On minimizers and critical points for anisotropic isoperimetric problems

Robin Neumayer

Abstract Anisotropic surface energies are a natural generalization of the perimeter functional that arise, for instance, in scaling limits for certain probabilistic models on lattices. We survey two recent results concerning isoperimetric problems with anisotropic surface energies. The first is joint work with Delgadino, Maggi, and Mihaila and provides a weak characterization of critical points in the anisotropic isoperimetric problem. The second is joint work with Choksi and Topaloglu and describes energy minimizers in an anisotropic variant of a model for atomic nuclei.

1 Introduction

The Euclidean isoperimetric problem, in which one minimizes the perimeter among sets of a fixed volume, is one of the most classical problems in mathematics and its study dates back over two millennia. In the language of modern calculus of variations, it is the minimization problem

$$\inf\{P(E) : |E| = 1\}, \tag{1}$$

where $E \subset \mathbb{R}^n$ is a set of finite perimeter and $P(E)$ is the distributional perimeter; see [27]. Modulo translations, the unique minimizer of (1) is the ball of volume one. Rephrased in a scaling invariant way, this fact gives the isoperimetric inequality:

$$P(E) \geq n|B|^{1/n}|E|^{(n-1)/n},$$

with equality if and only if E is a translation or dilation of the unit ball B.

Robin Neumnayer
Institute for Advanced Study, e-mail: neumayer@ias.edu

Name of Second Author
Name, Address of Institute e-mail: name@email.address

Anisotropic surface energies are a natural generalization of the perimeter functional, which frequently arise in models for equilibrium shapes of crystalline materials and in scaling limits of probabilistic models on lattices [5, 2]. Given a *surface tension* $f : \mathbb{R}^n \to \mathbb{R}$, i.e. a convex, positively one-homogeneous function with $f|_{S^{n-1}} > 0$, the *anisotropic surface energy* of a smooth open set $E \subset \mathbb{R}^n$ is given by

$$\mathscr{F}(E) = \int_{\partial E} f(\nu_E)\, d\mathscr{H}^{n-1}$$

where ν_E is the outer unit normal to E. The definition extends to sets of finite perimeter by integrating over the reduced boundary $\partial^* E$ and taking ν_E to be the measure theoretic outer unit normal. One can then study the corresponding minimization problem

$$\inf\{\mathscr{F}(E) : |E| = 1\}. \tag{2}$$

This problem is known as the Wulff problem, so named for the Russian crystallographer George Wulff who in 1901 conjectured the form of energy minimizers [32]. The unique minimizer, modulo translations, is given by the *Wulff shape*

$$K = \bigcap_{\nu \in S^{n-1}} \{x \cdot \nu < f(\nu)\};$$

see [4, 15, 16, 30, 31]. This bounded convex set K plays the role of the ball in the anisotropic setting. As with the Euclidean isoperimetric problem, one can express this minimality in the scaling invariant form

$$\mathscr{F}(E) \geq n|K|^{1/n}|E|^{(n-1)/n}, \tag{3}$$

with equality if and only if $E = rK + x$ for some $r > 0$ and $x \in \mathbb{R}^n$.

Of course, the case when f is the Euclidean norm corresponds to the classical notion of perimeter, in which case the Wulff shape is a ball. A less trivial example comes from the class of *smooth, elliptic* surface tensions: those that are smooth on $\mathbb{R}^n \setminus \{0\}$ and are (λ-)elliptic in the sense that

$$\lambda\, \mathrm{Id} \leq \nabla^2 f(\nu) \leq \lambda^{-1}\, \mathrm{Id} \qquad \text{on } \nu^\perp \ \ \forall \nu \in \mathbb{S}^{n-1}.$$

The Wulff shapes for such norms are smooth and uniformly convex. From an analytic perspective, the surface energies arising from smooth elliptic norms share many desirable properties with the perimeter functional. However, many examples of surface tensions are not smooth nor elliptic; in fact, every bounded convex set is the Wulff shape for some surface energy. In typical applications, the physically relevant surface tensions are *crystalline surface tensions*, those that are the maximum of finitely many linear functions. Wulff shapes corresponding to crystalline norms are convex polyhedra.

So, from both applied and theoretical perspectives, an important question is to understand *which structural aspects of anisotropic isoperimetric problems are dic-*

tated by the smoothness and ellipticity of the surface tension, and which are preserved when these assumptions are relaxed.

One example of a property that is *independent* of smoothness and ellipticity assumptions is seen through the work of [14]. The main result there states that the deviation of a set from achieving equality in (3) quadratically controls the distance of a set to a homothety of the Wulff shape. More specifically,

$$\frac{\mathscr{F}(E)-n|K|^{1/n}|E|^{(n-1)/n}}{n|K|^{1/n}|E|^{(n-1)/n}} \geq c \inf_{x\in\mathbb{R}^n}\left\{\frac{|E\Delta(rK+x)|}{|E|} : |rK|=|E|\right\}^2 .$$

Remarkably, the constant c depends only on the dimension. This can been seen as a uniform convexity property of the energy profile of $\mathscr{F}(E)$ near the global minimizer: after modding out by translations and dilations, the energy $\mathscr{F}(E)$ grows from its global minimum quadratically in the symmetric difference, with a modulus of convexity is independent of the surface energy.

On the other hand, the following example provides a property that is *dependent* on ellipticity. If f is an elliptic surface tension[1], then for any set of finite perimeter E and any half-space H intersecting E nontrivially (i.e. $|E\cap H|>0$ and $|E\setminus H|>0$), one has

$$\mathscr{F}(E) > \mathscr{F}(E\cap H). \tag{4}$$

In particular, if one considers the Plateau problem with respect to $\mathscr{F}(E)$ and with boundary data, say, a copy of S^{n-2} that is contained in a hyperplane, then the unique solution is given by the $(n-1)$-dimensional ball contained in this hyperplane. One sees (4) from the following simple calibration argument. For simplicity, say that E is smooth. Let ν_H be the outer unit normal to H and let $-x_0\in\mathbb{R}^n$ be the slope of a supporting hyperplane to the convex function f at ν_H. The ellipticity of f ensures that the hyperplane with slope x_0 is a supporting hyperplane to f at exactly one $\nu_0\in S^{n-1}$, and that $f(\nu)>\nu\cdot x_0$ for every other $\nu\in S^{n-1}$. So

$$\mathscr{F}(E)-F(E\cap H) = \int_{\partial E\setminus H} f(\nu_E) - \int_{\partial H\cap E} f(\nu_H) \tag{5}$$

$$> \int_{\partial E\setminus H} x_0\cdot\nu_E - \int_{\partial H\cap E}(-x_0)\cdot\nu_H = \int_{\partial R} x_0\cdot\nu_R, \tag{6}$$

where we let $R=E\setminus H$. Now, by the divergence theorem we see that the right-hand side is equal to zero, which establishes (4). In contrast, in the absence of ellipticity assumptions on f, one can construct examples of quite dramatic failure of uniqueness for Plateau's problem. For instance, considering $f(\nu)=\|\nu\|_{\ell^\infty}$ in $\mathbb{R}^2$, the line segment joining $(-1,0)$ and $(1,0)$ has the same energy as the segment joining $(-1,0)$ and $(0,1)$ union the segment joining $(0,1)$ and $(1,0)$.

Here, we survey two results concerning minimizers and critical points of anisotropic isoperimetric problems. The first, which is joint work with Delgadino, Maggi, and

[1] One can actually assume slightly less; f needs only to be strictly convex is directions orthogonal in tangential directions to its level sets.

Mihaila [11], points toward a further properties of the energy profile of $\mathscr{F}(E)$ that are independent of smoothness and ellipticity. The second, which is joint work with Choksi and Topaloglu [7], demonstrates a variational problem in which the character of energy minimizers depends crucially on smoothness and ellipticity assumptions on the surface tension.

2 Critical points in the Wulff problem

Suppose f is a smooth elliptic surface tension. Then the first variation of the surface energy $\mathscr{F}(E)$ for a variation with initial velocity $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ is given by

$$\delta\mathscr{F}(E)[X] = \int_{\partial^* E} \mathrm{div}_\tau(\nabla f(\nu_E))\, X \cdot \nu_E .$$

Here $\mathrm{div}_\tau X$ denotes the tangential divergence. Notice that when f is the Euclidean norm, this is the usual first variation of perimeter, and we call

$$H_E^f := \mathrm{div}_\tau(\nabla f(\nu_E))$$

the *anisotropic mean curvature of E* in analogy with the isotropic case. If E is a critical point of $\mathscr{F}(E)$ with respect to variations that preserve area, then

$$H_E^f = \mathrm{const} \quad \text{ on } \partial^* E.$$

In such a case, this constant is given by

$$H_0 := \frac{(n-1)\mathscr{F}(E)}{n|E|}.$$

A celebrated theorem of Aleksandrov [1] says that the only smooth, bounded, connected, embedded hypersurfaces in $\mathbb{R}^n$ of constant mean curvature are spheres. Or, in the language of the present setting, if a smooth, bounded, connected set is a critical point of the perimeter among variations that preserve volume, then E is a ball. For smooth elliptic norms, the analogous result was shown in [20]: a smooth bounded connected set E with constant anisotropic mean curvature H_E^f is a translation or dilation of the Wulff shape.

For a generic surface tension, the interpretation of what constant anisotropic mean curvature means is a subtle issue in itself; if a surface tension is not C^1, then the first variation is not even a linear functional. Following the common theme in analysis, one may interpret "constant anisotropic mean curvature" via approximation by smooth objects. Given any surface tension f, we approximate f point-wise by a sequence of smooth λ_h-elliptic norms $\{f_h\}$. (Note that necessarily $\lambda_h \to 0$ if f fails to be smooth or elliptic.) We quantify the L^2 deficit of a smooth set from having constant $H_E^{f_h}$ with the scaling invariant quantity

$$\delta_{f_h}(E) = \left(\fint_{\partial E} \left| \frac{H_E^{f_h}}{H_0} - 1 \right|^2 \right)^{1/2}.$$

A natural definition for a set E to have constant f-mean curvature is for E to be approximated in L^1 by smooth, bounded sets E_h with $\delta_{f_h}(E_h) \to 0$. The following theorem, proven in [11], roughly says that such a set must be the union of Wulff shapes.

Theorem 1. *Let f be any surface tension and let K be the corresponding Wulff shape. Let $\{f_h\}$ to be a sequence of smooth λ_h-elliptic norms approximating f in a point-wise sense. Suppose $\{E_h\}$ is a sequence of smooth, bounded open sets normalized to have $H_0 = n$ that satisfy $H_{E_h}^{f_h} \geq \varepsilon$ on ∂E_h, $\sup \operatorname{diam}(E_h) < \infty$, and $\mathscr{F}_h(E_h) \leq L \mathscr{F}_h(K)$. If*

$$\lambda_h^{-2}\, \delta_h(E_h) \to 0$$

and $E_h \to E$ in L^1, then

$$E = \bigcup_{i=1}^{M} (K + x_i), \tag{7}$$

where $M \leq L$ and the $K + x_i$ are pairwise disjoint.

The fact that E is a finite union of Wulff shapes, instead of just one, is an instance of the type of bubbling phenomenon exhibited in many geometric variational problems, and is an artifact of only using first order information. On the other hand, while our proof requires $\delta_h(E_h)$ to converge to zero faster than the loss of ellipticity of f_h, we expect that this assumption is purely technical and the result should likely hold if one simply assumes that $\delta_h(E_h) \to 0$.

We refer the reader to [11] for the proof of Theorem 1, and here we attempt only indicate some of the key ideas. The starting point for proving Theorem 1 is an anisotropic version of the Heintze-Karcher inequality, which states the following. Let f be a smooth, elliptic surface tension and let E be a smooth, bounded, connected set with $H_E^f > 0$. Then

$$\int_{\partial E} \frac{n-1}{H_E^f} f(\nu_E)\, d\mathscr{H}^{n-1} \geq n|E|, \tag{8}$$

with equality if and only if $E = rK + x$. Notice that (8) implies the result of [20] (indeed, this is the method of proof in [20]). Indeed, if H_f^E is constant, then it is not difficult to show that this constant must be

$$H_f^E = \frac{(n-1)\mathscr{F}(E)}{n|E|}.$$

Plugging this into the left-hand side of (8), we immediately see that such a set achieves equality in (8) and thus is a homothety of the Wulff shape. A further key point is that, provided $H_E^f \geq \varepsilon$, the scale-invariant deficit $\delta(E)$ from having constant anisotropic mean curvature controls the deficit from equality in (8). As such, a crucial part of the proof of Theorem 1 is a quantitative analysis of (8).

To derive quantitative estimates for sets almost achieving equality in (8), it is fruitful to trace through a PDE proof of the inequality, which is due to Ros in the isotropic case [29]. In this proof, we consider the solution to the equation

$$\begin{cases} L_f u = 1 & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases} \tag{9}$$

where the elliptic operator L_f is given by

$$L_f u = \operatorname{div}\left(\nabla f^2/2(\nabla u)\right).$$

When $E = K$, the solution is given by

$$u_K(x) = \frac{f_*(x)^2}{2n}.$$

where $f_*(x) = \sup\{x \cdot \nu : f(\nu) < 1\}$ is the dual norm to f.

The above discussion holds only for smooth and elliptic surface tensions. In the proof of Theorem 1, we solve the equation (9) with $f = f_h$ and $E = E_h$. The main idea of the proof is to show that these solutions u_h this function converge to a sum of (translations of) the model function u_K corresponding to the limit surface energy, and from there deduce that the support of this limit function is the L^1 limit of E_h and takes the form (7). To this end, we first establish Lipschitz estimates allowing us to produce a C^0 limit u of the sequence $\{u_h\}$. We then prove quantitative estimates from (8) that allow us to show that u is supported on a countable union of disjoint Wulff shapes $r_iK + x_i$, possibly of different radii r_i. Some of the more difficult analysis comes into showing that all the radii are equal to one. In this step, we establish a family of Pohozaev-type identities involving integral quantities of ∇u_h. With a somewhat delicate Young measure argument, we can pass these identities to the limit function ∇u, despite having only weak-$*$ convergence in L^∞ for the gradients. Then, pairing these identities with a scaling argument allows us to conclude the $r_i = 1$ for all i.

3 Minimizers in the anisotropic liquid drop model

Gamow's liquid drop model [19] is among the principal models, along with the shell and cluster models, used to describe atomic nuclei [10]. (None of these models come from first principles, and none individually can be used to describe all observed phenomena.) In its simplest form, the liquid drop model assumes that the nucleus of an atom minimizes an energy comprising the sum of a perimeter term and a Coulombic self-interaction term:

$$\inf\{P(E)+\mathscr{V}(E):\ |E|=m\}. \tag{10}$$

Here, for a fixed parameter $\alpha \in (0,n)$, $\mathscr{V}(E)$ is a nonlocal repulsion term defined by

$$\mathscr{V}(E)=\int_E\int_E \frac{dx\,dy}{|x-y|^{\alpha}}.$$

The physical case is $n=3$ and $\alpha=1$, corresponding to a Coulombic potential in three dimensional space. This model predicts that nuclei of small mass are spherical and nuclei of sufficiently large mass do not exist.

While the liquid drop model was introduced in the 1930s, the variational problem (10), its study in the calculus of variations community has mostly been concentrated in the past decade. Due primarily in large part to the work of Knüpfer and Muratov in [22, 23], along with important contributions [3, 8, 18, 21, 25, 12], the state of the art for global minimizers of (10) is as follows: For any $n \geq 2$, we have:

1. for all $\alpha \in (0,n)$ there exists $m_1 > 0$ such that if $m \leq m_1$, then the problem admits a minimizer;
2. for all $\alpha \in (0,n)$ there exists $m_0 > 0$ such that if $m \leq m_0$, then the minimizer is uniquely (modulo translations) given by the ball of mass m; and
3. for all $\alpha \in (0,2)$ there exists $m_2 > 0$ such that if $m > m_2$, then no minimizer exists.

It is conjectured in [9] that $m_0 = m_1 = m_2$ when $n=3$ and $\alpha=1$. While the conjecture remains open, it was shown in [3] that $m_0 = m_1 = m_2$ in any dimension for α sufficiently small. It also remains open whether the nonexistence result (iii) can be extended to $\alpha \in [2,n)$.

Nuclei can exhibit distortions from a spherical shape, and some of the physics literature [26] suggests that instead this is due to the fact that *"nuclei may possess anisotropic surface tension."* This motivates the replacement of the perimeter functional by an anisotropic surface energy in [7], leading to the minimization problem

$$\inf\{\mathscr{F}(E)+\mathscr{V}(E):\ |E|=m\}. \tag{11}$$

The properties (i) and (iii) for (10) are, at their core, consequences of the inhomogeneous scaling of the energy $P(E)+\mathscr{V}(E)$ with respect to dilations. As the anisotropic surface energy scales in the same way as perimeter, it comes as no surprise that in [7] we readily establish analogous existence and nonexistence properties for (11). More interesting is the question of what form property (ii) should take in the setting of (11). Given that the Wulff shape plays the role of the ball for the anisotropic surface energy, is natural to wonder whether if it is a minimizer of (11). We show in [7] that the answer depends in a crucial way on the regularity and ellipticity of the surface tension:

Theorem 2. *Fix $n \geq 2$ and $\alpha \in (0, n-1/3)$, and $m > 0$. Suppose f is a smooth elliptic surface tension with Wulff shape K. Then K is a critical point of* (11) *if and only if f is the Euclidean norm.*

Theorem 3. *Let $n = 2$ and $f(\nu) = \|\nu\|_{\ell^1}$. There exists m_2 such that if $m \leq m_2$ then the Wulff shape is the unique minimizer of* (11).

This result is an instance where, for smooth, elliptic nature of the problem is notably different than the isotropic case, and the character of minimizers depends crucially on the regularity and ellipticity of the surface tension.

The proof of of Theorem 2 makes use of a first variation argument, and in fact we prove a stronger statement: if the Wulff shape is a critical point of (11) for any mass m, then f is the Euclidean norm. Indeed, a critical point satisfies

$$H_E^f + v_E = \text{const} \quad \text{on } \partial^* E \tag{12}$$

where $v_E(x) = \int_E |x-y|^{-\alpha}\, dx$ is the first variation of $\mathscr{V}(E)$. One directly computes that $H_K^f = n-1$. So, if K satisfies (12), then

$$v_K = \text{const} \qquad \text{on } \partial K$$

Theorem 2 then follows from the following characterization:

Proposition 1. *Fix $n \geq 2$ and $\alpha \in (0, n-1/3)$. Let K be a smooth set with v_K =const on ∂K. Then K is a ball.*

Proposition 1 was established for the Coulombic case $\alpha = n-2$ in [17] and was extended to $\alpha \in (0, n-1)$ in [24]. Both proofs use the method of moving planes; see also [28]. The case when $\alpha \geq n-1$ is significantly more delicate, principally due to the fact that the Riesz potential v_K is merely Hölder continuous in this case. Our proof of Proposition 1 in the subtler case $\alpha \in [n-1, n-1/3)$ pairs the method of moving planes on integral forms in the spirit of [24, 6] with some new reflection arguments and estimates on how the Riesz potential v_K grows compared to its reflection across a hyperplane.

In the setting of Theorem 3, the Wulff shape is a square in $\mathbb{R}^2$ with sides aligning with the coordinate axes. The proof of Theorem 3 makes use of a structure theorem proven in [13] for crystalline surface energies in $\mathbb{R}^2$. This result says that suitably defined quasi-minimizers of such a crystalline surface energy must be convex polyhedra with whose set of normal vectors is contained in the set of normal vectors of the Wulff shape. Pairing this result with a compactness argument, we are able to deduce that any minimizer of (11) for small enough mass in the setting of Theorem 3 is a rectangle with side aligning with the coordinate axes. From here, the study of minimizers of (11) essentially reduces to a one-dimensional variational problem, and the analysis can be done in a quite explicit way.

References

1. Aleksandrov, A.D.: Uniqueness theorems for surfaces in the large. V. Vestnik Leningrad. Univ. **13**(19), 5–8 (1958)

2. Auffinger, A., Damron, M., Hanson, J.: 50 years of first-passage percolation, *University Lecture Series*, vol. 68. American Mathematical Society, Providence, RI (2017)
3. Bonacini, M., Cristoferi, R.: Local and global minimality results for a nonlocal isoperimetric problem on $\mathbb{R}^N$. SIAM J. Math. Anal. **46**(4), 2310–2349 (2014)
4. Brothers, J.E., Morgan, F.: The isoperimetric theorem for general integrands. Michigan Math. J. **41**(3), 419–431 (1994). DOI 10.1307/mmj/1029005070. URL http://dx.doi.org.ezproxy.lib.utexas.edu/10.1307/mmj/1029005070
5. Cerf, R.: The Wulff crystal in Ising and percolation models, *Lecture Notes in Mathematics*, vol. 1878. Springer-Verlag, Berlin (2006). Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With a foreword by Jean Picard
6. Chen, W., Li, C., Ou, B.: Classification of solutions for an integral equation. Comm. Pure Appl. Math. **59**(3), 330–343 (2006). DOI 10.1002/cpa.20116. URL https://doi.org/10.1002/cpa.20116
7. Choksi, R., Neumayer, R., Topaloglu, I.: Anisotropic liquid drop models. Preprint available at arXiv:1810.08304
8. Choksi, R., Peletier, M.: Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional. SIAM J. Math. Anal. **42**(3), 1334–1370 (2010). DOI 10.1137/090764888. URL http://dx.doi.org/10.1137/090764888
9. Choksi, R., Peletier, M.: Small volume-fraction limit of the diblock copolymer problem: II. Diffuse-interface functional. SIAM J. Math. Anal. **43**(2), 739–763 (2011). DOI 10.1137/10079330X. URL http://dx.doi.org/10.1137/10079330X
10. Cook, N.: Models of the Atomic Nucleus: Unification Through a Lattice of Nucleons. Springer Berlin Heidelberg (2010). URL https://books.google.com/books?id=CwRGogWF5-oC
11. Delgadino, M.G., Maggi, F., Mihaila, C., Neumayer, R.: Bubbling with L^2-Almost Constant Mean Curvature and an Alexandrov-Type Theorem for Crystals. Arch. Ration. Mech. Anal. **230**(3), 1131–1177 (2018). DOI 10.1007/s00205-018-1267-8. URL https://doi-org.turing.library.northwestern.edu/10.1007/s00205-018-1267-8
12. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. Comm. Math. Phys. **336-1**, 441–507 (2015)
13. Figalli, A., Maggi, F.: On the shape of liquid drops and crystals in the small mass regime. Arch. Ration. Mech. Anal. **201**(1), 143–207 (2011). DOI 10.1007/s00205-010-0383-x. URL http://dx.doi.org/10.1007/s00205-010-0383-x
14. Figalli, A., Maggi, F., Pratelli, A.: A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math. **182**(1), 167–211 (2010). DOI 10.1007/s00222-010-0261-z. URL http://dx.doi.org/10.1007/s00222-010-0261-z
15. Fonseca, I.: The Wulff theorem revisited. Proc. Roy. Soc. London Ser. A **432**(1884), 125–145 (1991). DOI 10.1098/rspa.1991.0009. URL http://dx.doi.org.ezproxy.lib.utexas.edu/10.1098/rspa.1991.0009
16. Fonseca, I., Müller, S.: A uniqueness proof for the Wulff theorem. Proc. Roy. Soc. Edinburgh Sect. A **119**(1-2), 125–136 (1991). DOI 10.1017/S0308210500028365. URL http://dx.doi.org.ezproxy.lib.utexas.edu/10.1017/S0308210500028365
17. Fraenkel, L.E.: An introduction to maximum principles and symmetry in elliptic problems, *Cambridge Tracts in Mathematics*, vol. 128. Cambridge University Press, Cambridge (2000). URL https://doi.org/10.1017/CBO9780511569203
18. Frank, R.L., Lieb, E.H.: A compactness lemma and its application to the existence of minimizers for the liquid drop model. SIAM J. Math. Anal. **47**(6), 4436–4450 (2015)
19. Gamow, G.: Mass defect curve and nuclear constitution. Proc. R. Soc. Lond. A **126**(803), 632–644 (1930). DOI 10.1098/rspa.1930.0032. URL http://rspa.royalsocietypublishing.org/content/126/803/632
20. He, Y., Li, H., Ma, H., Ge, J.: Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. Indiana Univ. Math. J. **58**(2), 853–868 (2009). DOI 10.1512/iumj.2009.58.3515. URL http://dx.doi.org/10.1512/iumj.2009.58.3515
21. Julin, V.: Isoperimetric problem with a Coulomb repulsive term. Indiana Univ. Math. J. **63**(1), 77–89 (2014). URL https://doi.org/10.1512/iumj.2014.63.5185

22. Knüpfer, H., Muratov, C.B.: On an isoperimetric problem with a competing nonlocal term I: The planar case. Comm. Pure Appl. Math. **66**(7), 1129–1162 (2013). URL https://doi.org/10.1002/cpa.21451
23. Knüpfer, H., Muratov, C.B.: On an isoperimetric problem with a competing nonlocal term II: The general case. Comm. Pure Appl. Math. **67**(12), 1974–1994 (2014). URL https://doi.org/10.1002/cpa.21479
24. Lu, G., Zhu, J.: An overdetermined problem in Riesz-potential and fractional Laplacian. Nonlinear Anal. **75**(6), 3036–3048 (2012). URL https://doi.org/10.1016/j.na.2011.11.036
25. Lu, J., Otto, F.: Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model. Comm. Pure Appl. Math. **67**(10), 1605–1617 (2014). URL https://doi.org/10.1002/cpa.21477
26. Mackie, F.D.: Anisotropic surface tension and the liquid drop model. Nuclear Physics A **245**(1), 61–86 (1975). DOI https://doi.org/10.1016/0375-9474(75)90082-2. URL http://www.sciencedirect.com/science/article/pii/0375947475900822
27. Maggi, F.: Sets of finite perimeter and geometric variational problems, *Cambridge Studies in Advanced Mathematics*, vol. 135. Cambridge University Press, Cambridge (2012). URL https://doi.org/10.1017/CBO9781139108133. An introduction to geometric measure theory
28. Reichel, W.: Characterization of balls by Riesz-potentials. Ann. Mat. Pura Appl. (4) **188**(2), 235–245 (2009). URL https://doi.org/10.1007/s10231-008-0073-6
29. Ros, A.: Compact hypersurfaces with constant higher order mean curvatures. Rev. Mat. Iberoamericana **3**(3-4), 447–453 (1987)
30. Taylor, J.E.: Existence and structure of solutions to a class of nonelliptic variational problems. In: Symposia Mathematica, Vol. XIV (Convegno di Teoria Geometrica dell'Integrazione e Varietà Minimali, INDAM, Roma, Maggio 1973), pp. 499–508. Academic Press, London (1974)
31. Taylor, J.E.: Unique structure of solutions to a class of nonelliptic variational problems. In: Differential geometry (Proc. Sympos. Pure. Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 1, pp. 419–427. Amer. Math. Soc., Providence, R.I. (1975)
32. Wulff, G.: Zur frage der geschwindigkeit des wachsturms und der auflösungder kristallflächen. Z. Kristallogr. **34**, 449–530 (1901)