\geq t^{2/2^*} \lambda(M) + \lambda(S^n) (1-t)^{2/2^*} \left(\sum \frac{\nu_j}{1-t} \right)^{2/2^*} \\ &= t^{2/2^*} \lambda(M) + \lambda(S^n) (1-t)^{2/2^*} \end{aligned}$$

The final equality holds because $\sum \nu_j = 1 - t$. Now, since $\lambda(S^n) > \lambda(M)$ and again applying Jensen's inequality, we have

$$\begin{aligned} \lambda(M) &\geq t^{2/2^*} \lambda(M) + \lambda(S^n) (1-t)^{2/2^*} \\ &\geq \lambda(M) \{t^{2/2^*} + (1-t)^{2/2^*}\} \end{aligned} \quad (3.3)$$

$$\geq \lambda(M). \quad (3.4)$$

Given that the left-hand side is equal to the right-hand side above, it follows that we can equality in both (3.3) and (3.4). Equality in (3.4) implies that $t = 0$ or $t = 1$. If $t = 0$, then we have strict inequality in (3.3). Therefore $t = 1$. This establishes the existence of a minimizer $u \in W^{1,2}(M)$.

Since $|\nabla u| = |\nabla|u||$ for a.e. $x \in M$, we may assume without loss of generality that $u \geq 0$. Results in elliptic regularity theory (see [Tru68]) show that u is smooth, and then the maximum principle ensures that u is positive. Thus our minimizer is indeed a conformal factor. \square

Let us now prove Lemma 3.2.

Proof of Lemma 3.2. We give the proof on a bounded subset of \mathbb{R}^n ; it is not hard to adapt the proof to M . Let

$$\begin{aligned} v_k &:= (u_k - u) \rightharpoonup 0 \quad \text{in } W^{1,2} \text{ and } L^{2^*} \\ \omega_k &:= (|u_k|^{2^*} - |u|^{2^*}) dx \rightharpoonup \omega \\ \tilde{\mu}_k &:= |\nabla v_k|^2 dx \rightharpoonup \tilde{\mu} \end{aligned}$$

One can show (exercise) that $\omega_k = |v_k|^{2^*} dx + o(1)$. Now, take any $\xi \in C_c^\infty(\mathbb{R}^n)$. Applying the Sobolev inequality, we have

$$\begin{aligned} \int \xi^{2^*} d\omega &= \lim \int |\xi v_k|^{2^*} dx \leq \liminf \frac{1}{\sigma_n^{2^*}} \left(\int |\nabla(\xi v_k)|^2 \right)^{2^*/2} \\ &= \liminf \frac{1}{\sigma_n^{2^*}} \left(\int \xi^2 |\nabla v_k|^2 \right)^{2^*/2} = \frac{1}{\sigma_n^{2^*}} \left(\int \xi^2 d\tilde{\mu} \right)^{2^*/2} \end{aligned}$$

The penultimate equality is an exercise that can be shown using Hölder's inequality and the compact embeddings of subcritical norms. Rearranging powers, what we have shown is a sort of “reverse Hölder's inequality” for the measures ω and $\tilde{\mu}$:

$$\sigma_n \left(\int \xi^{2^*} d\omega \right)^{1/2^*} \leq \left(\int \xi^2 d\tilde{\mu} \right)^{1/2} \quad \forall \xi \in C_c^\infty(\mathbb{R}^n). \quad (3.5)$$

Applied to ξ approximating the characteristic function of any open set Ω , (3.5) shows that $\tilde{\mu}$ controls ω nonlinearly:

$$\sigma_n^2 \omega(\Omega)^{2/2^*} \leq \tilde{\mu}(\Omega), \quad (3.6)$$

This scaling will force ω to be supported on a countable set of atoms. Indeed, since $\tilde{\mu}$ is a finite measure, it contains at most countably many atoms, say at $\{x_j\}$. For any point $x \in B_1 \setminus \bigcup \{x_j\}$, we can take any open set Ω containing x with $\tilde{\mu}(\Omega) \leq \sigma_n^2$, so that (3.6) yields

$$1 \geq \sigma_n^{-2} \tilde{\mu}(\Omega) \geq \omega(\Omega)^{2/2^*} \geq \omega(\Omega).$$

In other words, ω is absolutely continuous with respect to $\tilde{\mu}$ on $B_1 \setminus \bigcup \{x_j\}$. By the Radon-Nikodym theorem, $\omega = f\tilde{\mu}$ and for $\tilde{\mu}$ -a.e. x ,

$$f(x) = \lim_{r \rightarrow 0} \frac{\omega(B_r(x))}{\tilde{\mu}(B_r(x))} \stackrel{(3.6)}{\leq} \lim_{r \rightarrow 0} \sigma_n^{-2^*} \tilde{\mu}(B_r(x))^{2^*/2-1} = 0.$$

Hence, the support of ω is contained on $\bigcup \{x_j\}$ and so

$$\omega = \sum_{j \in J} \nu_j \delta_{x_j},$$

proving (3.1). To see (3.2), take any x_j and again use the reverse Hölder's inequality (3.5), now applied to a ξ with $\xi(x_j) = 1$, and $\xi = 0$ on $B_r(x_j)^c$, to find

$$\sigma_n^2 \nu_j^{2/2^*} \delta_{x_j} \leq \tilde{\mu}(x_j).$$

This concludes the proof. □

4 Proof of Theorem 1.2

We now prove Theorem 1.2, which is due to Aubin in [Aub76a]. The idea is to construct test functions which are pullbacks of a sequence of extremal functions in the Sobolev inequality that concentrate at a point. To this end, let us define

$$v_\alpha(x) = \alpha^{(2-n)/2} v(x/\alpha) \quad (4.1)$$

to be a one-parameter family of Sobolev extremals that concentrate at zero in Euclidean space as $\alpha \rightarrow 0$. After multiplying by a cutoff function and pulling back these functions with the concentration centered at a point that is not locally conformally flat, we show that the geometry forces $Q(v_\alpha) < \lambda(S^n)$. We need some background before we get into the proof.

4.1 The Weyl tensor

The Weyl tensor is a tensor whose coordinate expression is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) - \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il})$$

The Weyl tensor is the “traceless component of the Riemann tensor” in the sense that the full curvature tensor can be written in terms of the Weyl and Ricci tensors. The key facts about the Weyl tensor that we will use are:

1. The Weyl tensor is invariant under conformal changes of metric.
2. if $n = 3$, the Weyl tensor vanishes.
3. If $n > 3$, the Weyl tensor vanishes at a point if and only if (M, g) is locally conformally flat near p .

Recall that (M, g) is locally conformally flat near p if, up to a conformal change of metric, the curvature vanishes in a neighborhood of p .

4.2 Conformal normal coordinates

For any $p \in M$, we can choose normal coordinates in a neighborhood of p using the exponential map. In such coordinates, the metric is Euclidean at p and the Christoffel symbols vanish, so the metric is Euclidean up to first order. The Yamabe constant is conformally invariant, so we have the freedom to simplify the local geometry even more by first choosing a smart conformal metric and then using normal coordinates.

Theorem 4.1 (Conformal normal coordinates). *Let (M, g) be a closed Riemannian manifold and fix $p \in M$. For any $K \geq 2$, there exists a conformal metric \tilde{g} on M such that in \tilde{g} normal coordinates near p ,*

$$\det \tilde{g} = 1 + O(r^K).$$

Here we let $r = |x|$. Furthermore, if $K \geq 5$, then $\tilde{R} = O(r^2)$ and $-\Delta \tilde{R} = \frac{1}{6}|W|^2$.

The proof of Theorem 4.1 is by induction and is fairly involved. We will not prove it here, but refer the reader to [LP87, Theorem 5.1].

4.3 Proof of Theorem 1.2

Pick a point $p \in M$ where $|W(p)| \neq 0$. Without loss of generality, assume that $g = \tilde{g}$ where \tilde{g} is as in Theorem 4.1 and K is taken to be as large as we need in the proof. Let $\{x_i\}$ be conformal normal coordinates in $B_{2\epsilon}(p)$. Let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth cutoff function with $\eta = 1$ in B_ϵ and $\eta = 0$ in $B_{2\epsilon}$. We take as test functions

$$\varphi = v_\alpha \eta$$

where α small will be chosen later. In order to compute $Q(\varphi)$ in coordinates, let us make a few preliminary estimates. The following three expressions simply come from changing variables and Taylor expanding the integrands.

$$\int_{B_\epsilon} v_\alpha^{2^*} = 1 - \int_{B_{\epsilon/\alpha}^c} v_1^{2^*} = 1 - O\left(\int_{\epsilon/\alpha}^\infty r^{-2n} r^{n-1}\right) = 1 + O(\alpha^n) \quad (4.2)$$

$$\int_{B_\epsilon} |\nabla v_\alpha|^2 = \sigma_n^2 - \int_{B_{\epsilon/\alpha}^c} |\nabla v_1|^2 = \sigma_n^2 - O\left(\int_{\epsilon/\alpha}^\infty r^{-2n+2} r^{n-1}\right) = \sigma_n^2 + O(\alpha^{n-2}) \quad (4.3)$$

$$\int_{B_{2\epsilon} \setminus B_\epsilon} v_\alpha^2 = O(\alpha^{n-2}) \quad (4.4)$$

The third estimate could be sharpened a bit, but the first and, crucially, the second are completely sharp. This turns out to be an important point; see Remark 4.2 below.

By (4.2), we have

$$Q(\varphi) = \frac{\int c_n |\nabla(v_\alpha \eta)|^2 + \int R v_\alpha^2 \eta^2}{\left(\int_{B_{2\epsilon}} v_\alpha^{2^*} \eta^{2^*}\right)^{2/2^*}} = \left(\int c_n |\nabla(v_\alpha \eta)|^2 + \int R v_\alpha^2 \eta^2\right) (1 + O(\alpha^n))$$

In the first equality, there are also errors in α coming from the volume form, but we will ignore them since we can take them to be as high of order as we wish thanks to Theorem 4.1. By (4.3), (4.4) and Hölder's inequality,

$$\int c_n |\nabla(v_\alpha \eta)|^2 = \int c_n |\nabla v_\alpha|^2 \eta^2 + \int c_n \eta v_\alpha \nabla v_\alpha \cdot \nabla \eta + \int c_n v_\alpha |\nabla \eta|^2 = \sigma_n^2 c_n + O(\alpha^{n-2}).$$

Hence, recalling (2.3), we have shown that

$$Q(\varphi) = \sigma_n^2 c_n + \int R v_\alpha^2 \eta^2 + O(\alpha^{n-2}).$$

Now, using a Taylor expansion of R (which vanishes at first order because we are in normal coordinates) and making use of Theorem 4.1, we find that

$$\begin{aligned} \int R v_\alpha^2 \eta^2 &= \int_{B_\epsilon} R v_\alpha^2 + O(\alpha^{n-2}) = \int_0^\epsilon \int_{\partial B_r} \left(\frac{1}{2} R_{,ij}(p) x^i x^j + O(r^3)\right) v_\alpha^2 + O(n-2) \\ &= \int_0^\epsilon \{-C r^2 |W(p)|^2 + O(r^3)\} v_\alpha^2 r^{n-1} + O(\alpha^{n-2}) \\ &= -C |W(p)|^2 \int_0^\epsilon r^{n+1} v_\alpha^2 + O(\alpha^{n-2}). \end{aligned}$$

A direct computation shows that

$$\int_0^\epsilon r^{n+1} v_\alpha^2 = \begin{cases} \alpha^4 & \text{if } n > 6 \\ \alpha^4 \log(1/\alpha) & \text{if } n = 6 \\ \alpha^{n-2} & \text{if } n < 6 \end{cases}$$

Thus we have shown that

$$Q(\varphi) \leq \begin{cases} \lambda(S^n) - c|W(p)|^2 \alpha^4 + O(\alpha^{n-2}) & \text{if } n < 6 \\ \lambda(S^n) - c|W(p)|^2 \alpha^4 \log(1/\alpha) + O(\alpha^{n-2}) & \text{if } n = 6 \\ \lambda(S^n) + O(\alpha^{n-2}) & \text{if } n > 6 \end{cases}$$

In particular, since $|W(p)| > 0$ and $n \geq 6$, we see that

$$\lambda(M) \leq Q(\varphi) < \lambda(S^n).$$

concluding the proof of Theorem 1.2.

Remark 4.2. Observe why this proof does not do through for higher dimensions. The Sobolev extremals are not compactly supported, so when we multiply by a cutoff function and pull the functions back to M , we necessarily introduce some errors between $Q(\varphi)$ and $\lambda(S^n)$ coming from the tails of these functions. The tails of the v_α decay polynomially, with a dimensional dependence on the polynomial decay of v_α . When $n < 6$ the error from the cutoffs of v_α (in (4.2), (4.3), (4.4)) is too large to allow us to capture any geometry in the Taylor expansion of the energy.

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which is due to Schoen in [Sch84]. For omitted proofs in this section we refer the reader to [Sch84, LP87]. We begin with some background.

5.1 Stereographic projections

To begin, let us recall our analysis of $\lambda(S^n)$ in Section 2. In order to study this variational problem explicitly, we used stereographic projection to transfer the problem to Euclidean space. One particular reason this was fruitful was that Euclidean space has zero scalar curvature, so the variational problem (1.2) simply became the one corresponding to the Sobolev inequality.

Let us take the following perspective on the choice to use the stereographic projection map in Section 2. As before, let $\Psi : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the stereographic projection map as defined in (2.1), and let g_{euc} and g_0 respectively denote the Euclidean metric on \mathbb{R}^n and the round metric on S^n . Define G to be the conformal factor relating g_0 to the metric \hat{g} on S^n obtained by pulling back the Euclidean metric by Ψ , that is:

$$\hat{g} = \Psi^*(g_{euc}) = G^{4/(n-2)} g_0.$$

Since (\mathbb{R}^n, g_{euc}) has zero scalar curvature, we observe by (1.1) that

$$-c_n \Delta_{g_0} G + R_0 G = 0 \quad \text{on } S^n \setminus \{N\}.$$

Here $R_0 = n(n-1)$ is the scalar curvature of (S^n, g_0) . And in fact, one can show that

$$-c_n \Delta_{g_0} G + R_0 G = \delta_N \quad \text{on } S^n.$$

In other words, G is the Green's function at N for the conformal Laplacian $-c_n \Delta + R$ on the round sphere. The key idea of Schoen in [Sch84] was that, as long as the conformal Laplacian on (M, g) has a positive Green's function G , we can perform a sort of "stereographic projection" in the following way. Fix $p \in M$ and let $\hat{g} := G^{4/(n-2)} g$ on $\hat{M} := M \setminus \{p\}$. Then we call (\hat{M}, \hat{g}) , along with the map $\sigma : M \setminus \{p\} \rightarrow M$, the stereographic projection of M from p . By construction and (1.1), we have

$$R_{\hat{g}} = 0 \quad \text{on } \hat{M}.$$

If (M, g) is a closed Riemannian manifold with $\lambda(M) \geq 0$, then the conformal Laplacian has a positive Green's function. Recall that in Example 1.4, we showed that the Yamabe problem is easily solved if $\lambda(M) \leq 0$. Therefore we may assume the existence of a positive Green's function.

Given a fixed point $p \in M$, without loss of generality, we may assume that g is the conformal metric in which normal coordinates give the conformal normal coordinates of Theorem 4.1. Let $\{x^i\}$ be conformal normal coordinates for (M, g) at p . In the setting of Theorem 1.3, we have the following expansion for the Green's function G with singularity at p .

Theorem 5.1. *Let $n = 3, 4, 5$ or assume M is conformally flat in a neighborhood of p . Let $\{x^i\}$ be conformal normal coordinates at p . Then*

$$G = r^{2-n} + A + O''(r) \quad \text{as } r \rightarrow 0.$$

Here, A is a constant and $r = |x|$.

The notation $O''(r)$ indicates corresponding scale invariant decay for two derivatives. That is, $f(r) = O''(r)$ if $f(r) = O(r)$, $f'(r) = O(1)$, and $f''(r) = O(1/r)$. We refer to the survey [LP87, Lemma 6.4] for the proof of Theorem 5.1. Theorem 5.1 gives rise to an expansion of the metric \hat{g} in related coordinates. Indeed, consider the inverted normal coordinates $\{z^i\}$ given by $z^i = r^{-2} x^i$ on $U \setminus \{p\}$. Letting $\rho = |z|$, we can derive from Theorem 5.1 that

$$\hat{g}_{ij}(z) = (1 + A\rho^{2-n} + O''(\rho^{1-n}))^{4/(n-2)} (\delta_{ij} + O''(\rho^2)) \quad (5.1)$$

when $n = 3, 4, 5$ or M is locally conformally flat. We recall that a Riemannian manifold (N, g) is said to be asymptotically flat to order $\tau > 0$ if there is a decomposition $N = N_0 \cup N_\infty$ such that N_0 is compact and N_∞ is diffeomorphic to $\mathbb{R}^n \setminus B_r$ for some $r > 0$ with

$$g_{ij} = \delta_{ij} + O''(\rho^{-\tau}) \quad \text{as } \rho \rightarrow \infty.$$

Here $\rho = |z|$ where $\{z^i\}$ are coordinates induced by the aforementioned diffeomorphism. (However, being asymptotically flat turns out to be independent of the choice of diffeomorphism.) So, (5.1) in particular implies that (\hat{M}, \hat{g}) is asymptotically flat to order τ with

$$\tau = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } n = 4, 5, \\ n-2 & \text{if } M \text{ is locally conformally flat.} \end{cases} \quad (5.2)$$

5.2 Constructing the test functions

In view of (5.1), we are able to understand the geometry of (\hat{M}, \hat{g}) well near infinity. With this in mind, the test functions we construct in order to prove Theorem 1.3 will “localize” an extremal function for the Sobolev inequality near infinity. To this end, we define

$$\varphi_\alpha(z) = \begin{cases} v_\alpha(z) & \text{if } \rho \geq R \\ v_\alpha(R) & \text{if } \rho \leq R \end{cases} \quad (5.3)$$

where v_α are the scalings of Sobolev extremal functions defined in (4.1). Instead of taking the parameter α to be small (and hence v_α to be concentrated) as before, we will now consider α large and hence v_α spread out.

We wish to compute $Q(\varphi_\alpha)$ asymptotically as $\alpha \rightarrow \infty$. Since φ_α is constant on spheres and concentrates on large spheres for α large, it is natural to expect that $Q(\varphi_\alpha)$ will depend on some average behavior of \hat{g} over large spheres. And indeed, we will find that the leading order term in the expansion of $Q(\varphi_\alpha)$, will involve a quantity μ , called the distortion coefficient, which is defined in the following way. Set

$$h(\rho) = \frac{1}{n\omega_n\rho^{n-1}} \int_{S_\rho} d\sigma_\rho$$

where S_ρ is a geodesic sphere of radius ρ with respect to \hat{g} and $d\sigma_\rho$ is the volume element on S_ρ induced by \hat{g} . The function $h(\rho)$ measures the ratio of the volumes of geodesic spheres of radius ρ in (\hat{M}, \hat{g}) to those in Euclidean space. When $n = 3, 4, 5$ or M is locally conformally flat, making use of Theorem 5.1 and (5.1), one comes to the expansion

$$h(\rho) = 1 + (\mu/k)\rho^{-k} + O''(\rho^{-k-1})$$

where k depends on the dimension. The constant μ is the *distortion coefficient of \hat{g}* . In the following theorem, we see that the distortion coefficient appears at leading order in the expansion of $\lambda(S^n) - Q(\varphi_\alpha)$.

Theorem 5.2. *Suppose $n = 3, 4, 5$ or M is locally conformally flat. Let φ_α be as defined in (5.3). Then there exists a positive dimensional constant C such that*

$$Q(\varphi_\alpha) \leq \lambda(S^n) - C\mu\alpha^k + O(\alpha^{-k-1}) \quad (5.4)$$

as $\alpha \rightarrow \infty$.

It turns out that when $n = 3, 4, 5$ or that M is locally conformally flat in a neighborhood of p , the distortion coefficient μ coincides up to a factor of two with a quantity called the *mass* $m(\hat{g})$, which arises in general relativity. A version of the positive mass theorem of Schoen and Yau states the following.

Theorem 5.3 (Positive Mass Theorem). *Suppose $n \geq 3$ and (N, g) is asymptotically flat to order $\tau > (n - 2)/2$ with nonnegative scalar curvature. Then $m(g) \geq 0$. Furthermore, $m(g) = 0$ if and only if (N, g) is isometric to $(\mathbb{R}^n, g_{\text{euc}})$.*

We now can conclude the proof of Theorem 1.3. Indeed, if (\hat{M}, \hat{g}) is isometric to (\mathbb{R}^n, g_{euc}) , then necessarily (M, g) is conformally equivalent to (S^n, g_0) . On the other hand, suppose that (\hat{M}, \hat{g}) is not isometric to (\mathbb{R}^n, g_{euc}) . Recalling (5.2) and that $R_{\hat{g}} = 0$ by construction, we can apply Theorem 5.3 to find that $\mu > 0$. Then, by taking α sufficiently large in (5.4), we find that $\lambda(M) < \lambda(S^n)$ and the proof of Theorem 1.3 is complete.

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