Talbot 2017:

Obstruction theory for structured ring spectra

Workshop mentored by Maria Basterra and Sarah Whitehouse

(notes from talks given at the workshop)

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Disclaimer and Acknowledgements

These are notes we took during the 2017 Talbot Workshop (with the exception of talks 12 and 14, which are independent writeups by their respective authors). We, not the speakers, bear responsibility for mistakes. If you do find any errors, please report them to:

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Part I: Introduction

Talk 0: Preliminaries (Dylan Wilson)

This "lecture" is not one of the planned talks, but rather is based on Dylan's answers to background questions, given during an informal questions seminar early in the workshop.

0.1. What is a spectrum? Here is a bad answer.

Definition 0.1. A spectrum is a sequence of spaces $\{X_n\}$ together with maps $\Sigma X_n \to X_{n+1}$.

What is a *map* between these? A first guess is that a map between spectra is a bunch of maps $X_n \to Y_n$ making various diagrams commute, either strictly or up to homotopy. There's a way to make that work, but it kind of obscures what's going on in the most accessible example: stable maps between finite complexes. After all, whatever stable homotopy theory is, it had better tell us that the stable maps $S^n \to S^0$ are computed as $\operatorname{colim} \pi_{n+k} S^k$. Taking $X_m = \Sigma^m S^n$ and $Y_m = S^m$ tells us that our first guess does *not* produce this answer. (Exercise!)

Remark 0.2. If we take as our starting point that spectra should represent cohomology theories, then the first guess is more reasonable, though there are still some subtleties about the difference between the category of spectra and the category of cohomology theories.

So what's our next guess? Let $X = \{X_n\}$ and $Y = \{Y_n\}$ be spectra. What is the set of stable maps, $\{X, Y\}$? You could say it's $\operatorname{colim}[X_n, Y_n]$, where, given $X_n \to Y_n$, we suspend it – but we only have $X_{n+1} \leftarrow \Sigma X_n \to \Sigma Y_n \to Y_{n+1}$, so we don't get a map $X_{n+1} \to Y_{n+1}$, which is what we need to form the colimit!

But what if X was very nice, say $X = \{\Sigma^n X_0\} = \Sigma^\infty X_0$ (this is called a suspension spectrum)? Then we could actually form this colimit, because I have the identity map $X_{n+1} = \Sigma X_n$. This idea of a map still doesn't work, but for a more subtle reason: you can define this category, but it turns out to be useless.

What's the problem? If X_0, Y_0 are finite cell complexes, then it's OK to take $\{\Sigma^\infty X_0, \Sigma^\infty Y_0\} = \operatorname{colim}[\Sigma^n X_0, \Sigma^n Y_0]$. In this case, one checks by adjunction that

$$\{\Sigma^\infty X_0, \Sigma^\infty Y_0\} = \operatorname{colim}[X_0, \Omega^n \Sigma^n Y_0].$$

Since X_0 is a finite complex (and in particular compact) we can rewrite this last bit as

$$\operatorname{colim}[X_0, \Omega^n \Sigma^n Y_0] = [X_0, \operatorname{colim} \Omega^n \Sigma^n Y_0].$$

When X_0 is not compact, not every map $X_0 \to \operatorname{colim} \Omega^n \Sigma^n Y_0$ is defined at some finite stage. Whatever our definition is for maps between spectra, it should specialize to

$$\{\Sigma^\infty X_0, \Sigma^\infty Y_0\} \cong [X_0, \operatorname{colim} \Omega^n \Sigma^n Y_0]$$

in the case of suspension spectra. The intuition here is that a spectrum is supposed to be a sort of colimit of all this data $X = \{X_n\}$, although that's not really precise. A precise statement, if we know what a homotopy colimit is, would be that we want:

$$X = \operatorname{hocolim} \Sigma^{-n}\Sigma^{\infty} X_n.$$

This essentially forces the definition of maps between spectra in general.

The idea behind the different (and correct) definition is that everything should be a filtered colimit of its finite subcomplexes. Now I'll just take an arbitrary sequence, and think of just the finite subcomplexes in there. And I know what maps are between those. For maps in general, I have to do one of these limit-colimit things, i.e., choose a map for every finite subcomplex that's compatible. If you do this, then you get the right answer for stable phenomena that we already knew about. (Except that I haven't mentioned how to deal with 'non-connective' phenomena, but that can be another day...)

The miracle is that this method of looking at stable phenomena makes it into an additive world. It's as if stabilization and abelianization are related[1]...

Remark 0.3. In practice, we don't really compute maps between spectra as some type of limit of colimits. Instead, we observe that our very first guess wasn't that far off: for finite complexes, every stable map $X_0 \to Y_0$ comes from compatible maps $\Sigma^n X_0 \to \Sigma^n Y_0$ *eventually*. We can think of an 'eventually defined' map as a zig-zag of 'levelwise compatible maps':

$$\{X_0, \Sigma X_0, ...\} \leftarrow \{*, *, ..., *, \Sigma^N X_0, \Sigma^{N+1} X_0, ...\} \to \{Y_0, \Sigma Y_0, ..., \Sigma^N Y_0, \Sigma^{N+1} Y, ...\},$$

where the left arrow is morally an equivalence. Now, for an arbitrary map of spectra $X \to Y$, it may not be the case that this even restricts to a map $X_n \to Y_n$. Instead, the map is only defined on some subcomplex $X'_n \subset X_n$. All together we have a zig-zag

$$\{X_n\} \leftarrow \{X'_n\} \to \{Y_n\}$$

of 'levelwise compatible maps'. Since each closed cell in X_n is compact, the stable map is eventually defined on some suspension Σ^k of this cell (or rather, on its image in X_{n+k}). That means that every cell is 'stably' contained in X', and we declare the left arrow to be an equivalence. Maps between (cellular) spectra can now be *defined* as homotopy classes of zig-zags where the left arrow is one of these 'cofinal inclusions'. This is the approach of Boardman and Adams. Bonus useless fact: the whole (derived) mapping space $\operatorname{map}(X, Y)$ can be computed as the classifying space of the category of such zig-zags.

Remark 0.4. The definition of spectra as a *sequence* of spaces together with various maps privileges the filtered category $\mathbb{Z}_{\geq 0}$ above all others. This causes trouble later on: the smash product of spectra $\{X_n\}$ and $\{Y_n\}$ wants to be the data $\{X_n \wedge Y_m\}_{n,m}$, but that's not allowed since it's indexed on a grid $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. In our hearts, since we only care about 'the limit', we know we should be able to choose any sequence $(n_0, m_0), (n_1, m_1), ...$ with $(n_i, m_i) \to (\infty, \infty)$ and restrict attention to this sub-data. But now we've made a choice! And we have to check the choice didn't matter. Then we have to check that when we smash together three things, the choices are suitably associative and commutative up to homotopy. This is an endeavor, but ultimately do-able (see Adams's blue book for a very informative discussion that most people tell you to skip but I think you should read).

[1] See Arpon's talk, which is Talk 16.

Later on, people cleverly replaced $\mathbb{Z}_{\geq 0}$ with other indexing categories such as the category of finite sets and injections (symmetric spectra) or the category of real inner product spaces (orthogonal spectra) to make this step easier to handle before passage to the homotopy category. In the latter case, the independence of 'reindexing' is elegantly encapsulated by the computation that the space of isometric embeddings $\mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty \times \mathbb{R}^\infty$ is contractible. A good reference for this story is Mandell-May-Schwede-Shipley 'Model categories of diagram spectra'. They discuss how to build spectra indexed on any suitable category $\mathcal{I}$ with associated loop and suspension functors Σ^C, Ω^C for $C \in \mathcal{I}$.

Remark 0.5. One final point, for those who have thought about this a bit more. If we use a different indexing category, we need to be sure we recover the correct homotopy type for the zeroth space of a spectrum. More precisely, suppose we're using spectra indexed by $\mathcal{I}$ and there is a canonical functor $\mathbb{Z}_{\geq 0} \to \mathcal{I}$ (e.g. $n \mapsto \mathbb{R}^n$ and $n \leq m$ maps to the standard coordinate embedding, in the case of orthogonal spectra.) Then we'd better check that the map

$$\mathrm{hocolim}_{n\geq 0}\, \Omega^n X_n \to \mathrm{hocolim}_{C\in\mathcal{I}}\, \Omega^C X_C$$

is a weak equivalence.

When $\mathcal{I}$ is the *topological* category of inner product spaces and isometric embeddings, this is always the case because the inclusion $\mathbb{Z}_{\geq 0} \to \mathcal{I}$ is *homotopy cofinal.* (Here it is very important that we use topological categories, homotopy colimits, and say homotopy cofinal, otherwise the statement is false!).

When $\mathcal{I}$ is the category of finite sets and injections, the above map is *not* always a weak equivalence, and the corresponding cofinality claim is false. However, it turns out to be a weak equivalence if the maps $\Omega^n X_n \to \Omega^{n+1} X_{n+1}$ are connected enough, by a theorem of Bökstedt. For those familiar with the homotopy theory of symmetric spectra, this wrinkle is the reason you can't always check a map of symmetric spectra is a weak equivalence by computing stable homotopy groups, without a non-trivial cofibrancy condition.

0.2. What is BP? Recall the dual Steenrod algebra $\mathcal{A}_*$. There's a beautiful theorem of Milnor, that says that, when $p > 2$:

$$\mathcal{A}_* \simeq \mathbf{F}_p[\xi_1, \xi_2, \cdots] \otimes \Lambda(\tau_0, \tau_1, \cdots)$$

where $|\xi_i| = 2(p^i - 1)$.

You can ask the following question: is there some spectrum Y such that $H_*Y \simeq \mathbf{F}_p[\xi_1, \xi_2, \cdots]$?

Whatever this spectrum is, we built it by killing the τ_i. In homotopy theory, nothing ever really dies: if you kill a class, it lives on in the cell you used to kill it. So this spectrum Y would know about some "secondary" phenomena related to the τ_i[2]. One of example of a secondary operation you already know is the Bockstein.

[2] This means "invisible to algebra". Suppose I'm in the cohomology of a space, and apply a primary operation (composite of Steenrod squares), and I get zero. I must have gotten zero for some *reason.* The collection of all the cohomology classes that die, each for their own special reason, conspire together to have a bit of extra structure: a new operation which records their cause of death. This operation isn't in the Steenrod algebra because the Steenrod algebra acts on everything, not just dead things. This is the idea behind a secondary operation.

What's the Bockstein? We have a SES $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. This means that I have $\cdots \to H^*(-;\mathbb{Z}/2) \to H^*(-;\mathbb{Z}/4) \to H^*(-;\mathbb{Z}/2) \xrightarrow{\beta} H^{*+1}(-;\mathbb{Z}/2) \to \cdots$. This β is the Bockstein. So if β of something is zero, I can lift that something to an element in cohomology with $\mathbb{Z}/4$-coefficients. Now, I have another SES $0 \to \mathbb{Z}/2 \to \mathbb{Z}/8 \to \mathbb{Z}/4 \to 0$, and so I can use the Bockstein again to lift to something in $\mathbb{Z}/8$-cohomology. (Notice, though, that I made a choice. Such choices are the cause of great difficulty in utilizing secondary and higher operations in practice.)

For those of you who know about Massey products, I can form $F(H\mathbf{F}_p, H\mathbf{F}_p)$, which is an A_∞-algebra. Any A_∞-algebra gives Massey products, and Massey products in $F(H\mathbf{F}_p, H\mathbf{F}_p)$ are secondary operations.

Anyway, this spectrum Y should, in theory, turn problems about secondary operations into cohomological questions. The answer is yes, as answered by Brown and Peterson. This spectrum is called BP in their honor.

Here's a nice table, which I stole from Steve Wilson:

	S	BP	$H\mathbf{F}_p$
$\pi_*(-)$	?!?!	$\mathbb{Z}_{(p)}[v_1, v_2, \cdots]$, $\lvert v_i\rvert = 2(p^i - 1)$	$\mathbf{F}_p$
$H_*(-;\mathbf{F}_p)$	easy ($\mathbf{F}_p$)	$\mathbf{F}_p[\xi_1, \xi_2, \cdots]$	annoying/harder $\widehat{A}_*$

Note that the v_i are chosen noncanonically.

BP is so awesome; for instance, everything is concentrated in even degrees, so it's complex oriented. This gives a map $MU \to BP$. In particular, you get a formal group law over BP_*; what is this? It turns out that just like MU_* is the Lazard ring classifying formal group laws, the graded ring BP_* classifies special types of formal group laws, that are called *p-typical formal group laws*. Quillen proved this (in three different ways!). He was able to construct BP using this idea, namely that over a $\mathbb{Z}_{(p)}$-algebra every formal group law is p-typical (up to isomorphism). This gives a self-map of $MU_{(p)}$ known as the Quillen idempotent, which we'll discuss at some point.

Talk 1: Overview Talk (Sarah Whitehouse)

This talk will have three parts:

- Overview of structured ring spectra
- What is this extra structure good for?
- How do we get structured objects and how can we tell how much structure we already have?

1.1. History. Spectra were first introduced in the 50's. The idea is to distill out stable phenomena – independent of dimension. We've known how to define the stable homotopy category for a long time – that's Boardman's thesis in 1964. (Stable homotopy theory was born in Warwick!) This is described in Adams' blue book of Chicago lecture notes, among other places.

There were some drawbacks; on the underlying point-set level spectra, you have a smash product but it is only associative, commutative, and unital upon passage to the homotopy category. It means you can't do various constructions you want to do (e.g. categories of module spectra) on the point-set level. There was a long search for a good point-set level category in which to do stable homotopy theory. What are the key requirements? You want a point-set level construction where $\wedge$ is associative, commutative, and unital; that is a closed model category; and whose homotopy category is the one we've already agreed is the stable homotopy category.

Theorem 1.1 (Lewis, 1991)**.** *There does not exist a category of spectra* Sp *such that:*

(1) there is a symmetric monoidal smash product $\wedge$,

(2) there is an adjunction $\Sigma^\infty : \mathrm{Top}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty$,

(3) the sphere spectrum $\Sigma^\infty S^0$ is a unit for $\wedge$,

(4) Σ^∞ is colax monoidal (i.e. there is a map $\Sigma^\infty(X \wedge Y) \to \Sigma^\infty X \wedge \Sigma^\infty Y$ that's natural in X and Y),

(5) there is a natural weak equivalence $\Omega^\infty\Sigma^\infty X \xrightarrow{\simeq} \varinjlim \Omega^n\Sigma^n X$.

The point is that this is a rather modest-looking list of requirements.

In the late 90s and early 00s, we suddenly got a whole bunch of good model categories of spectra:

- EKMM (Elmendorf-Kriz-Mandell-May), 1997, S-modules $\mathcal{M}_S$
- HSS (Hovey-Shipley-Smith), 2000, symmetric spectra Sp^Σ
- MMSS (Mandell-May-Schwede-Shipley), 2001, orthogonal spectra Sp^0

These fail to satisfy the requirements in the theorem in different ways, and have different advantages and disadvantages. All of these have a symmetric monoidal smash product, and the sphere is a unit in all of them. For now, I'll stick with the model of S-modules. There it's (5) that fails, and so you'll want to replace Ω^∞ and Σ^∞ with other constructions, but that causes (3) to fail.

Since we have a good $\wedge$, we can define associative monoids in S-modules, a.k.a. S-algebras. These correspond to what used to be called A_∞-ring spectra. We also have commutative monoids in S-modules, called commutative S-algebras; these correspond to the older notion of E_∞-ring spectra. A_∞ is also known as E_1. In between, there is a notion of E_n-ring spectra for each natural number n. You should think of A_∞ as homotopy associative up to all higher coherences, and as you move up the E_n ladder, you're getting commutativity up to more and more higher coherences until you get E_∞, which means homotopy associative and commutative up to *all* higher coherences.

1.2. Higher homotopy associativity and commutativity. Let's go back to based, connected spaces. The whole story starts with the space of based loops ΩX. Given three loops a, b, c, we notice that $a(bc)$ and $(ab)c$ aren't *the same*, but they are homotopic (you spend different amounts of time going around the various loops – $a(bc)$ spends half the time on a and a quarter of the time on b and c each, whereas $(ab)c$ spends half the time on c and a quarter of the time on a and b each). If I have a d as well, there are 5 ways of associating them, and if you draw them in the right order

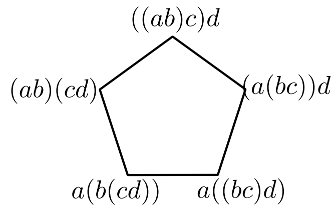


FIGURE 1. Stasheff pentagon

you get the "Stasheff pentagon", where each line segment in the boundary can be thought of as a homotopy between two groupings. You can also deform across the interior of the pentagon because this is all just about how quickly you go around the loops. The same happens for compositions of more loops.

Suppose you have a space Y with a multiplication. If all you have is naïve homotopy associativity for triples, $a(bc) \simeq (ab)c$, say it is an A_3 space. If there are also homotopies as indicated in the Stasheff pentagon for any four points, then say Y is an A_4 space. If $Y = \Omega X$, you also have all the possible higher homotopies between homotopies, and so you say that ΩX is A_∞. Roughly speaking, loop spaces are the same as A_∞ spaces – this is the recognition principle. You can encode this information using an A_∞ operad, such as one built with the Stasheff associahedra. We'll hear more about this in the next talk.

If you have a double loop space $\Omega^2 X$, then you're starting to have some homotopy commutativity built in. This can be described by the little 2-cubes operad or little 2-disks operad. The double loop space $\Omega^2 X$ was already naïvely homotopy commutative, but if you have $\Omega^n X$, then this is an E_n-space, and this structure can be encoded by the little n-cubes operad. In the limit, you have $\Omega^\infty X$, which is an E_∞ space, and this structure is described by the ∞-cubes operad, but also by other E_∞ operads such as the Barratt-Eccles operad and the linear isometries operad. Again, we'll learn more about these in the next talk.

Let's summarize this in a table:

Space	Structure	Operads
ΩX	A_∞-space $= E_1$-space	little intervals operad, Stasheff associahedra
$\Omega^2 X$	E_2-space	little 2-cubes/disks
$\vdots$	$\vdots$	$\vdots$
$\Omega^n X$	E_n-space	little n-cubes/disks
$\vdots$	$\vdots$	$\vdots$
$\Omega^\infty X$	E_∞-space	∞-cubes, linear isometries, Barratt-Eccles

As soon as Y has an associative product, you get an algebra structure on $H_*(Y, \mathbb{F}_p)$. As you go up the E_n hierarchy, you get more and more structure on homology. For example, by the time you get to E_∞, you have Dyer-Lashof operations on homology.

1.3. What is this extra structure good for?

Module categories (Mandell, 2012). Suppose I have an actual (discrete) algebra R. If it's just associative, and I have a right module and a left module, I can form a tensor product, but that won't be a module. If R is commutative, the tensor product is again an R-module (and you can say fancier things about it being a symmetric monoidal structure).

structure on spectrum R	R-mod$=$ R-module spectra	$D_R = \mathrm{Ho}(R\text{-mod})$
$E_1 = A_\infty$	product pairing $M \wedge_R N$ for $M \in R^{\mathrm{op}}$-mod, $N \in R$-mod	product pairing
E_2	R^{op}-mod $=$ R-mod, monoidal structure on R-mod $- \wedge_R -$	$D_{R^{\mathrm{op}}} \simeq D_R$, monoidal structure
E_3		braided monoidal structure
E_4		symmetric monoidal category
$\vdots$		
E_∞	symmetric monoidal category	symmetric monoidal category

Everything here is at least A_∞, and here "R-mod" means just using that structure. In theory, if you have more structure, you could look at different module categories. Mandell's theorems are about what happens when you take into account the A_∞-structure.

The point is that you have to go to E_∞ in order to get a symmetric monoidal structure on the point-set level module category (second column); on the homotopy category you get this with an E_4 structure (third column).

"Brave new algebra". Now that we can talk about commutativity and associativity, we should view structured ring spectra as some vast generalization of ordinary algebra; we can generalize constructions (and maybe even theorems, sometimes) from algebra, or algebraic geometry if you're very brave, to spectra. Examples:

- Galois theory in the sense of Rognes,
- duality results (Dwyer, Greenless, Iyengar),
- invariants: algebraic K-theory [of a spectrum], THH (a generalization of Hochschild homology in algebra), TC, TAQ (a generalization of André-Quillen homology).

We might even be able to get new algebraic results – ordinary algebra is embedded via the Eilenberg-Maclane construction. There are actually statements about pure algebra whose only known proof is via structured ring spectra.

Extra structure. If you put in extra structure such as an E_∞-ring, then you get out extra structure such as Dyer-Lashof operations on e.g. homology. These are really useful. You can also show things are not E_∞ by showing one of the requisite Dyer-Lashof operations can't exist.

1.4. How much structure do we have?

How to get highly structured ring spectra. This is what's coming in Talk 3. We'll talk about various machines for generating highly structured ring spectra, such as multiplicative infinite loop space machines, Thom spectra, free objects and standard operadic constructions that produce algebras over an operad. Once you're in a nice category, there are some "new from old" constructions such as localization. There's also a notion of building E_∞-ring spectra out of "E_∞-cells".

Obstruction theory. Suppose you have something with an A_∞-structure, and you want to know whether you can build in more structure. The general approach is to build up the structure in stages. In all these theories you're going to get obstructions to existence and uniqueness, living in some kind of cohomology, of a lift from one stage to the next. In one instance the n^{th} stage is going to be E_n. For Γ-homology, we've got a different notion of stage.

I'll first talk about a motivating example, namely BP, the Brown-Peterson spectrum. There's a secret prime that isn't written in the notation. Naïvely, think about MU, fix a prime p, think about the p-local version $MU_{(p)}$; it turns out that $MU_{(p)}$ splits as a bunch of pieces which we call BP:

$$MU_{(p)} \simeq \bigvee \Sigma^{2i} BP.$$

MU is enormously important, and BP is in some sense a more convenient version of $MU_{(p)}$ (why would you carry around a bunch of copies of information if you could just work with a small piece that contains it all?). It is known that MU is E_∞ (the structure is really natural), and localizations of E_∞-ring spectra are E_∞. It's a really old question about whether BP is an E_∞-ring spectrum. The issues are with commutativity; we've known for a while that it's an A_∞-ring spectrum (and it's good as an algebra over MU).

1.5. Outline of what's coming. We'll list the obstruction theories and what's coming in which talk.

talks	obstruction theory	context	who	key ingredients	results about BP
4-7	$H\Gamma$ (Γ-cohomology)	E_∞-algebras	Robinson, Whitehouse	particular E_∞-operad and a filtration of it giving a notion of "n-stage"	Richter: BP has at least a $(2p^2+2p-2)$-stage
8-12	TAQ (Topological André-Quillen cohomology)	comm. S-algebras	Basterra, Kriz	Postnikov towers, k-invariants	Basterra-Mandell: BP is at least E_4
13-15	Goerss-Hopkins obstruction theory	any good category of spectra	Goerss-Hopkins	simplicial resolutions	[lots of applications, including to spectra related to BP]

Talks 16-17 are about comparison results for the various obstruction theories. Talks 18-19 are about limits to the structure of BP. We know that you can't have an E_∞-map $BP \to MU$. So if BP did have an E_∞-structure, it wouldn't be nicely related to MU. Recently, (indeed, shortly after we wrote the Talbot syllabus), Lawson posted a preprint proving that BP at $p = 2$ doesn't have an E_{12}-structure!

There are still lots of things we don't know: for a start, the problem of whether BP is E_∞ is open at odd primes, and even at $p = 2$ we don't know the maximum n for which BP is E_n (though maybe we don't care as much about that one). Lawson conjectures BP is not E_∞ for odd primes as well. His proof for $p = 2$ uses secondary Dyer-Lashof operations and he thinks the same idea should work for odd primes but it might be painful to get the details right.

The final talk will be by Maria Basterra, on future directions.

Talk 2: Operads (Calista Bernard)

Let $(M, \otimes, I)$ be a closed, symmetric monoidal category. Examples to keep in mind are:

- $(\text{Set}, \times, \{*\})$
- $(\text{Top}^{\text{nice}}, \times, \{*\})$
- $(\text{Vect}_k, \otimes, k)$

Definition 2.1. An operad $\mathcal{O}$ is a collection of objects of M, written $\{\mathcal{O}(n)\}_{n\geq 0}$ with

(i) composition law

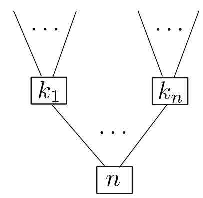
$$\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_n) \xrightarrow{\gamma} \mathcal{O}(k_1 + \cdots + k_n)$$

(ii) right action of Σ_n on $\mathcal{O}(n)$

(iii) a unit in $\mathcal{O}(1)$

where γ satisfies an associativity condition and behaves nicely w.r.t. Σ_n action and the unit.

Think of $\mathcal{O}(n)$ as "n-ary operations". Think of the symmetric group action as permuting the inputs, and composition as plugging in n functions in k_1, …, k_n variables, respectively, into a function in n variables to get a function in $k_1 + \cdots + k_n$ variables. Draw this as follows:



If $\mathcal{O}$ just satisfies (i) and (iii), then say $\mathcal{O}$ is a *non-symmetric operad*. A morphism of operads is a collection of maps $\{\mathcal{O}(n) \to \mathcal{O}'(n)\}$ preserving the operad structure.

Example 2.2 (Associative operad)**.** Over spaces, $\mathrm{As}(n) = \Sigma_n$ (the disjoint union of $n!$ points). In general, you can do this over any symmetric monoidal category: take $n!$ copies of the unit. The composition

$$\Sigma_n \times (\Sigma_{k_1} \times \ldots \times \Sigma_{k_n}) \xrightarrow{\gamma} \Sigma_k$$

sends $(\sigma; \tau_1, \ldots, \tau_n)$ to the permutation of $(a_1, \ldots, a_{k_1}, \ldots, a_{k-k_n}, \ldots, a_k)$ that does τ_1 to $(a_1, \ldots, a_{k_1})$, τ_2 to $(a_{k_1+1}, \ldots, a_{k_1+k_2})$ etc., and then does σ to the resulting thing. That is, if we partition $(a_1, \ldots, a_k)$ into n boxes according to the k_i, τ_i permutes the elements of the ith box, and σ permutes the boxes. Σ_n acts by right multiplication.

There is also a non-symmetric associative operad, defined by $\mathrm{As}(n) = *$. Think of this as there being a single multiplication μ, a single ternary operation $\mu(-, \mu(-,-)) = \mu(\mu(-,-),-)$, etc. indicating that all the higher associativities hold on the nose. The symmetric version, above, also keeps track of the fact that, once you also allow switching factors, there are $2! = 2$ binary operations ($(a,b) \mapsto ab$ and $(a,b) \mapsto ba$), $3! = 6$ ternary operations ($(a,b,c) \mapsto (ab)c$, $(a,b,c) \mapsto (ba)c$, …), and so on; a priori there aren't any relationships between these and so the space of them is discrete, with $n!$ points.

Example 2.3 (Commutative operad)**.** Over spaces, define the (symmetric) operad Comm such that $\mathrm{Comm}(n) = *$ (or over any symmetric monoidal category, it's just the unit). Define the Σ_n-action to be trivial. (As opposed to the (symmetric version of the) associative operad, which had a different n-ary operation for every permutation of the inputs, here we have the same n-ary operation no matter how you act on it by Σ_n.)

How does this help?

Definition 2.4. If $\mathcal{O}$ is an operad over a category M (i.e. such that $\mathcal{O}(n) \in M$), then an algebra over $\mathcal{O}$ is an object $X \in M$, together with maps $\mathcal{O}(n) \otimes X^{\otimes n} \to X$ that are compatible with the operad structure on $\mathcal{O}$.

This gives a way of parametrizing n-ary operations on X.

Exercise 2.5.

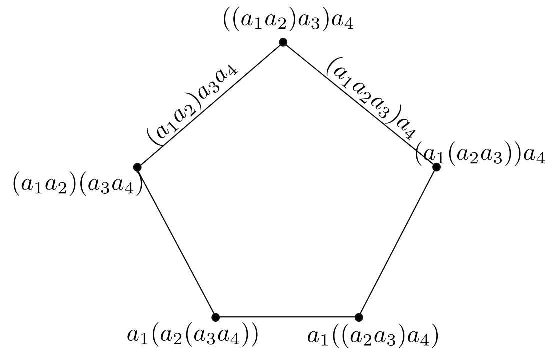
(1) Algebras over As are associative monoids.

(2) Algebras over Comm are commutative monoids.

Definition 2.6. An A_∞-operad is an operad $\mathcal{O}$ over spaces together with a morphism $\mathcal{O} \to \mathrm{As}$ such that the levelwise map $\mathcal{O}(n) \to \mathrm{As}(n)$ is a Σ_n-equivariant homotopy equivalence.

Example 2.7 (Stasheff's associahedra)**.** I will state this as a non-symmetric A_∞-operad. Define K_i to be a cell complex homeomorphic to $I^{\times i-2}$ (where $I = [0, 1]$) that keeps track of ways to bracket a word with i letters in it. Define $K_0 = * = K_1$.

- $K_2 = *$ corresponding to one way to bracket $(a_1 a_2)$
- K_3 is an interval with endpoints $(a_1 a_2)a_3$ and $a_1(a_2 a_3)$.
- K_4 is the pentagon



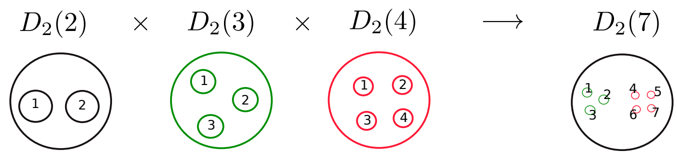
- The rest can be defined inductively.

The composition

$$K_n \times (K_{k_1} \times \ldots \times K_{k_n}) \to K_{k_1 + \cdots + k_n}$$

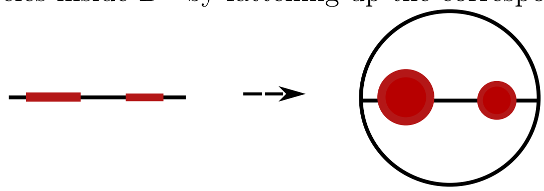
takes a bracketing ψ_n of a word with n letters and gives a bracketing of $a_1 \ldots a_{k_1} \ldots a_{k-k_n} \ldots a_k$ where you use the k_1 bracketing on $a_1 \ldots a_{k_1}$, the k_2 bracketing on $a_{k_1+1} \ldots a_{k_1+k_2}$, etc, and then apply ψ_n to the result (where the n "letters" are (1) the bracketing of $a_1 \ldots a_{k_1}$, (2) the bracketing of $a_{k_1+1} \ldots a_{k_1+k_2}$, etc.). The claim is that this extends to a map of cell complexes. This is a non-symmetric operad so I'm not going to define an action of the symmetric group. Since K_i is contractible, there is a morphism $K \to \mathrm{As}$, and therefore K is an A_∞ operad. If X is a K-algebra, $\underbrace{K_2}_{*} \times X^2 \to X$ makes X an H-space, $\underbrace{K_3}_{I} \times X^3 \to X$ gives a homotopy between $(a_1 a_2)a_3$ and $a_1(a_2 a_3)$, etc.

Example 2.8 (Little n-disks operad)**.** Let $D_n = \mathrm{sEmb}(\bigsqcup_k D^n, D^n)$ where sEmb, standard embeddings, means on each component it looks like $x \mapsto \lambda x + c$. The composition is defined as in the following example:



The symmetric group acts by permuting the labels.

Note: we have a morphism $D_n \hookrightarrow D_{n+1}$ from an embedding $D^n \hookrightarrow D^{n+1}$. For example, I can embed an interval ($= D^1$) into D^2 by putting the interval as the equator, and I can map intervals inside D^1 to circles inside D^2 by fattening up the corresponding intervals.



This generalizes to other n. Let $D_\infty = \operatorname{colim}_n D_n$.

Theorem 2.9 (May, Boardman-Vogt)**.** *Every n-fold based loop space is a D_n-algebra. If Y is a D_n-algebra and $\pi_0 Y$ is a group, then there exists a space X such that $Y \simeq \Omega^n X$.*

Proving the first statement isn't that hard, but the second statement is harder; May does it by constructing a delooping involving the 2-sided bar construction.

Definition 2.10. An E_n-operad (for $1 \leq n \leq \infty$) is an operad $\mathcal{O}$ over spaces with a weak equivalence of operads (i.e. on each arity I have a zigzag of weak homotopy equivalences) $\mathcal{O} \to D_n$ such that the Σ_k-action on $\mathcal{O}(k)$ is free.

Remark 2.11. D_1 is weakly equivalent to As (contract all the intervals to points and then you get configurations of n points in an interval), so E_1 is "the same" as A_∞.

Remark 2.12. $D_\infty(n)$ is contractible. (You can write down a specific map showing that $D_k(n)$ is contractible in $D_{k+1}(n)$.)

Being an E_∞-operad is the same as being weakly equivalent to a point in each arity, and having a free Σ_n-action.

Example 2.13 (Barratt-Eccles)**.** Let $\mathcal{E}(n) = E\Sigma_n$ (the total space for the classifying space). Earlier when defining the associative operad, I defined a map $\Sigma_n \times \Sigma_{k_1} \times \ldots \times \Sigma_{k_n} \to \Sigma_{k_1+\cdots+k_n}$; the corresponding map

$$\Sigma_n \ltimes (\Sigma_{k_1} \times \ldots \times \Sigma_{k_n}) \to \Sigma_{k_1+\cdots+k_n}$$

is a group homomorphism. This induces a map

$$E(\Sigma_n \ltimes (\Sigma_{k_1} \times \ldots \times \Sigma_{k_n})) \to E\Sigma_{k_1+\cdots+k_n}$$

which gives a map

$$E\Sigma_n \times E\Sigma_{k_1} \times \ldots \times E\Sigma_{k_n} \to E\Sigma_{k_1+\cdots+k_n}.$$

Since $E\Sigma_n$ is contractible with a free Σ_n action, $\mathcal{E}$ is an E_∞-operad.

Example 2.14 (Linear isometries operad)**.** Let $\mathcal{L}(n)=\{\text{linear isometries } (\mathbb{R}^\infty)^n \to \mathbb{R}^\infty\}$. The operad composition is multi-composition. (This is just a sub-operad of the endomorphism operad.) Σ_n acts by permuting the inputs. We can topologize $\mathbb{R}^\infty$ as the limit of $\mathbb{R}^n$'s, which gives rise to a topology on $\mathcal{L}(n)$. It turns out that $\mathcal{L}(n)$ is contractible (this is a consequence of Kuiper's theorem).

For example, BO is an algebra over this operad, because you can look at the infinite Grassmannian and define the operad map using direct sums of subspaces of $\mathbb{R}^\infty$.

Simplicial spectra over simplicial operads. The last thing I want to talk about will be useful for Goerss-Hopkins obstruction theory. In the reference in the syllabus, they work over simplicial sets, but for simplicity we'll work over spaces.

Definition 2.15. Let $\mathcal{O}$ be the category of operads over spaces. Let $s\mathcal{O}$ be simplicial objects of $\mathcal{O}$. A *simplicial operad* is an object of $s\mathcal{O}$.

Given $C \in \mathcal{O}$ and a spectrum X in our modern category of spectra, we can define the *free C-algebra generated by X* as

$$C(X) = \bigvee_{n\geq 0} C(n)_+ \wedge_{\Sigma_n} X^{\wedge n}.$$

(You can always make a free algebra in this way using the coproduct and product.) Then X is a C-algebra iff there is a map $C(X) \to X$ satisfying some conditions.

If $X = \{n \to X_n\}$ is a simplicial spectrum (simplicial object in the category of spectra) that is an algebra over a simplicial operad $T = \{n \mapsto T_n\}$, then X_n is an algebra over T_n for all n.

We can define a geometric realization functor

$$|\ \ | : s\mathcal{O} \to \mathcal{O}$$

where $|T| = \coprod_{n\geq 0} \Delta^n \times T_n/\sim$. Here $\Delta^n \times T_n$ is the operad with k^{th} arity $\Delta^n \times T_n(k)$. Similarly, there is a geometric realization

$$|\ \ | : s\,\text{Spectra} \to \text{Spectra}$$

where $|X| = \bigvee_{n\geq 0} \Delta^n_+ \wedge X_n/\sim$.

Theorem 2.16. *If T is a simplicial operad and X is a simplicial spectrum that is an algebra over T, then $|X|$ has a natural structure as an $|T|$-algebra.*

The proof is basically a definition chase, plus showing that you have a map $T(X) \to X$ that behaves nicely under realization.

Talk 3: Examples of structured ring spectra (Jens Kjaer)

3.1. Linear isometries operad. I'm going to follow May's style of E_∞-ring spectra. To each $V \subset \mathbb{R}^\infty$ (inclusion as a finite-dimensional inner product space) we associate $E(V) \in \mathrm{Top}_*$ and structure maps $S^W \wedge E(V) \to E(W \oplus V)$ (here S^W is the 1-point compactification of the representation W, and we need $W \cap V = 0$).

What does it mean for this to be E_∞? Given $g \in \mathcal{L}(n)$ (linear isometries $\oplus_n \mathbb{R}^\infty \to \mathbb{R}^\infty$), we should have maps $g_* : E_{V_1} \wedge \ldots \wedge E_{V_n} \to E_{g(V_1 \times \ldots \times V_n)}$ that are compatible with the operadic structure.

The sphere spectrum is an example of an algebra over the linear isometries operad: $\mathbb{S}(V) = S^V$. The map $S^{V_1} \wedge \ldots \wedge S^{V_n} \to S^{g(V_1 \times \ldots \times V_n)}$ is just smashing and re-indexing.

There aren't a lot of other examples that are easy to write out on the point-set level.

3.2. Thom spectra. How else can we make E_∞ ring spectra? Use Thom spectra. If V is a finite-dimensional inner product space, and $F(V)$ is the set of self homotopy equivalences of S^V, then form $F = \mathrm{colim}_{V \subset \mathbb{R}^\infty} F(V)$. This is a group under composition, so I can form the classifying space BF. This classifies spherical fibrations: if I have a map $X \to BF(V)$, this is the same data as a sphere bundle $\xi \to X$. Given a sphere bundle $\xi \to X$, I can associate the Thom space $\mathrm{Thom}(\xi)$, which you should think of as a sphere bundle where all the fibers look like S^V; write $\mathrm{Thom}(\xi) = \xi / X \times \infty$.

Given a map $j : X \to BF$, I will show how to construct an associated spectrum. There's a map $BF(V) \to BF$, so I can define X_V as the pullback

$$\begin{array}{ccc} X_V & \xrightarrow{j_V} & BF(V) \\ \downarrow & & \downarrow \\ X & \xrightarrow{j} & BF \end{array}$$

The map j_V gives rise to a sphere bundle χ_V over X_V. If $V \hookrightarrow W$ is an inner product space, you get a map $BF(V) \to BF(W)$. There are canonical sphere bundles γ_V and γ_W over $BF(V)$ and $BF(W)$, respectively. I get

$$\begin{array}{ccc} \gamma_V \oplus \varepsilon_{W-V} & \longrightarrow & \gamma_W \\ \downarrow & & \downarrow \\ BF(V) & \longrightarrow & BF(W) \end{array}$$

Applying Thom space machinery, I get a map $\Sigma^{V^\perp} \mathrm{Thom}(\gamma_V) \to \mathrm{Thom}(\gamma_W)$. So I get $\Sigma^{V^\perp} \mathrm{Thom}(\chi_V) \to \mathrm{Thom}(\chi_W)$. Then define $\mathrm{Thom}(j)(V) = \mathrm{Thom}(\chi_V)$.

Right now this is just an ordinary spectrum, but I claim it can be given the structure of an E_∞-ring spectrum via an action of the linear isometries operad. If $g \in \mathcal{L}(n)$, I get an induced map $g_* : BF(V_1) \times \ldots \times BF(V_n) \to BF(V)$ (where $V = g(V_1 \times \ldots \times V_n)$). These fit into an operadic picture; there's some compatibility between how all the g's act.

I will be less than precise in the following. Assume X is an $\mathcal{L}$-algebra. Assume $X \to BF$ is a map of $\mathcal{L}$-algebras. This means I have commutative diagrams

$$\begin{array}{ccc} \mathcal{L}(n) \times X^n & \longrightarrow & \mathcal{L}(n) \times BF^{\times n} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BF \end{array}$$

This ensures the following diagram commutes.

$$\begin{array}{ccc} X_{V_1} \times \ldots \times X_{V_n} & \longrightarrow & BF(V_1) \times \ldots \times BF(V_n) \\ \downarrow g_* & & \downarrow g_* \\ X_V & \longrightarrow & BF(V) \end{array}$$

Drawing all the total spaces we have a commutative diagram

$$\begin{array}{ccccccc} \chi_{V_1} \times \ldots \times \chi_{V_n} & & \longrightarrow & & \gamma_{V_1} \times \ldots \times \gamma_{V_n} & & \\ & \searrow & & & & \searrow & \\ & & X_{V_1} \times \ldots \times X_{v_n} & \longrightarrow & & & BF(V_1) \times \ldots \times BF(V_n) \\ \downarrow & & & & \downarrow & & \\ \chi_V & & \longrightarrow & & \gamma_V & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & X_V & \longrightarrow & & & BF(V) \end{array}$$

This gives rise to a map

$$\mathrm{Thom}(\chi_{V_1}) \wedge \ldots \wedge \mathrm{Thom}(\chi_{V_n}) \to \mathrm{Thom}(\chi_V).$$

One thing you should be concerned about is that you get an E_∞-Thom spectrum if $X \to BF$ is a map of E_∞-spaces.

I have a map $O \to F$ (take the 1-point compactification), which gives rise to a map $BO \to BF$. I assert that this is an E_∞-map, which works for this choice of operad. If you take the Thom spectrum of this, you get MO. Similarly, there is a map $BU \to BF$, and the Thom spectrum of this is MU. You can get the sphere spectrum this way by taking the trivial map.

3.3. Commutative spectra. If you have a highly structured model of the category of spectra (symmetric spectra, orthogonal spectra, S-modules), then a commutative monoid will, by translation machinery, give you an E_∞-ring spectrum. Different machines have different things that are easy to write down.

Let's focus on symmetric spectra. Recall, a symmetric spectrum is a sequence of pointed simplicial sets X_n where Σ_n acts on X_n, along with structure maps $S^m \wedge X_n \to X_{m+n}$ which is $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$-equivariant (on the left, write $S^m = (S^1)^{\wedge m}$ and permute the coordinates).

Example 3.1. If A is a commutative ring, define the Eilenberg-MacLane spectrum $HA_n = A \otimes \widehat{\mathbb{Z}}[S^n]$ (here $\widehat{\mathbb{Z}}[S^n]$ is the free reduced abelian simplicial group on the simplicial space S^n, and $A \otimes \widehat{\mathbb{Z}}[S^n]$ means it's tensored levelwise with A). You can work out the maps

$$HA_n \wedge HA_m = A \otimes \widehat{\mathbb{Z}}[S^n] \wedge A \otimes \widehat{\mathbb{Z}}[S^m] \to A \otimes \widehat{\mathbb{Z}}[S^{n+m}];$$

use the multiplication on A. This gives that HA is a commutative spectrum, and therefore an E_∞-ring.

3.4. Pairs of operads. Assume E is an E_∞-ring spectrum. Then $\Omega^\infty E$, an infinite loop space, is an E_∞-space since the little cubes operad $\mathcal{C}_\infty$ acts on $\Omega^\infty(-)$. Since E was already E_∞, there's also an action of $\mathcal{L}$ on E, which gives rise to an action of $\mathcal{L}$ on $\Omega^\infty E$. So now $\Omega^\infty E$ is an E_∞-space for two different reasons – there are two different actions of an E_∞-operad. There is a deep and subtle interaction of these two.

Definition 3.2. $(\mathcal{G}, \mathcal{C})$ is an *operad pair* if $\mathcal{G}$ and $\mathcal{C}$ are operads, and there are structure maps

$$\mathcal{G}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \to \mathcal{C}(j_1 + \cdots + j_k).$$

These are subject to all compatibility diagrams which I won't write here.

The idea is that $\mathcal{G}$ is multiplication and $\mathcal{C}$ is addition.

Examples of operad pairs are really hard. May claims that the following are the only known examples:

- $(\mathrm{Comm}, \mathrm{Comm})$ because there is exactly one map from a point to a point
- $(\mathcal{L}, \mathcal{K})$ where $\mathcal{K}$ is the Steiner operad (think of this as like little disks or little cubes, with the good qualities of both and none of the bad qualities)

  Let me pretend this actually worked for the little disks operad, and give a sketch of how this goes. Look at $\mathrm{Isom}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}, \mathbb{R}^{n_1 + \cdots + n_k})$; note that if you let all the n's go to ∞ you get the linear isometries operad. Then you get a map

  $$\mathrm{Isom}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}, \mathbb{R}^{n_1 + \cdots + n_k}) \times \mathcal{C}_{n_1}(j_1) \times \ldots \times \mathcal{C}_{n_k}(j_k) \to \mathcal{C}_{n_1 + \cdots + n_k}(j_1 + \cdots + j_k)$$

  which is producting all the little cubes together and then applying the isometry. This doesn't actually work for subtle basepoint issues, which is why you need the Steiner operad instead.

Definition 3.3. Let $(\mathcal{G}, \mathcal{C})$ be an operad pair.

- X is a $(\mathcal{G}, \mathcal{C})$-*space* if X is a $\mathcal{G}$-algebra and a $\mathcal{C}$-algebra, and $CX \to X$ is a $\mathcal{G}$-algebra map. (Here C is the monad that comes from $\mathcal{C}$.)
- $(\mathcal{G}, \mathcal{C})$ is an E_∞-*pair* if $\mathcal{G}$ and $\mathcal{C}$ are E_∞.
- X is an E_∞-*ring space* if it is a $(\mathcal{G}, \mathcal{C})$-space for an E_∞-pair.

Example 3.4. $\Omega^\infty E$ is an $(\mathcal{L}, \mathcal{K})$-space, and hence an E_∞-ring space.

How can we get other examples of infinite loop spaces? One way to get loop spaces is from permutative categories.

Definition 3.5. Let $(C, \oplus, 0)$ be a symmetric monoidal category. This is called *permutative* if:

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (note this is strict)

(2) $a \oplus 0 = a = 0 \oplus a$

(3) some compatibility between the unit being strict and the symmetric monoidal structure

Let $\underline{E\Sigma_n}$ be the category whose objects are the elements of Σ_n, and $\mathrm{Hom}(\sigma, \tau) = *$ for every σ, τ.

Let $\mathcal{C}$ be a permutative category, so we get a functor $\underline{E\Sigma_n} \times \mathcal{C}^{\times n} \to \mathcal{C}$; taking nerves yields a map $E\Sigma_n \times B\mathcal{C}^{\times n} \to B\mathcal{C}$ giving $B\mathcal{C}$ the structure of an E_∞-space. (To any E_∞-space X you can associate a spectrum, and the 0^{th} space ΓX is an infinite loop space. The claim is that $B\mathcal{C}$ is an E_∞-space and $\Gamma B\mathcal{C}$ is an infinite loop space. Idea: moving from $B\mathcal{C}$ to $\Gamma B\mathcal{C}$ is like adding the inverses. Think of this is as the algebraic K-theory $K(\mathcal{C})$.)

Example 3.6. Let $(\mathcal{C}, \oplus, 0) = (\mathrm{Proj}_R^{\text{fin},\cong}, \oplus, 0)$; then $K\mathcal{C} = K(R)$.

Definition 3.7 (Bipermutative categories)**.** A *bipermutative category* is $(\mathcal{C}, \oplus, \otimes, 0, 1)$ such that $(\mathcal{C}, \oplus, 0)$ and $(\mathcal{C}, \otimes, 1)$ are permutative, and

$$d_\ell : (a \otimes b) \oplus (a' \otimes b) \to (a \oplus a') \otimes b$$
$$d_r : (b \otimes a) \oplus (b \otimes a') \to b \otimes (a \oplus a')$$

satisfy compatibilities.

You should be able to conclude that $B\mathcal{C}$ is an E_∞-ring space. May tried to do this using the Barratt-Eccles operad, but it turns out that this doesn't act on itself, and there's a subtle basepoint issue, so you have to do something more complicated.

Why was I talking about E_∞-ring spaces when I started out by talking about E_∞-ring spectra?

Proposition 3.8. *If X is an E_∞-ring space, given as a $(\mathcal{G}, \mathcal{C})$-space, for an E_∞ pair $(\mathcal{G}, \mathcal{C})$, then the associated spectrum to the $\mathcal{C}$-delooping is an E_∞-ring spectrum.*

In particular, if $\mathcal{C}$ is a bipermutative category, then $K\mathcal{C}$ (the algebraic K-theory of $\mathcal{C}$) can be delooped to an E_∞-ring spectrum. This is the primary example.

Example 3.9. Take $\mathcal{C} = (\mathrm{Vect}_{\mathbb{C}}^{\cong\text{-classes,fin}}, \oplus, \otimes, 0, \mathbb{C})$. This isn't bipermutative, but it can be strictified to a bipermutative category $\mathcal{C}'$. Then $K\mathcal{C}' = kU$, connective complex K-theory. One way of getting non-connective K-theory KU is to take connective K-theory kU and invert the Bott element. Our theory of E_∞-ring spectra is rich enough to allow that.

If you have an E_∞-ring space and you add a disjoint basepoint and take the suspension spectrum, then you get an E_∞-ring spectrum.

PART II: Γ-HOMOLOGY

Interlude: why do we care whether things are E_∞, etc.? (Dylan Wilson)

This is Dylan's answer to the above question, delivered during an informal questions seminar during the workshop.

Most of the rest of the workshop is about techniques for showing whether ring spectra are E_∞, E_n, etc. But first let's collect a few bits of intuition about why this is a useful endeavor.

Reason 1: multiplicative structures lead to power operations, and those are useful for computation. Suppose I have a homotopy commutative multiplication on a space X; that means there is a homotopy (possibly lots!) between $X \times X \xrightarrow{\mu} X$ and $X \times X \xrightarrow{\text{switch}} X \times X \xrightarrow{\mu} X$. Pick one, and precompose with the diagonal map to get a homotopy from $X \xrightarrow{\Delta} X \times X \xrightarrow{\mu} X$ to $X \xrightarrow{\Delta} X \times X \xrightarrow{\text{switch}} X \times X \xrightarrow{\mu} X$. Pick an element $x \in X$; both of these maps take $x \mapsto x^2$, but the homotopy might do something crazy – the homotopy gives you a loop on x^2. So you started with a point, and the homotopy gave you a loop (a 1-dimensional thing). Similarly, I could also start with some simplicial subcomplex in X; the same procedure gives rise to something of dimension 1 higher. If we're lucky, this takes boundaries to boundaries, and the map from n-chains to $(n+1)$-chains lifts to a map $H_*(X; \mathbb{F}_2) \to H_{*+1}(X, \mathbb{F}_2)$. Again: this map depends on the homotopy!

Given an H-space X and a chosen witness of homotopy commutativity, you get a map $Q : H_*(X, \mathbb{F}_2) \to H_{*+1}(X, \mathbb{F}_2)$. This is an example of a Dyer-Lashof or power operation. Upshot: structure on X gives structure on H_*X. This is *great*.

Here's an example: try to compute any spectral sequence. Just as knowing the fact that cohomology has a ring structure vastly simplifies the computation of the Serre spectral sequence, the use of power operations allows computations for spectral sequences that respect those operations. It is always good when we can propagate differentials from known ones!

Here's another example: the sphere is an E_∞-ring spectrum. Bruner, in his thesis, used the E_∞-ring structure on the sphere to get power operations on the Adams spectral sequence and get tons of differentials. (This has even earlier precedents in work of Kahn, Milgram, and others.)

That was reason 1 why we care about things being E_∞. Reason 2 is that we don't care. But we want a map $X \to Y$, say, or a spectrum. If X and Y have no structure, then there are lots of maps. Suppose you have some Thom spectrum e.g. $MString$, and you want to build a cobordism invariant that lands in some spectrum X. If you just know what it does on homotopy groups, you have no hope – there are just so many maps $MString \to X$ that you'll never find the "right" one. Suppose we think of $MString$ as a cell complex, and try to build a map by defining it on the n-skeleton, one n at a time. There are lots of choices, and at every stage there's a significant possibility that you'll just choose the wrong one. But maybe there's extra structure, and you declare that you want your map to respect this structure.

There are fewer choices of maps that respect the structure, but if you can find a map that respects this structure it's much more likely to be a good one.

For example, $C_2 = \{1, \tau\}$ acts on kU by complex conjugation; suppose you want to find the automorphism $kU \xrightarrow{\tau} kU$ in $[kU, kU]$. There's an entire Adams spectral sequence converging to $[kU, kU]$; there are lots of differentials, extensions, and nonzero groups for obstructions to lie in. But Goerss-Hopkins obstruction theory builds it for free: there are no nonzero groups for obstructions to live in.

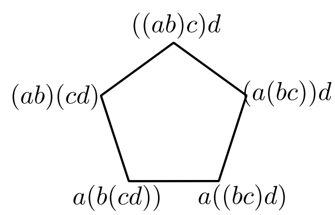
Talk 4: Robinson's A_∞ obstruction theory (Foling Zou)

Plan:

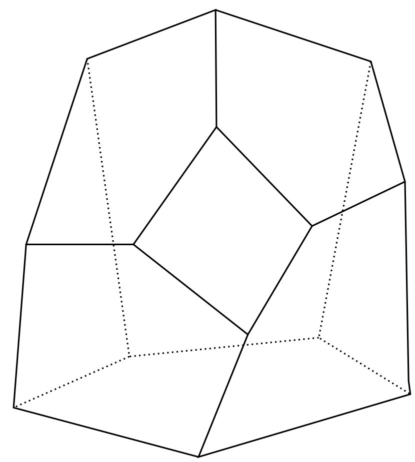
- Associahedra and definition of an A_n-structure on ring spectra
- Robinson's obstruction theory
- Application: show $K(n)$ admits (uncountably many) A_∞-structures
- $\widehat{E(n)}$ admits an A_∞-structure

4.1. A_n and Stasheff associahedra. We work in a symmetric monoidal category of spectra. A ring spectrum is a monoid in the homotopy category. That is to say, it is a spectrum that has a homotopy associative multiplication and a homotopy unit. For some purposes, we would like more coherence structures on the homotopy associative multiplication, which are called A_n structures that we are going to define.

Recall that $K_0 = K_1 = *$, and K_n is an $(n-2)$-dimensional polyhedron such that every $(n-2-i)$-dimensional cell corresponds to a way to put i parentheses in n letters. For example, $K_2 = *$, K_3 is an interval with endpoints given by $(ab)c$ and $a(bc)$, K_4 is the pentagon



and K_5 looks like



This has 6 pentagons and 3 squares. In general, for $1 \le j \le r$, there is a "face map" $\partial_j^{r,s} : K_r \times K_s \to K_{r+s-1}$ picking out the face corresponding to putting 1 pair of parentheses of s letters starting at the j^{th} spot; it is the map

$$\partial_j^{r,s} : K_r \times (K_1)^{j-1} \times K_s \times (K_1)^{r-j} \to K_{r+s-1}$$

given by regarding $j-1$ 1-letter words, one s-letter word, then $r-j$ 1-letter words as an r-letter word. Down to the cells, the product of an i-dimensional cell of K_r and a j-dimensional cell of K_s corresponds to $r-2-i$ parentheses and $s-2-j$ parentheses, so the composition procedure above gives a word with $(r-2-i)+(s-2-j)+1 = (r+s-1)-2-(i+j)$ parentheses, corresponding exactly to an $(i+j)$-dimensional cell in K_{r+s-1}. You can extend this to an operad map.

$\mathcal{K}(n) = K_n$ forms a (non-Σ) operad. We can define a truncated sub-operad $\mathcal{K}_n$ of $\mathcal{K}$ for each n by setting $\mathcal{K}_n(j) = K_j$ for $j \le n$, and then closing up under the structure maps.

Definition 4.1. An A_n-*structure* on a ring spectrum E is defined to be an algebra structure over $\mathcal{K}_n$. Note that in this definition, for $n \ge 3$ we require our ring spectrum to be homotopy associative but we require a strict unit.

Definition 4.2. An $\widehat{A}_n$-structure on E is the same as an A_n-structure, except you don't require the unit to be strict. That is, you have

- $\mu_m : \mathcal{K}_{m+} \wedge E^{(m)} \to E$ for $2 \le m \le n$;
- μ_2 has a two-sided homotopy unit;
- The following diagram commutes:

$$\begin{array}{ccc}
(K_r \times K_s)_+ \wedge E^{(r+s-1)} & \xrightarrow{\cong} & K_{r+} \wedge E^{(j-1)} \wedge ((K_s)_+ \wedge E^{(s)}) \wedge E^{(r-j)} \\
 & & \downarrow{\scriptstyle 1\wedge\mu_s\wedge 1} \\
\downarrow{\scriptstyle \partial_j^{r,s}} & & K_{r+} \wedge E^{(j-1)} \wedge E \wedge E^{(r-j)} \\
 & & \downarrow{\scriptstyle \mu_r} \\
K_{r+s-1+} \wedge E^{(r+s-1)} & \xrightarrow{\mu_{r+s-1}} & E
\end{array}$$

For example, $\widehat{A}_2$ means you have a multiplication and homotopy unit; $\widehat{A}_3$ is that you have a homotopy between two ways of multiplying three letters, which is $\widehat{A}_2$ plus homotopy associativity of the multiplication; and $\widehat{A}_4$ is that you can fill in the pentagon of homotopies, which is homotopy between the homotopies.

In this language, a ring spectrum is a spectrum with an $\widehat{A}_2$ structure that is extendable to $\widehat{A}_3$.

In Robinson's original paper, he studied the obstruction to $\widehat{A}_n$ structure and he proved in fact in a latter paper that an $\widehat{A}_n$ ring spectrum can always be given an A_n structure.

Robinson's obstruction theories. Goal: Given an $\widehat{A}_{n-1}$-structure for E (for $n \geq 3$), we want to extend this to an $\widehat{A}_n$-structure on E, or understand obstructions to such an extension. So we want to come up with $\mu_n : K_{n+} \wedge E^{(n)} \to E$, where the restriction to the boundary is already defined , $c_n : (\partial K_n)_+ \wedge E^{(n)} \to E$.

Since $K_n \cong D^{n-2}$, we have (passing to homotopy classes) $c_n \in E^0(S^{n-3} \wedge E^{(n)}) \cong E^{3-n}(E^{(n)})$. Just like the classical case of obstruction theory:

(1) We can extend the map from the boundary of a disk to the inner part iff the map is trivial (i.e., here, $c_n = 0$).

(2) If c_n is not trivial, can we change the $\widehat{A}_{n-1}$-structure to make it trivial?

We would like to make some assumptions to study this group: assume a perfect universal coefficient theorem and Künneth theorem hold for E. To be more precise, what we need is

$$E^*(E^{(n)}) \cong \mathrm{Hom}_{E_*}((E_*E)^{\otimes n}, E_*).$$

This is a pretty strong restriction but it is satisfied by the examples we'll talk about today. (For example, it holds for Landweber-exact things.) Then $c_n \in \mathrm{Hom}^{3-n}_{E_*}((E_*E)^{\otimes n}, E_*)$. Denote $(E_*E, E_*) = (\Lambda, R)$; there will be an augmentation $\varepsilon : \Lambda \to R$ because of the ring structure on E.

Definition 4.3. The *unnormalized Hochschild complex* is given by:

$$C^{n,*}(\Lambda|R; R) \cong \mathrm{Hom}^*_R(\Lambda^{\otimes n}, R).$$

There is a differential on this complex $\delta : C^n \to C^{n+1}$ given by

$$\begin{aligned}(\delta\theta)(\lambda_0 \otimes \ldots \otimes \lambda_n) = \varepsilon(\lambda_0)\theta(\lambda_1 \otimes \ldots \otimes \lambda_n) + \sum_{i=1}^{n} (-1)^i \theta(\lambda_0 \otimes \ldots \otimes \lambda_{i-1}\lambda_i \otimes \ldots \otimes \lambda_n) \\ + (-1)^n \theta(\lambda_0 \otimes \ldots \otimes \lambda_{n-1})\varepsilon(\lambda_n).\end{aligned}$$

The homology of this complex is called Hochschild cohomology:

$$HH^{n,m}(\Lambda; R) = H^n(C^{*,m}(\Lambda|R; R)).$$

Then notice $c_n \in C^{n,3-n}(\Lambda|R; R)$.

If we study the A_n-structure instead of the $\widehat{A}_n$-structure, then the obstruction c_n will lie in the reduced Hochschild complex

$$\widetilde{C}^{n,3-n}(\Lambda|R;R) = \mathrm{Hom}_R^{3-n}(\overline{\Lambda}^{\otimes n}, R),$$

where $\overline{\Lambda}$ is the augmentation ideal. There is a normalization theorem saying that the homology of the normalized complex is the same as the homology of the un-normalized complex, so it doesn't matter which one we use for obstruction theory.

Proposition 4.4. *Suppose $n \geq 4$ and the previous assumptions. Then if we alter the $\widehat{A}_{n-1}$-structure on E by a class $a \in C^{n-1,3-n}(\Lambda|R;R)$ fixing the $\widehat{A}_{n-2}$ structure, then the obstruction class c_n is altered by the boundary δa.*

First let's see why the difference is a class $a \in C^{n-1,3-n}(\Lambda|R;R)$. Suppose we fix the $\widehat{A}_{n-2}$ structure on E and we want to change the A_{n-1}-structure. Then on $K_{n-1+} \wedge E^{(n-1)} \to E$, we fix the map on the boundary, and alter the map on the inside. So the difference is a map $(K_{n-1} \cup_{\partial K_{n-1}} K_{n-1})_\wedge E^{(n-1)} \to E$. But $K_{n-1} \cong D^{n-3}$ and so the difference lies in $S^{n-3} \wedge E^{(n-1)} \to E$, a.k.a. it represents a class in $E^0(\Sigma^{n-3}E^{(n-1)}) \cong E^{3-n}(E^{(n-1)}) \cong C^{n-1,3-n}(\Lambda|R;R)$.

Then the reason that the obstruction is altered by the boundary of a is that now we're looking at a map $c_n : \partial K_{n+} \wedge E^n \to E$, and the only face maps that involve K_{n-1} are $\partial_j^{n-1,2} : K_{n-1} \times K_2 \to K_n$ for $1 \leq j \leq n-1$ and $\partial_j^{2,n-1} : K_2 \times K_{n-1} \to K_n$ for $1 \leq j \leq 2$. If you trace through the difference on these faces, you exactly get the $n+1$ terms in the formula for (δa).

Theorem 4.5. *Let E be a ring spectrum satisfying the above assumptions, and suppose E has an $\widehat{A}_{n-1}$-structure with $n \geq 4$. Then*

- *c_n is a cocycle in $C^{n,3-n}(\Lambda|R;R)$*
- *The $\widehat{A}_{n-2}$-structure on E can be refined to an $\widehat{A}_n$-structure iff the class of c_n is zero in $HH^{n,3-n}(\Lambda;R)$.*

The reason is that, as we observed in the proposition, if you change the $\widehat{A}_{n-1}$-structure, you can change the obstruction by a coboundary. So we only care about the obstruction group modulo coboundaries.

The upshot is the following:

Theorem 4.6. *With assumptions, given an $\widehat{A}_{n-1}$-structure on E, the $\widehat{A}_{n-2}$ can be extended to an $\widehat{A}_n$-structure iff $[c_n] = 0 \in HH^{n,3-n}(\Lambda|R;R)$.*

Example: Morava K-theory. Fix a prime p and an integer $n \geq 1$. Then Morava K-theory is a ring spectrum $K(n)$ such that $K(n)_* \cong \mathbb{F}_p[v_n^{\pm}]$, where $|v_n| = 2(p^n - 1)$. Classically, $K(n)$ is constructed by some bordism theory with singularities. It's proved that it's a ring spectrum. This is work of Mironov, Morava, and Shimada.

It is a theorem of Yagita (for odd p) that

$$K(n)_*K(n) \cong \Sigma(n) \otimes \Lambda_{K(n)_*}(\tau_1, \dots, \tau_{n-1})$$

where $|\tau_j| = 2p^j - 1$. With this information, Robinson computed (for odd p)

$$HH^{**}(K(n)_*K(n), K(n)_*) \cong K(n)_*[\alpha_1, \dots, \alpha_{n-1}]$$

where $|\alpha_j| = (1, 2p^j - 1)$. So that these cohomology groups HH^{**} are concentrated in even total degrees. Since the obstruction classes lie in odd total degree, given any $\widehat{A}_{n-1}$-structure, there are no obstructions to extending the underlying $\widehat{A}_{n-2}$-structure to $\widehat{A}_n$. This means that we can always alter the $\widehat{A}_{n-1}$-structure to make it extendable.

Theorem 4.7. *If p is odd and $n \geq 1$ then $K(n)$ admits an $\widehat{A}_\infty$-structure.*

At $p = 2$, the assumptions aren't satisfied (and also the computation doesn't work). Robinson's assumptions are implicitly assuming E is homotopy commutative, which is not the case when $p = 2$.

Angeltveit overcomes this obstacle; his method works in a more general setting than Robinson's and also works at $p = 2$.

A_n-structures on homomorphisms. Let E, F be A_N-ring spectra, and suppose we have a ring map $\varphi : E \to F$. Question: can φ be given an A_N-structure?

Definition 4.8. An A_n-*structure* ($n \leq N$) on φ is an A_n-structure on the mapping cylinder $M_\varphi = (I_+ \wedge E) \cup_\varphi F$ that restricts to the given A_n-structures on E and F.

Robinson claimed that, homotopically, the obstruction theory on M_φ is the same as constructing maps $(K_n \times I)_+ \wedge E^{(n)} \to F$.

Using the same logic as before, we have an obstruction theory for extending structures on maps:

Theorem 4.9. *Let $\varphi : E \to F$ be as above. Then*

- *The obstruction to extending an A_{n-2}-structure on φ to A_n is $b_n \in C^{n,2-n}(E_*E; F_*)$*
- *The A_{n-2}-structure extends to A_n iff $[b_n] = 0 \in HH^{n,2-n}(E_*E, F_*)$.*

With the calculations before, one can conclude the following observations about the obstruction groups for uniqueness for $K(n)$ when p is odd:

$$\dim_{\mathbb{F}_p} HH^{k,2-k}(K(n)_*K(n), K(n)_*) = \begin{cases} 0 & k \leq p-1 \\ 1; & k = p \\ \neq 0; & \textit{for infinitely many } k. \end{cases}$$

Indeed, every element in $HH^{k,2-k}(K(n)_*K(n), K(n)_*)$ can be realized as a difference class of two A_∞-structures, because you can take a cocycle representing the class and alter the A_n-structure of a fixed A_∞-structure by the cocycle. Being a cocycle means that the altered

A_n-structure is extendable to an A_{n+1}-structure, and since all obstruction groups of $K(n)$ vanish, extendable to A_{n+1} will imply extendable to A_∞.

Robinson concluded from these calculations the following description about the higher structures on $K(n)$:

Theorem 4.10. *Let p be odd with $n \geq 1$. Then*

- *$K(n)$ has exactly one A_{p-1}-structure which can be extended;*
- *$K(n)$ has p A_p-structures which can be extended;*
- *$K(n)$ has uncountably many A_∞-structures, in the sense that the identity map $K(n) \to K(n)$ is not A_∞ if the $K(n)$'s are given different A_∞-structures. However, Angeltveit proved that there is a weak equivalence $K(n) \to K(n)$ that carries one A_∞-structure to another.*

Completed Johnson-Wilson theory. Let $E(n)$ be the Johnson-Wilson spectrum, with $E(n)_* \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm}]$. Let $I_n = (p, v_1, \ldots, v_{n-1})$. So $E(n)/I_n = K(n)$, and we can form an inverse limit diagram

$$\cdots \to E(n)/I_n^2 \to E(n)/I_n = K(n)$$

and define $\widehat{E(n)} = \operatorname{holim}_k E(n)/I_n^k$.

Theorem 4.11 (Baker, 2000)**.** *The assumptions for Robinson obstruction theory are satisfied for $\widehat{E(n)}$. Furthermore,*

$$HH^{r,*}(\widehat{E(n)}_* \widehat{E(n)}, \widehat{E(n)}_*) = \begin{cases} \widehat{E(n)}_* & r = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$HH^{r,*}(\widehat{E(n)}_* \widehat{E(n)}, K(n)_*) = \begin{cases} K(n)_* & r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.12. *$\widehat{E(n)}$ admits an A_∞-structure that is compatible with its underlying canonical ring structure. The obstructions to non-uniqueness also vanish, so this is a unique A_∞-structure. Moreover, the natural map $\widehat{E(n)} \to K(n)$ can be given an A_∞-structure.*

Small issue: actually $\widehat{E(n)}$ satisfies the universal coefficient theorem for *continuous* cohomology, and so you need to consider *continuous* Hochschild cohomology... It works out, anyway.

Talk 5: Robinson's E_∞ obstruction theory (J.D. Quigley)

Let V be a homotopy commutative, associative ring spectrum.

Definition 5.1. An E_∞-*ring structure* on V is a morphism of operads $\mathcal{C} \to \operatorname{End}(V)$ where $\mathcal{C}$ is an E_∞ operad.

"Recall" the endomorphism operad $\mathrm{End}(V)$ is defined so $\mathrm{End}(V)_n = \mathrm{Hom}(V^{\wedge n}, V)$. We should assume V is in some "nice" category of spectra like S-modules or symmetric spectra.

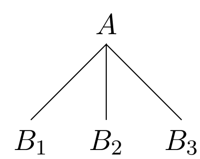
We want to mimic Robinson's obstruction theory for A_∞-rings, but in the E_∞ setting. Our first attempt might be to choose $\mathcal{C} = \mathcal{E}$, the Barratt-Eccles operad – recall this is the operad whose n^{th} level is $E\Sigma_n$. In the A_n case, we extensively used the boundary of the Stasheff associahedron operad. What is the boundary here?

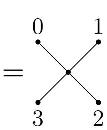
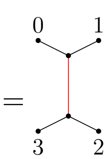
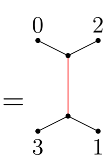
Claim 5.2. *There's not a great notion of boundary for the Barratt-Eccles operad. In particular, there is a notion of boundary of an E_∞ operad, but the inclusion $\partial\mathcal{E}_n \hookrightarrow \mathcal{E}_n$ is not a Σ_n-equivariant cofibration.*

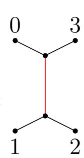
For example, if you want to induct up the skeleton of a cell complex, you want the inclusions of skeleta to be cofibrations.

So we need a different operad. Robinson and Whitehouse use instead the tree operad (though sometimes that is used to refer to a further refinement to be defined later). Define $\widetilde{T}$ to be the operad where $\widetilde{T}_n$ is the space of isomorphism classes of trees with $(n+1)$-labelled leaves, and internal edges (i.e. those whose endpoints aren't leaves) of length in $(0, 1]$.

For example, $\widetilde{T}_3$ looks like:



where the vertices A, B_1, B_2, B_3 are given by trees $A =$  , $B_1 =$  , $B_2 =$  ,

and $B_3 =$  , where the red edges denote internal edges of length 1. (It looks like there should be 6 permutations, but e.g. B_1 is isomorphic to the tree that looks like B_1 except with 2 and 3 swapped.) The edge from B_1 to A corresponds to shrinking the one internal edge to a point. Say a tree is *fully grown* if at least one internal edge has length 1. For example, in the trees above the fully grown ones are the B_i's.

Exercise 5.3. Draw the subspace of fully grown trees in $\widetilde{T}_4$.

Let $\partial\widetilde{T}_n$ denote the subspace of fully grown trees in $\widetilde{T}_n$.

Facts 5.4.

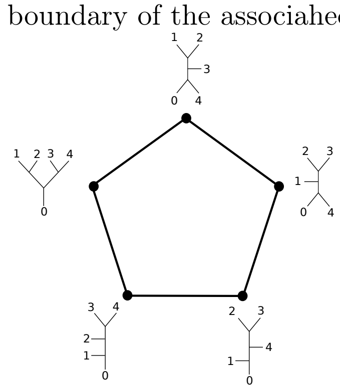
(1) There is a homotopy equivalence $\partial\widetilde{T}_n \simeq \bigvee_{(n-1)!} S^{n-3}$ and the representation of Σ_{n-1} on $H_{n-3}(\partial\widetilde{T}_n)$ is isomorphic to the regular representation.

(2) $\widetilde{T}_n/\partial\widetilde{T}_n$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-2$ and its homology is isomorphic as a Σ_n-module to $\varepsilon \cdot \mathrm{Lie}_n$, , where ε denotes the sign representation. (There are some different notational conventions here. In some of the sources, the signed action of permutations is considered to be built in to Lie_n^*, in which case the ε doesn't appear.)

(3) The integral representation of Σ_{n+1} on $H_{n-3}(\partial\widetilde{T}_n)$ has character

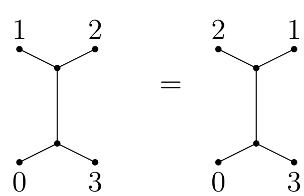
$$\varepsilon \cdot (\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n - \mathrm{Lie}_{n+1}).$$

Remark 5.5. For (1), you should really be thinking of this as $\partial\widetilde{T}_n \simeq \bigvee_{(n-1)!} \partial K_n$, where e.g. if $n = 4$, one of the copies of the boundary of the associahedron K_4 comes from:



(so all the trees represented by the points on the boundary have at least one internal edge of length 1) and the other copies come from permuting the entries.

The Lie operad is an operad in k-modules where Lie_n consists of bracketings of n letters, modulo anti-commutativity (or commutativity if permutations act with signs) and the Jacobi relation. Since there's an obvious way to associate a binary tree to a bracketing, you can relate this to the free k-module on binary trees with n leaves, modulo the Jacobi relation. In fact, the relationship involves building in the sign representation. So for example, if you're working with rooted trees with root 0, the relation $[[1, 2], 3] = -[[2, 1], 3]$ corresponds to the permutation $(1, 2)$ acting with its sign -1 and the relation of (isomorphism classes of) trees:



So we have:

(1) $\{\widetilde{T}_n\}$ form an operad $\widetilde{T}$

(2) $\partial\widetilde{T}_n \hookrightarrow \widetilde{T}_n$ is a Σ_n-equivariant cofibration

(3) $\widetilde{T}$ is not E_∞ (the action isn't free)

We're going to fix this to make it an E_∞ operad, by combining it with the Barratt-Eccles operad: define

$$\mathcal{T} = \mathcal{E} \times \widetilde{T}.$$

The claim is that this is cofibrant and E_∞.

The boundary is

$$\partial \mathcal{T}_n = E\Sigma_n \times \partial \widetilde{T}_n.$$

There is a filtration

$$\mathcal{T}_n^r = E\Sigma_n^r \times \widetilde{T}_n.$$

Shift around degrees to define

$$\nabla^r \mathcal{T}_m = \mathcal{T}_m^{r-m} = E\Sigma_m^{r-m} \times \widetilde{T}_m$$

(you have to define what happens around when $r-m<0$; this is probably like the definition of the truncated stages of the associahedron operad in the last talk). We can define a boundary

$$\partial \nabla^r \mathcal{T}_m = \nabla^r \mathcal{T} \cap \partial \mathcal{T}_m.$$

Definition 5.6. An *n-stage* for an E_∞-structure on V is a sequence of maps

$$\mu_m : \nabla^n \mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)} \to V$$

which on a restricted domain satisfy requirements for a morphism of operads $\mathcal{T} \to \mathrm{End}(V)$.

If you could lift an n-stage to an $(n+1)$-stage and keep doing that forever, you would get exactly the structure maps for an E_∞-ring spectrum.

Exercise 5.7. Show that a 2-stage corresponds to having a multiplication, and a 3-stage corresponds to being commutative and associative up to homotopy.

Let's try to do the obstruction theory, and see how far we can get.

We want to extend an n-stage, given by a collection of maps $\{\mu_m\}$, and we want to get an $(n+1)$-stage. We need, for all $2 \leq m \leq n+1$, an extension of

$$\mu_m : \nabla^n \mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)} \to V$$

over $\nabla^{n+1}\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)}$. Through sheer willpower, you can show that an n-stage actually determines an extension of $\partial\nabla^{n+1}\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)}$, using a shuffle deformation retract onto some smaller Σ_i's.

Just like in the A_∞ case, there's going to be an obstruction living in the V-cohomology of some space. The obstruction to extending μ_m lies in

$$V^1((\nabla^{n+1}\mathcal{T}_m/(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m)) \ltimes_{\Sigma_m} V^{(m)}).$$

Let $R = V_*$ and $\Lambda = V_*V$. We need to assume a universal coefficient isomorphism: $V^*(V^{(m)}) \cong \mathrm{Hom}_R(\Lambda^{\otimes m}, R)$ for all m. Using this we can identify the group where the obstruction lives as

$$\begin{aligned} V^1(((E\Sigma_m^{n-m+1}/E\Sigma_m^{n-m}) \wedge (\widetilde{T}_m/\partial\widetilde{T}_m)) \ltimes_{\Sigma_m} V^{(m)}) \\ = V^1(((\bigvee_{\Sigma_m^{n-m+2}} S^{n-m+1}) \wedge (\bigvee_{(m-1)!} S^{m-2})) \ltimes_{\Sigma_m} V^{(m)}) \\ \cong \mathrm{Hom}_R^{2-n}(\varepsilon \cdot \mathrm{Lie}_m^* \otimes R[\Sigma_m^{n-m+1}] \otimes \Lambda^{\otimes n}, R) \end{aligned} \tag{5.1}$$

(Here in the first step we have identifyied the layers of the bar construction, giving the first wedge of spheres, with free Σ_m-action. And we have used the previous results about the space of trees, giving the second wedge of spheres with its action via the $\varepsilon \cdot \mathrm{Lie}_m$ representation. In the second step we have used the universal coefficient isomorphism; the total dimension of the spheres produces a degree shift of $n-1$; the dual representation $\varepsilon \cdot \mathrm{Lie}_m^*$ appears because of the Hom.)

Definition 5.8. Let Γ be the category of finite pointed sets and basepoint-preserving maps. A *left* Γ*-module* is a functor $F : \Gamma \to k$-modules. (Here k is a commutative unital ring.)

Given such a Γ-module, the machinery of Segal and Bousfield-Friedlander gives you a spectrum called $\|F\|$. Define

$$\pi_k F := \pi_k \|F\|.$$

Example 5.9. There is a right Γ-module $t : \Gamma^{op} \to k$-Mod given by $t(S) = \mathrm{Hom}_{\mathrm{Set}_*}(S, k) = \mathrm{Hom}_k(kS, k)$. There's also a dual $t^*(S) = k[S]/k[0]$ (think of this as reduced chains).

Theorem 5.10 (Pirashvili)**.** $\pi_* F \cong \mathrm{Tor}_*^{\Gamma}(t, F)$

(This is the homology of some sort of bar construction.)

Example 5.11. Let R be an associative graded commutative ring, Λ a graded commutative R-algebra and G a Λ-module. Define the *Loday functor*

$$\begin{aligned} \mathcal{L}(\Lambda|R; G) : \Gamma \to \Lambda\text{-Mod} \\ [n] \mapsto \Lambda^{\otimes_R n} \otimes_R G \end{aligned}$$

This should remind you of the cyclic bar construction. You can make this a Γ-module.

Definition 5.12. The Γ-*homology* of Λ relative to R with coefficients in G, is

$$H\Gamma_*(\Lambda|R; G) := \pi_*(\mathcal{L}(\Lambda|R; G)).$$

Similarly, define $H\Gamma^*(\Lambda|R; G) := \pi^* \mathrm{Hom}_\Lambda(\mathcal{L}(\Lambda; R), G)$

Define

$$\Xi_{p,q} = \varepsilon \cdot \mathrm{Lie}_{q+1}^* \otimes k[\Sigma_{q+1}^p] \otimes F([q+1]).$$

The rows in this are the bar construction on the symmetric group, so the horizontal differential is the bar differential. The vertical differential is quite complicated; in the analogue for the associative case it's related to the boundary map for associahedra and thus to the Hochschild differential; the version here builds in the interaction with permutations. It can be found in p. 335 of Robinson's E_∞ obstruction theory paper.

There is a noncanonical isomorphism

$$H_*(\mathrm{Tot}\,\Xi(F)) \cong \pi_* F.$$

Definition 5.13. The Γ-*cotangent complex* is

$$\mathcal{K}(\Lambda|R) = \Xi\mathcal{L}(\Lambda|R; R).$$

This is some bicomplex; Robinson shows

$$H\Gamma^*(\Lambda|R; R) \cong H_* \,\mathrm{Tot}\,\Xi(\mathrm{Hom}_R(\mathcal{L}(\Lambda|R; R), R)).$$

If you work through what all of this means, you end up identifying

$$(5.1) \cong \mathrm{Hom}_\Lambda^{n-m+1,m-1,2-n}(\mathcal{K}(\Lambda|R), R).$$

This tells us where the obstructions to existence of an extension from the n-stage to an $(n+1)$-stage live.

You get the obstructions to uniqueness by understanding exactly what the differentials are doing, and comparing them to the filtration of the operad.

Theorem 5.14 (Robinson)**.** *Suppose given an $(n-1)$-stage μ for V which can be extended to an n-stage. Then there is a natural Γ-cohomology class*

$$[\theta] \in H\Gamma^{n,2-n}(\Lambda|R; R),$$

the vanishing of which is necessary and sufficient for μ to be extendable to an $(n+1)$-stage. If $H\Gamma^{n,1-n} = 0$ for all n, this is unique up to homotopy.

(Actually there's a more precise version of the uniqueness statement that allows you to say something about the uniqueness of extension to each stage.)

5.1. Appendix: Analogy with ordinary homology (by Dylan Wilson). *This is Dylan's response during an informal questions seminar, when asked to give more intuition for Γ-homology.*

The main goal is to get tools to study and build algebras and maps between them. I claim there's an analogy with spaces as follows:

Spaces	E_∞-algebras
X	A
Sing	Loday functor
singular chains	Γ-module associated to A, M
ordinary homology	Γ-homology

In the world of spaces, you have a space X and look at the singular construction $\mathrm{Sing} : \Delta^{op} \to \mathrm{Set}$ sending $n \mapsto \mathrm{Hom}(|\Delta^n|, X)$. On the algebra side, let A be an E_∞ algebra; we have a functor Fin $\to$ Sp sending $S \mapsto A^{\wedge S}$. This is called the Loday functor. It is symmetric monoidal up to coherent homotopy.

On the space side, I can make $\mathbb{Z}[\mathrm{Sing}(X)] \otimes_{\mathbb{Z}} M$. The E_∞ analogue is the functor $\mathrm{Fin}_* \to \mathrm{Sp}$ $\to$ *k-mod?* sending $S_+ \mapsto A^{\otimes S} \otimes M$. This is the Γ-module associated to A and M.

On the space side, given $\mathbb{Z}[\mathrm{Sing}(X)] \otimes_{\mathbb{Z}} M$ I could either take $\pi_*(-)$ or the homology of the normalized chain complex on this. On the algebra side, I want to get a spectrum $\|A^\Gamma\|$ and compute its homotopy groups, or equivalently get a chain complex (the Γ-cotangent complex) and compute its homology.

Talk 6: Γ-homology I: Properties and calculations (Robin Elliott)

Outline:

(1) Properties

(2) Calculations

(3) Application

Let Λ be a commutative algebra over a commutative ring R, and M a Λ-module (E_∞ could replace commutative). We will consider $H\Gamma(\Lambda|R;M)$.

The point is: given a homotopy associative and commutative ring spectrum E, take $\Lambda = E_*E$, $R = E_*$, $M = E_*$. The obstructions to lifting an n-stage live in $H\Gamma^*(\Lambda|R;M)$.

6.1. Properties.

I'll write most of the properties for homology, but they all work for cohomology as well. I'll switch notation temporarily so B is a commutative A-algebra, and M is a B-module.

(A) (LES in coefficients) Given $0 \to M' \to M \to M'' \to 0$ we get a LES
$$\cdots \to H\Gamma_s(B|A;M') \to H\Gamma_s(B|A;M) \to H\Gamma_s(B|A;M'') \to H\Gamma_{s-1}(B|A;M') \to \cdots$$
This comes from the SES on the Γ cotangent complex.

(B) (Transitivity LES) Given inclusions $A \subset B \subset C$ of algebras and M a C-module, there exists a LES
$$\cdots \to H\Gamma_{s+1}(C|B;M) \to H\Gamma_s(B|A;M) \to H\Gamma_s(C|A;M) \to H\Gamma_s(C|B;M) \to \cdots.$$

(C) (Flat base change) Suppose we have a flat map $A \to B$ and a map $A \to C$. Let M be an $B \otimes_A C$-module. Then
$$H\Gamma_*(B \otimes C|C;M) \cong H\Gamma_*(B|A;M).$$

(D) (Vanishing for étale algebras) If B is an étale A-algebra and M is any B-module, then $H\Gamma_*(B|A;M) = 0$.

(E) (Flat additivity) Let B and C be flat A-algebras, and M a $B \otimes_A C$-module. Then
$$H\Gamma_*(B \otimes_A C|A;M) \cong H\Gamma_*(B|A;M) \oplus H\Gamma_*(C|A;M).$$

(F) If B contains $\mathbb{Q}$ and M is any B-module then
$$H\Gamma_*(B|A;M) \cong AQ_*(B|A;M)$$

where AQ is André-Quillen homology (will be defined in a few talks). In general, there is a map between these.

(G) If X is a chain complex of flat k-modules, and $\mathcal{C}$ is an E_∞-operad, there is a free E_∞-algebra on X given by

$$FX = X \oplus \bigoplus_{n \geq 1} \mathcal{C}(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

Then

$$H\Gamma_*(FX|k; M) \cong H_*(X; M).$$

There are different methods of constructing Γ-homology and Γ-cohomology, and these properties come from constructing it in a different way. You can get better properties for strictly commutative algebras.

6.2. Calculations.

Theorem 6.1. *Let k be a commutative ring with unit and G a discrete abelian group. Let $k[G]$ denote the group ring. Note k is a $k[G]$ module where each $g \in G$ acts as 1. Then*

$$H\Gamma_*(k[G]|k; k) \cong Hk_*HG.$$

The idea is we want to use $H\Gamma$ being the stable homotopy of the relevant Γ-module. For the group ring, you can explicitly write down what the Loday functor is in a slightly nicer form. Then there's a comparison to a cubical construction that Eilenberg did in the 50's. This is the one hard hands-on calculation.

We're only going to use this for the case $G = \mathbb{Z}$.

Proposition 6.2. *Let A be a commutative augmented k-algebra, G an abelian group such that $k[G]$ is étale over A. Then*

$$H\Gamma_*(A|k; k) \cong H\Gamma_*(k[G]|k; k) \cong Hk_*HG.$$

Proof. We have $k \subset A \subset k[G]$ and the second inclusion is étale. So we get a transitivity LES

$$\cdots \to \underbrace{H\Gamma_{s+1}(k[G]|A; A)}_{0} \to H\Gamma_s(A|k; k) \xrightarrow{\cong} H\Gamma_s(k[G]|k; k) \to \underbrace{H\Gamma_s(k[G]|A; k)}_{0} \to \ldots$$

□

Corollary 6.3. $k[\mathbb{Z}] = k[x, x^{-1}]$ *is étale over* $k[x]$*, and so*

$$H\Gamma_*(k[x]|k; k) \cong Hk_*H\mathbb{Z}.$$

Definition 6.4. Let L be a K-algebra for a field K. L is *étale over* K if $L = \prod_{\text{finite}} L_i$ where L_i is a separable extension of K. More generally, if A is a ring, then B is étale over A if B is flat over A and of finite presentation and $\Omega_{B/A} = 0$. (Or, you could say that the field definition is true at every geometric point.)

Definition 6.5. If A is a commutative k-algebra which is essentially of finite type,[3] then say A is *smooth* if for all $\mathfrak{p} \in \operatorname{Spec} A$, there exists $f \notin \mathfrak{p}$ such that we can factorize $k \to A_f$ as $k \to k[x_1, \dots, x_m] \xrightarrow{\psi} A_f$ where ψ is étale.

(Geometrically, "at every point you look like a polynomial algebra.")

Theorem 6.6. *Let A be smooth and augmented over k. Then*

$$H\Gamma_*(A|k;k) = \bigoplus_{\dim_k \Omega^1_{A/k} \otimes k} Hk_* H\mathbb{Z}.$$

Proof. We have $A \subset A \subset A_f$ where the second inclusion is étale. By the same argument as before, $H\Gamma_*(A_f|k;k) \cong H\Gamma_*(A|k;k)$. The triple $k \to k[x_1, \dots, x_n] \xrightarrow{\psi} A_f$ has ψ is étale (by smoothness) and so the LES gives

$$H\Gamma_*(A|k;k) \cong H\Gamma_*(k[x_1, \dots, x_m]|k;k).$$

□

6.3. Application: Lubin-Tate spectra from Honda formal group laws.

Goal 6.7. There is a unique E_∞-structure on Lubin-Tate spectra from Honda formal group laws.

(Why care? Honda formal group laws are particularly nice formal group laws that we can actually write down more than we normally can.) By Theorem 5.14, we just need to show that $H\Gamma^s(E_*E|E_*;E_*) = 0$ for all $s \geq 0$.

Fix a height n and let E denote the Lubin-Tate spectrum for the Honda formal group law of height n. Here are some properties of this spectrum:

(1) $E_* = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^\pm]$ (for $|u_i| = 0$ and $|u| = 2$) has a maximal ideal $\mathfrak{m} := (p, u_1, \dots, u_{n-1})$, and $E_*/\mathfrak{m} = \mathbb{F}_{p^n}[u^\pm]$.

(2) E_*E is flat over E_*.

(3) $E_*E/\mathfrak{m} \cong E_0E/\mathfrak{m} \otimes_{E_0/\mathfrak{m}} E_*/\mathfrak{m}$.

(4) $E_0E/\mathfrak{m} = \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}[a_0, a_1, a_2, \dots]/(a_0^{p^n-1} - 1, a_1^{p^n} - a_1, a_2^{p^n} - a_2, \dots)$

Reference: Rezk's notes on the Hopkins-Miller theorem (e.g. 17.4 for the last fact).

Lemma 6.8. *To show vanishing, it is sufficient to show $H\Gamma^*(E_*E|E_*;E_*/\mathfrak{m}) = 0$.*

Proof. We have the SES

$$0 \to \mathfrak{m}^d/\mathfrak{m}^{d+1} \to E_*/\mathfrak{m}^{d+1} \to E_*/\mathfrak{m}^d \to 0.$$

[3] A is essentially of finite type if it is a localization of some finitely generated k-algebra

This gives a LES in the coefficients of $H\Gamma^*(E_*E|E_*;-)$. We know that $\mathfrak{m}^d/\mathfrak{m}^{d+1}$ is finite-dimensional over $E_*/\mathfrak{m}$, so by the assumption in the statement we have

$$H\Gamma^s(E_*E|E_*;\mathfrak{m}^d/\mathfrak{m}^{d+1}) = 0, \qquad \text{for all } s \geq 0 \text{ and } d \geq 0.$$

Then by induction and assumption for $d = 1$, the LES shows that $H\Gamma^*(E_*E|E_*;E_*/\mathfrak{m}^d) = 0$ for all $d \geq 1$. Now we have a SES

$$0 \to E_* \to \prod_{d\geq 1} E_*/\mathfrak{m}^d \overset{1-s}{\to} \prod_{d\geq 1} E_*/\mathfrak{m}^d \to 0$$

where s is the shift map. Now the LES gives $H\Gamma^*(E_*E|E_*;E_*) = 0$. □

Proof that $H\Gamma^*(E_*E|E_*;E_*/\mathfrak{m}) = 0$. I claim that

$$H\Gamma^s(E_*E|E_*;E/\mathfrak{m}) \cong H\Gamma^s(E_*E/\mathfrak{m}|E_*/\mathfrak{m};E_*/\mathfrak{m})$$

by flat base change along

$$\begin{array}{ccc} E_* & \longrightarrow & E_*/\mathfrak{m} \\ {\scriptstyle \text{flat}}\downarrow & & \downarrow \\ E_*E & \longrightarrow & E_*E/\mathfrak{m} \end{array}$$

We can also do flat base change along

$$\begin{array}{ccc} E_0/\mathfrak{m} & \xrightarrow{\text{flat}} & E_*/\mathfrak{m} \\ \downarrow & & \downarrow \\ E_0E/\mathfrak{m} & \longrightarrow & E_*E/\mathfrak{m} \end{array}$$

(we know it's flat by explicitly writing $E_*/\mathfrak{m} \cong \mathbb{F}_{p^n}[u^{\pm}]$ and $E_0/\mathfrak{m} \cong \mathbb{F}_{p^n}$). So the above is $H\Gamma^*(E_0E/\mathfrak{m}|E_0/\mathfrak{m};E_*/\mathfrak{m})$. We have that $E_0/\mathfrak{m} \cong \mathbb{F}_{p^n}$, and $E_0E/\mathfrak{m}$ is a tensor product of étale algebras, and so this is zero. □

Talk 7: Γ-cohomology II: Application: KU has a unique E_∞ structure (Pax Kivimae)

Theorem 7.1. *Let E be a homotopy commutative ring spectrum that satisfies*

$$E^*(E^{\wedge k}) = \mathrm{Hom}_{E_*}(E_*E^{\otimes k}, E_*).$$

In this case, E has a 3-stage structure automatically, and the obstruction to extending an n-stage structure on E to an $(n+1)$-stage lies in $H\Gamma^{n,2-n}(E_*E|E_*,E_*)$. Obstructions to uniqueness lie in $H\Gamma^{n,1-n}(E_*E|E_*,E_*)$.

Here are two results that we won't prove.

Theorem 7.2 (Richter)**.** *BP has a $(2p^2+2p-2)$-stage structure.*

Theorem 7.3. *If E has an n-stage structure for $n > p$, there exist Dyer-Lashof operations Q_i on $H_n(E;\mathbb{F}_p)$ for $i \leq n-p$.*

So you get some cool operations on BP that you can try to compute. But I'm not going to talk more about this. The main goal of this talk is the following application of Theorem 7.1:

Theorem 7.4 (Baker-Richter)**.** *If $E = KU$, KO, the Adams summand $E(1)$, or completed Johnson-Wilson theory $\widehat{E}(n)$, then E has a unique E_∞-structure.*

I'm going to focus on the case of KU. By the obstruction theory result, our main goal is to show:

Goal 7.5. $H\Gamma^n(KU_*KU|KU_*; KU_*) = 0$ for $n > 1$.

I'll need some properties of $H\Gamma^*$, which are corollaries of the properties about $H\Gamma_*$ Robin talked about:

(1) For a SES $0 \to M \to M' \to M'' \to 0$ we get a long exact sequence on $H\Gamma^*$.

(2) $H\Gamma^*(A|k; M) = 0$ if A/k is formally étale (remove "finite presentation" in the definition (6.4) of étaleness).

(3) If B is flat over A, then $H\Gamma^*(B \otimes_A C|C; M) = H\Gamma^*(B|A; M)$.

(4) If B and C are flat over A, then $H\Gamma^*(B \otimes_A C|A; M) = H\Gamma^*(B|A; M) \oplus H\Gamma^*(C|A; M)$.

(5) If A contains $\mathbb{Q}$, then $H\Gamma^*(B|A; M) = AQ^*(B|A; M)$.

Remark 7.6. If B is smooth over A then $AQ^*(B|A; M) = 0$ for $* > 0$.

7.1. Continuous $H\Gamma$ (and general lemmas about Γ-homology).

You can modify Γ-homology to form continuous Γ-homology in the same way that you have to modify group cohomology to make continuous group cohomology for profinite groups.

If A, B, and M all also have a topological structure (e.g. the $\mathfrak{m}$-adic topology on a profinite thing), then $\Xi(B|A; M)$ inherits a topology, and we can consider *continuous* linear maps $\mathrm{Hom}_{\mathrm{cts}}$, and define

$$H_c^*(B|A; M) := H^*(\mathrm{Hom}_{\mathrm{cts}}(\Xi(B|A); M)).$$

You want to commute cohomology with limits, but you can only do that via a Milnor exact sequence:

Lemma 7.7. *Let $\mathfrak{m} \subset A$ be a maximal ideal. Equip A and B with the $\mathfrak{m}$-adic topology. If M is a complete, Hausdorff module over A, we have the following SES.*

$$0 \to \varprojlim{}^1(H\Gamma^{n-1}(B/\mathfrak{m}^k; A/\mathfrak{m}^k; M/\mathfrak{m}^k)) \to H\Gamma_c^*(\widehat{B}|\widehat{A}; M) \to \varprojlim H\Gamma^*(B/\mathfrak{m}^k|A/\mathfrak{m}^k; M/\mathfrak{m}^k) \to 0.$$

In all the cases we care about here, the $\lim^1$ term will vanish; this is a consequence of a general vanishing theorem that only makes assumptions on the ($\mathfrak{m}$-adic) topology on these things (see Theorem 1.1 in the paper).

Lemma 7.8. *Assume the same setup as the previous lemma. Suppose A is Noetherian and $B^\wedge_{\mathfrak{m}}$ is countably free over $A^\wedge_{\mathfrak{m}}$. Then*

$$H\Gamma^*_c(B^\wedge_{\mathfrak{m}}|A^\wedge_{\mathfrak{m}}; M) = H\Gamma^*(B^\wedge_{\mathfrak{m}}|A^\wedge_{\mathfrak{m}}; M) = H\Gamma^*(B|A; M).$$

Now we introduce the main hammer that we use to hit these things with.

Theorem 7.9. *Let A be an augmented $\mathbb{Z}$-algebra. Suppose:*

(1) $A^\wedge_p$ is free on a countable basis over $\mathbb{Z}_p$ for all p.

(2) A/p^k is formally étale over $\mathbb{Z}/p^k$ for all p, k.

Then

$$H\Gamma^*(A|\mathbb{Z}; \mathbb{Z}) = H\Gamma^{*-1}(A \otimes \mathbb{Q}|\mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z}).$$

Here $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$. By the Chinese remainder theorem, we have $\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$.

Exercise 7.10. $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a $\mathbb{Q}$-vector space.

Proof of Theorem 7.9. Consider the SES $0 \to \mathbb{Z} \to \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}/\mathbb{Z} \to 0$. If we knew that $H\Gamma^*(A|\mathbb{Z}; \widehat{\mathbb{Z}}) = 0$, then the LES associated to this SES would show that $H\Gamma^*(A|\mathbb{Z}; \mathbb{Z}) = H\Gamma^{*-1}(A|\mathbb{Z}; \widehat{\mathbb{Z}}/\mathbb{Z})$. Since $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a $\mathbb{Q}$-vector space, this is $H\Gamma^{*-1}(A \otimes \mathbb{Q}|\mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z})$.

So we just have to show that $H\Gamma^*(A|\mathbb{Z}; \widehat{\mathbb{Z}}) = 0$. By Lemma 7.8, we have

$$H\Gamma^*(A|\mathbb{Z}; \widehat{\mathbb{Z}}) = \prod_p H\Gamma^*(A|\mathbb{Z}; \mathbb{Z}_p) = \prod_p H\Gamma^*_c(A_p|\mathbb{Z}_p; \mathbb{Z}_p).$$

By Lemma 7.7, it suffices to show that $H\Gamma^*(A/p^k|\mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0$, and this follows by the second assumption in the theorem together with property (2) of $H\Gamma$. ☐

We're going to apply this to show that $H\Gamma^*(KU_*KU|KU_*; KU_*)$ is zero in the right degrees. First we need to know more about KU_*KU.

7.2. Structure of KU_*KU.

We know that KU is a 2-periodic spectrum made up of spaces $\mathbb{Z}\times BU, U, \mathbb{Z}\times BU, \ldots$ We have $KU_*(KU) = \operatorname{colim} \overline{KU}_{*+2n}(BU\times\mathbb{Z}) = \operatorname{colim} \overline{KU}_*(BU)$. Look at the AHSS

$$H_*(BU; KU_*) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots][\beta^\pm] \implies KU_*(BU).$$

Since $|\beta_i| = 2i$ and $|\beta| = 2$, all the odd parts of the spectral sequence vanish, and so the spectral sequence collapses. This shows that $\overline{KU}_*(BU)$ is even-graded and torsion free. In particular, $KU_*KU = KU_0KU \otimes KU_*$ and there is an injection $KU_*KU \hookrightarrow KU_*KU \otimes \mathbb{Q}$. Since the coefficient ring of K-theory is a Laurent polynomial ring, we can apply the general fact 7.11 below to show that $KU_*KU \otimes \mathbb{Q} \cong \mathbb{Q}[u^\pm, v^\pm]$ where $|u| = |v| = -2$.

Fact 7.11. $(E_*F) \otimes \mathbb{Q} = (\pi_*E \otimes \pi_*F) \otimes \mathbb{Q}$.

So we have an injection $KU_*KU \hookrightarrow \mathbb{Q}[u^\pm, v^\pm]$; which polynomials do you get? It turns out that you can describe KU_*KU as the set of polynomials $f(u,w) \in \mathbb{Q}[u^\pm, w^\pm]$ such that for all $k, h \in \mathbb{Z} \setminus \{0\}$, $f(kt, ht) \in \mathbb{Z}[\frac{1}{hk}, t^\pm]$.

The description of KU_*KU that will be most useful to us is the following:

Lemma 7.12. *Let $A_0 \subset \mathbb{Q}[w^\pm]$ be the $\mathbb{Z}$-module generated by $c_n(w) = \binom{w}{n} = \frac{w(w-1)\dots(w-(n+1))}{n!}$. Define $A_0^{st} = A_0[w^{-1}]$ and $A^{st} = A_0^{st} \otimes \mathbb{Z}[u^\pm]$. Then*

$$KU_0KU \cong A_0^{st}$$
$$KU_*KU \cong KU_0KU \otimes KU_* \cong A^{st}.$$

The idea is the following:

Step 1: Show that $KU_0(\mathbb{C}\mathrm{P}^\infty) \cong \{f(w) \in \mathbb{Q}[w] \,:\, f(w) \in \mathbb{Z}\} =: A'$ by first using the Chern character in cohomology $KU^0(\mathbb{C}\mathrm{P}^\infty) \to H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q}) = \mathbb{Q}[[x]]$ which sends $\eta - 1$ to $e^x - 1$ where η is the canonical bundle, and then dualizing (finding polynomials that pair integrally with $e^x - 1$).

Step 2: Use pure algebra to show that A' is generated by the $c_n(w)$'s (i.e. $A' \cong A_0$).

Step 3: Show that the natural map $KU_0(\mathbb{C}\mathrm{P}^\infty) = K_0(BU(1)) \to K_0(BU) = \mathbb{Z}[\beta_1, \beta_2, \dots]$ sends $c_n(w) \mapsto \beta_n$. So $KU_0(BU) \cong \mathbb{Z}[c_n(w) \,:\, n \geq 0]$.

Step 4: Show that the map $A_0 \to K_0(BU)$ induces a map from $A_0[w^{-1}] = \mathrm{colim}(A_0 \overset{w}{\to} A_0 \overset{w}{\to} \dots)$ to $KU_0KU = \mathrm{colim}(KU_0(BU) \overset{\mathrm{Bott}}{\to} KU_0(BU) \overset{\mathrm{Bott}}{\to} \dots)$, which is an isomorphism.

(This comes from 1.3 in Baker-Clarke-Ray-Schwartz, "On the Kummer congruences and the stable homotopy of BU".)

Remark 7.13. $\mathrm{Hom}(KU_0KU, R) = \mathrm{Aut}(F_{\mathrm{mult}})(R)$.

Now we can start working on Goal 7.5.

$$\begin{aligned} H\Gamma^*(KU_*KU|KU_*; KU_*) &\cong H\Gamma^*(KU_0KU \otimes KU_*|KU_*; KU_*) \\ &\cong H\Gamma^*(KU_0KU|\mathbb{Z}; KU_*) \\ &\cong H\Gamma^*(KU_0KU|\mathbb{Z}; \mathbb{Z}) \otimes KU_* \\ &\cong H\Gamma^*(A_0^{st}|\mathbb{Z}; \mathbb{Z}) \otimes KU_*. \end{aligned}$$

To get the obstruction theory to work, it suffices to show $H\Gamma^*(A_0^{st}|\mathbb{Z};\mathbb{Z}) = 0$ in the right degrees. Since $A_0 \to A_0^{st}$ is an étale extension, we can use properties (2) and (3) to reduce to showing:

Goal 7.14. $H\Gamma^*(A_0|\mathbb{Z};\mathbb{Z}) = 0$ for $* > 1$.

Checking the conditions in the theorem is a bit messy (see below), but if you assume for a moment the theorem applies, we have

$$H\Gamma^*(A_0|\mathbb{Z};\mathbb{Z}) = H\Gamma^{*-1}(A_0 \otimes \mathbb{Q}|\mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z}).$$

We know that there is an inclusion $A_0 \hookrightarrow \mathbb{Q}[w]$, and $c_1(w) = w$, so rationalizing this inclusion turns it into an isomorphism. So now we have that this group equals $H\Gamma^{*-1}(\mathbb{Q}[w]|\mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z})$. We could use the theorem mentioned last time to compute this; alternatively, we can use (5) to show that this is zero for $* \neq 1$. In particular, there are no obstructions to existence and uniqueness.

It remains to check the étale-ness condition in Theorem 7.9, namely:

Claim 7.15. *A_0/p^k is formally étale over $\mathbb{Z}/p^k$ for $k \geq 1$.*

We can express $c_n(w)$ in terms of a generating function: $(1+x)^w = \sum c_n(w)x^n$. Since $(1+x)^w(1+y)^w = (1+(x+y+xy))^w$ we have

$$\Big(\sum c_n(x)x^n\Big)\Big(\sum c_m(w)y^m\Big) = \sum c_\ell(w)(x+y+xy)^\ell.$$

The coefficient of x^ny^m is

$$c_n(w)c_m(w) = \binom{n+m+1}{n}c_{n+m}(w) + (\text{lower-degree terms}).$$

I'm a little confused here; I think the idea is that the "lower-degree terms" are supposed to go away mod p? Sarah has pointed out that for $p = n = 2, m = 1$, we get $c_4c_1 = 5c_5 + 4c_4$. – Eva

We will show that $A_{(p)}$ is étale over $\mathbb{Z}_{(p)}$. The claim is that there is an additive basis for $A_{(p)}$ over $\mathbb{Z}_{(p)}$ given by $\prod_{i=0}^n c_{p^i}^{\alpha_i}(w)$ where $m = \sum_{i=1}^n \alpha_i p^i$.

To show this, let's first do the simpler task of computing a basis for A_0/p over $\mathbb{Z}/p$. Recall we had generators $c_n(w) = \binom{w}{n} = \frac{w(w-1)\ldots(w-(n+1))}{n!}$ of A_0 as a $\mathbb{Z}$-module.

There is a surjection $\mathbb{F}_p[c_{p^n}(w)] \twoheadrightarrow A_0/p$. *Here I think you're using that $c_{p^n}(w)c_m(w) = c_{p^n+m}(w)\nu$ for a constant ν, which I don't know how to prove and is probably only true mod p.* Since $(1+x)^p \equiv 1+x^p \pmod p$, we have $((1+x)^w)^p = (1+x)^{wp} = (1+x^p+p(j(x)))^w$ for some polynomial j. We have

$$\Big(\sum c_n(w)x^n\Big)^p = \sum c_n(w)\big(x^p+p(j(x))\big)^n,$$

so, modulo p, we get $\sum c_n(w)^p x^{pn} \equiv \sum c_n(w)x^{pn}$ and

$$c_n(w)^p \equiv c_n(w) \pmod p.$$

In particular, the surjection $\mathbb{F}_p[c_{p^n}(w)] \twoheadrightarrow A_0/p$ factors through $\mathbb{F}_p[c_{p^n}(w)]/(c_{p^n}^p - c_{p^n})$, and the claim is that now it's an isomorphism. In particular, since $F_p[x]/(x^p - x)$ is étale over $\mathbb{F}_p$, we've verified étaleness for $k = 1$. For $k > 1$, you need to apply the infinite-dimensional Hensel's lemma to lift these generators to ones that satisfy the same relation modulo higher powers of p. The upshot is

$$A_0/p^k = (\mathbb{Z}/p^k)[c_{p^n,k}]/(c_{p^n,k}^p - c_{p^n,k})$$

for some new elements $c_{p^n,k}$.

Part III:
Topological André-Quillen homology

Interlude: Informal introduction to TAQ (Dylan Wilson)

This is Dylan's response to a request to provide more intuition behind TAQ, delivered during an informal questions seminar.

One way we can study a space is via its Postnikov tower:

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & \downarrow \\ & \nearrow & \tau_{\le 1}X \\ & \nearrow & \downarrow \\ X & \longrightarrow & \tau_{\le 0}X \end{array}$$

Given an E_∞-ring A, we will see that there is an analogue

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & \downarrow \\ & \nearrow & \tau_{\le 1}A \\ & \nearrow & \downarrow \\ A & \longrightarrow & \tau_{\le 0}A \end{array}$$

This has the same underlying spaces if you forgot the structure.

On the space side: if you're lucky (e.g. when X is simply connected), the fiber $K(\pi_n X, n)$ of $\tau_{\le n}X \to \pi_{\le n-1}X$ deloops to $K(\pi_n X, n+1)$. There is a homotopy pullback diagram

$$\begin{array}{ccccc} K(\pi_n X, n) & \longrightarrow & \tau_{\le n}X & \longrightarrow & E(\dots) \\ & & \downarrow & & \downarrow \\ & & \tau_{\le n-1}X & \longrightarrow & K(\pi_n X, n+1) \end{array}$$

and the obstruction to a lift

$$\begin{array}{ccccc} & & \tau_{\le n}X & & \\ & \nearrow & \downarrow & & \\ Y & \longrightarrow & \tau_{\le n-1}X & \longrightarrow & K(\pi_n X, n+1) \end{array}$$

is an element in $H^{n+1}(Y, \pi_n X)$. (The bottom row is a fiber sequence.)

On the algebra side, our tower looks like

$$\begin{array}{ccccc} & & \tau_{\leq 1}A & & \\ & \nearrow & \downarrow & & \\ A & \longrightarrow & \tau_{\leq 0}A & \longrightarrow & \Sigma H\pi_1 A \end{array}$$

and $\tau_{\leq n}A \to \tau_{\leq n-1}A$ is a square-zero extension, hence it is classified by derivations; these derivations are what we define TAQ to be.

Remark (Maria)**.** The map $\tau_{\leq 0}A \to \Sigma H\pi_1 A$ factors through $A \vee \Sigma H\pi_1 A$. If you want to lift an algebra map $B \to \tau_{\leq 0}A$ to any old map $B \to \tau_{\leq 1}A$, then you just need the composite $B \to \Sigma H\pi_1 A$ to be null. If you want to lift an algebra map to another *algebra* map, you need the stronger requirement that $B \to A \vee \Sigma H\pi_1 A$ be null.

Talk 8: Quillen cohomology (Eric Berry)

We're mainly going to talk about André-Quillen homology. This is really what motivated Quillen to invent model categories. The main idea here is to think of homology as the total left derived functor of abelianization. The outline is:

(1) Quillen homology

(2) André-Quillen (AQ) homology, which is just Quillen homology for commutative algebras

(3) Extension to algebras over operads

Let $\mathcal{C}$ be a category with binary products and a terminal object $*$.

Definition 8.1. An *abelian group object* in $\mathcal{C}$ is an object $X \in \mathcal{C}$, together with an operation, $\mu : X \times X \to X$, a unit map, $\varepsilon : * \to X$, and an inverse map, $inv : X \to X$, such that the expected diagrams commute.

Let $\mathcal{C}_{ab}$ denote the category of abelian group objects in $\mathcal{C}$. For example, if $\mathcal{C}$ is the category of sets, then $\mathcal{C}_{ab}$ is the category of abelian groups. An important observation is that if $\mathcal{C}$ is the category of commutative k-algebras, for some commutative ring k, then $\mathcal{C}_{ab}$ consists of only the trivial k-algebra.

Suppose $\mathcal{C}$ and $\mathcal{C}_{ab}$ have model structures, and suppose that the forgetful functor $\mathcal{C}_{ab} \to \mathcal{C}$ has a left adjoint that forms a Quillen adjunction. This left adjoint is called *abelianization.*

Definition 8.2. The *Quillen homology* of $X \in \mathcal{C}$ is the total left derived functor, $\mathbf{L}\,\mathrm{Ab}(X)$, of the abelianization of X.

Quillen was trying to attain a suitable homology theory for commutative algebras, but as we saw above, the only abelian group object is the trivial object. However, this issue can be resolved by augmenting.

Let k be a commutative ring, and let $A \in \mathrm{CAlg}_k$. Then, it turns out that the forgetful functor $(\mathrm{CAlg}_k/A)_{ab} \to \mathrm{CAlg}_k/A$, has a left adjoint, namely the abelianization, that provides for a suitable homology theory. To construct the abelianization functor, we need some definitions.

8.1. Derivations and differentials.

Definition 8.3. A *k-derivation* of A valued in M is a k-linear map $D : A \to M$ that satisfies the Leibniz rule:

$$D(ab) = aDb + bDa.$$

Let $\mathrm{Der}_k(A, M)$ denote all such derivations. It turns out that $\mathrm{Der}_k(A, -)$ is representable, i.e., there is some A-module $\Omega_{A/k}$ such that

$$\mathrm{Hom}_A(\Omega_{A/k}, M) = \mathrm{Der}_k(A, M).$$

Definition 8.4. This object $\Omega_{A/k}$ is called the *module of relative Kähler differentials.*

There are a few ways of constructing $\Omega_{A/k}$. The simplest way is to define $\Omega_{A/k}$ as the A-module generated by the symbols da, for $a \in A$, subject to the following relations:

- $d(a + a') = da + da'$;
- $d(aa') = ada' + a'da$;
- $d(\alpha a) = \alpha da$, for $\alpha \in k$.

Alternatively, one could define $\Omega_{A/k}$ to be I/I^2, where $I = \ker(A \otimes_k A \to A)$.

8.2. Square-zero extensions.

Definition 8.5. The object $A \ltimes M$ is the called the *square-zero extension of* A. As an abelian group, it's just $A \oplus M$. The ring structure is given by a map $A \ltimes M \times A \ltimes M \to A \ltimes M$, defined by

$$((a, m), (a', m')) = (aa', am' + a'm).$$

The algebra structure is given by the algebra structure of A.

Also, projection gives an augmentation $A \ltimes M \to A$, so $A \ltimes M \in \mathrm{CAlg}_k/A$.

It is easily seen that a square-zero extension is an abelian group object, and:

Fact 8.6. $(\mathrm{CAlg}_k/A)_{ab}$ *is precisely square-zero extensions.*

Another observation is:

Fact 8.7. $A \ltimes - : \mathrm{Mod}(A) \to (\mathrm{CAlg}_k/A)_{ab}$ *defines an equivalence.*

Let $B \in \mathrm{CAlg}_k/A$. Then

$$\mathrm{Hom}_{\mathrm{CAlg}_k/A}(B, A \ltimes M) \simeq \mathrm{Der}_k(B, M)$$

$$\simeq \mathrm{Hom}_B(\Omega_{B/k}, M)$$
$$\simeq \mathrm{Hom}_A(\Omega_{B/k} \otimes_B A, M). \tag{8.1}$$

By definition abelianization is the left adjoint in

$$\mathrm{Ab} : \mathrm{CAlg}_k /A \rightleftarrows (\mathrm{CAlg}_k /A)_{ab} : U;$$

because of Fact 8.7 we can also identify it as the left adjoint in

$$\mathrm{Ab} : \mathrm{CAlg}_k \rightleftarrows (\mathrm{CAlg}_k /A)_{ab} \simeq \mathrm{Mod}_A : A \ltimes -.$$

Therefore, (8.1) shows that $X \mapsto A \otimes_X \Omega_{X/k}$ (as a functor $\mathrm{CAlg}_k /A \to \mathrm{Mod}_A$) can be identified with abelianization. What we've just done is construct our abelianization functor $\mathrm{Ab} : \mathrm{CAlg}_k/A \to (\mathrm{CAlg}_k/A)_{ab}$ given by:

$$\mathrm{Ab}(X) = A \otimes_X \Omega_{X/k}.$$

8.3. Enter simplicial algebras. We need some appropriate model structures happening here.

Proposition 8.8. $s\mathrm{CAlg}_k$ *has a model structure, where* $f : X \to Y$

- *is a weak equivalence if it is one in* $s\,\mathrm{Set}$*;*
- *is a fibration if it is one in* $s\,\mathrm{Set}$*.*

You can explicitly describe cofibrations as retracts of "free maps", but I won't get into that.

Note that an object in $s\mathrm{CAlg}_k/A$ is just a factorization $ck \to X \to cA$, where $c(-)$ denotes the constant simplicial object.

Furthermore, this gives us the appropriate notion of resolution; you can take a cofibrant resolution:

Definition 8.9. An object $P \in s\mathrm{CAlg}_k/A$ is a *cofibrant resolution of* A if it's a cofibrant factorization.

For example, one can just take a simplicial set and apply "free" everywhere.

Definition 8.10. The *cotangent complex* of A is the simplicial A-module defined by

$$L_{A/k} := \mathbf{L}\,\mathrm{Ab}(A) = A \otimes_P \Omega_{P/k},$$

where P is a cofibrant resolution of A.

Now, we finally have:

Definition 8.11. The *André-Quillen homology of* A *with coefficients in* M is defined to be

$$D_*(A|k, M) = \pi_*(L_{A/k} \otimes_A M).$$

This is also denoted as $AQ_*(A|k, M)$. The *André-Quillen cohomology of* A *with coefficients in* M is defined to be

$$D^*(A|k, M) = H^*(\mathrm{Hom}_A(NL_{A/k}, M)).$$

where N is the normalization functor that takes a simplicial object to its normalized chain complex.

8.4. Properties.

(1) Transitivity exact sequence: if $A \to B \to C$ are maps of rings, then we have an exact sequence
$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$
AQ homology extends this to a long exact sequence. This is the property that Quillen was seeking.

(2) Flat base change also holds; this is like $H\Gamma$.

Note, if we did the above in the case of associative algebras, rather than commutative, we would recover Hochschild homology.

Now, our goal is to:

8.5. Generalize this to algebras over operads. This part is just laying the groundwork for what is needed in Goerss-Hopkins obstruction theory.

Let $\mathcal{O}$ be an operad in k-modules, let A be an $\mathcal{O}$-algebra, and let $M \in \text{Mod}(A)$ [4]. The main goal is to be able to define the cotangent complex again, because that's how you define André-Quillen homology. Thus, we need Kähler differentials and square-zero extensions. The first step is to define square-zero extensions.

Definition 8.12. The square-zero extension $A \ltimes M$ is, as a k-module, just $A \oplus M$. This needs to be a $\mathcal{O}$-algebra, so we need to have maps
$$\mathcal{O}(n) \otimes (A \ltimes M)^{\otimes n} \to A \ltimes M.$$
But, we have that
$$(A \oplus M)^{\otimes n} = A^{\otimes n} \oplus \bigoplus_{i=1}^{n} (A^{\otimes n-1} \otimes M) \oplus \text{crap},$$
where crap consists of everything that has more than one M term. We don't care about the crap since we only care about deformations. Tossing out all the crap, we get a projection into the non-crap stuff, and so we get a map
$$\mathcal{O}(n) \otimes (A \ltimes M)^{\otimes n} \to \mathcal{O}(n) \otimes (A^{\otimes n} \oplus \bigoplus_{i=1}^{n} (A^{\otimes n-1} \otimes M)) \xrightarrow{\text{algebra structure of } A,\text{ module structure of } M} A \ltimes M.$$

These, again, are going to turn out to be our abelian group objects:

Fact 8.13. *Objects of the form $A \ltimes M$ are the abelian group objects.*

[4] What's an A-module? It's just a family of Σ_{n-1}-equivariant maps $\mathcal{O}(n) \otimes A^{n-1} \otimes M \to M$ that satisfy unit and associativity conditions.

Now, we need some notion of derivations. We can use our knowledge of the commutative algebra case to just *define* this in terms of $A \ltimes M$.

Definition 8.14. *$\mathcal{O}$-derivations* are defined by $\mathrm{Der}_{\mathcal{O}}(B, M) := \mathrm{Hom}_{\mathrm{Alg}_{\mathcal{O}}/A}(B, A \ltimes M)$.

This, again, turns out to be representable, and we define:

Definition 8.15. The *module of Kähler differentials* is the representing object for $\mathrm{Der}_{\mathcal{O}}(B, -)$.

You can also try to write this down explicitly; they'll end up satisfying some form of the Leibniz rule.

Now, we want to derive this functor, and define homology to be the derived functor of Kähler differentials. In order to talk about derived things we need to talk about resolutions, which requires us to talk about simplicial algebras over simplicial operads.

Let $\mathcal{O}$ be a simplicial operad. Then $X \in s\mathcal{C}$ is a simplicial algebra over $\mathcal{O}$ if if for all n, X_n is an $\mathcal{O}_n$-algebra, compatibly.

Fact 8.16. *$s\mathrm{Alg}_{\mathcal{O}}$ has a model structure where $f : X \to Y$*

- *is a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is an isomorphism;*
- *is a fibration if the normalization $Nf : NX \to NY$ is surjective in positive degrees* [5].

Our appropriate notion of resolution is going to be a cofibrant factorization.

Definition 8.17. The *cotangent complex* of A is $L_{A/\mathcal{O}} := A \otimes_P \Omega_{P/\mathcal{O}}$ where P is a cofibrant resolution.

The AQ-homology of X is defined to be

$$D_*(X|\mathcal{O}) := \pi_*(X \otimes_P \Omega_{P/\mathcal{O}}) = \pi_* L_{X/\mathcal{O}}.$$

You can also write down what cohomology is.

Talk 9: Topological André-Quillen cohomology I (Daniel Hess)

Today, we'll focus on the construction and properties.

Let R be a commutative S-algebra, and let A be a commutative R-algebra. Our goal is to study Quillen (co)homology in the category $(\mathrm{CAlg}_R/A)_{ab}$, where CAlg_R/A is the category of commutative R-algebras with an augmentation to A.

9.1. Abelianization in CAlg_R/A.

[5] Here "normalization" is in terms of the normalized chain complex à la Dold-Kan.

Proposition 9.1. *There is an equivalence*

$$(\mathrm{CAlg}_R/A)_{ab} \simeq \mathrm{Mod}(A)$$

Proof. As before, we'll show that the abelian objects are square-zero extensions. I'll write down a backwards map: send $M \in \mathrm{Mod}(A)$ to $A \vee M \in \mathrm{CAlg}_R/A$ (so it has the zero multiplication). This is a little tricky; what I want is a map

$$(A \vee M) \wedge_R (A \vee M) \simeq (A \wedge_R A) \vee (A \wedge_R M) \vee (M \wedge_R A) \vee (M \wedge_R M) \to A \wedge M$$

This can be defined by sending $A \wedge_R A$ to A, $(A \wedge_R M) \vee (M \wedge_R A)$ to M, and $M \wedge_R M$ to $*$. This is the "square-zero" extension of A by M.

As in the classical case, it turns out that $A \vee M$ is an abelian object in CAlg_R/A (maps into it are derivations). It is also true that every abelian object in CAlg_R/A is a square-zero extension. Therefore $M \mapsto A \vee M$ is an equivalence. ☐

Given $B \in \mathrm{CAlg}_R/A$, what is its abelianization?

Recall that in the classical case, the answer is $A \otimes_R \Omega_R(B)$. The way you can define $\Omega_R(B)$ is as $Q_B(I_B(B \otimes_R B))$, where I_B is the augmentation ideal, and Q_B denotes the indecomposables. I'll now define what "augmentation ideal" and "indecomposable" mean in this context.

Definition 9.2 (Augmentation ideal)**.** Let $X \in \mathrm{CAlg}_A/A$. Define $I_A(X)$ via the pullback

$$\begin{array}{ccc} I_A(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{augmentation}]{} & A \end{array}$$

Definition 9.3 (Indecomposables)**.** Let $N \in \mathrm{NUCA}_A$ (the category of nonunital commutative algebras over A). Define $Q_A(N)$ as the pushout

$$\begin{array}{ccc} N \wedge_A N & \longrightarrow & * \\ \downarrow & & \downarrow \\ N & \longrightarrow & Q_A(N) \end{array}$$

Here are some remarks:

- If $B \in \mathrm{CAlg}_R/A$, I can consider $A \wedge_R B \in \mathrm{CAlg}_A/A$, with the augmentation given by $A \wedge_R B \xrightarrow{aug} A \wedge_R A \xrightarrow{\mu} A$.
- Both of these functors $I_A(-)$ and $Q_A(-)$ pass to homotopy categories, and I get "derived functors" $\mathbf{R}I_A(-)$ and $\mathbf{L}Q_A(-)$.

Proposition 9.4. *Let $A, B \in \mathrm{CAlg}_R/A$ be such that A is cofibrant and B is fibrant and cofibrant. Then*

$$\mathbf{L}Q_A(\mathbf{R}I_A(A \wedge_R^{\mathbf{L}} B)) = \mathbf{L}Q_A(\mathbf{R}I_A(A \wedge_B^{\mathbf{L}} B \wedge_R^{\mathbf{L}} B)) \simeq A \wedge_B^{\mathbf{L}} \mathbf{L}Q_B(\mathbf{R}I_B(B \wedge_R^{\mathbf{L}} B))$$

Proof. See Maria's paper. □

We'll write $\Omega_R(B) := \mathbf{L}Q_B(\mathbf{R}I_B(B \wedge^{\mathbf{L}}_R B))$.

Theorem 9.5. *The abelianization of $B \in h\mathrm{CAlg}_R/A$ is $A \wedge^{\mathbf{L}}_B \Omega_R(B)$.*

The proof follows from a few facts.

(1) I'll define $K_A : \mathrm{NUCA}_A \to \mathrm{CAlg}_A/A$ defined by $N \mapsto A \vee N$ (this is *not* the square-zero extension!). This is adjoint to I_A; it's also adjoint on homotopy categories, where *they give equivalences!*.

(2) We can also define $Z_A : \mathrm{Mod}(A) \to \mathrm{NUCA}_A$ given by $M \mapsto M$, given the zero multiplication. This is adjoint to Q_A, and the same on homotopy categories.

A subtle point is that the result of this theorem fails if we don't pass to homotopy categories first! This rests on the fact that I_A and K_A are equivalences on homotopy categories.

Definition 9.6. Let $A \in \mathrm{CAlg}_R/A$, and let $M \in \mathrm{Mod}(A)$. Then the *topological André-Quillen homology* is

$$TAQ_*(A, R; M) = \pi_*(\Omega_R(A) \wedge_A M)$$

and the *topological André-Quillen cohomology* is

$$TAQ^*(A, R; M) = \pi_{-*}F_A(\Omega_R(A), M)$$

9.2. Aside. Let Alg_R/A denote associative R-algebras augmented over A. It turns out that

- Abelian group objects are square-zero extensions.
- $(\mathrm{Alg}_R/A)_{ab} = {}_A\mathrm{Bimod}_A$.
- The abelianization of $B \in \mathrm{Alg}_R/A$ is

$$(A \wedge A^{op}) \wedge_{B \wedge B^{op}} \Omega^{assoc}_R(B),$$

  where $\Omega^{assoc}_R(B)$ is the module of Kähler differentials in the associative case, defined as the fiber of the multiplication $B \wedge_R B \to B$.

The key point is that in this associative case Quillen cohomology is "essentially" the same as THH. Why? I have a fiber sequence

$$\Omega^{assoc}_R(A) \to A \wedge_R A^{op} \to A.$$

I'll hit this with $F_{A\wedge_R A^{op}}(-, M)$, and I'll get

$$F_{A\wedge_R A^{op}}(\Omega^{assoc}_R(A), M) \leftarrow F_{A\wedge_R A^{op}}(A \wedge_R A^{op}, M) \leftarrow F_{A\wedge_R A^{op}}(A, M).$$

Now, $F_{A\wedge_R A^{op}}(A, M)$ is how you define THH, and $F_{A\wedge_R A^{op}}(A \wedge_R A^{op}, M) = M$. So the Quillen cohomology of A, $F_{A\wedge_R A^{op}}(\Omega^{assoc}_R(A), M)$ is called "topological derivations".

9.3. Back to TAQ. Let me first give an example.

Example 9.7. Let X be a cofibrant R-module. I can form its free commutative R-algebra $\mathbf{P}_R(X) = \bigvee_i X^{\wedge_R i}/\Sigma_i$. Then the following result can be found in one of Baker's papers:

$$\Omega_R(\mathbf{P}_R(X)) \simeq \mathbf{P}_R(X) \wedge_R X.$$

So that

$$TAQ_*(\mathbf{P}_R(X), R; M) \simeq \pi_*(X \wedge_R M) = X_*^R(M).$$

Here's the relationship between TAQ cohomology and ordinary cohomology. Remember that

$$TAQ^*(A, R; M) = \pi_{-*}F_A(\Omega_R(A), M).$$

We may consider the map

$$\xi : A \to A \vee \Omega_R(A) \to \Omega_R(A),$$

where the first map is the map in CAlg_R/A adjoint to the identity map of $\Omega_R(A)$ in $\mathrm{Mod}(A)$. Forgetting structure, the composite is a map of R-modules which induces a forgetful map to ordinary R-module cohomology

$$TAQ^*(A, R; M) = \pi_{-*}F_A(\Omega_R(A), M) \to \pi_{-*}F_R(\Omega_R(A), M) \to \pi_{-*}F_R(A, M) = M_R^*(A).$$

It's not known exactly what this maps does in general, but here's an example that allows you to identify the image.

Example 9.8 (Lazarev)**.** We have $TAQ^*(H\mathbf{F}_p, S; H\mathbf{F}_p) \to H^*(H\mathbf{F}_p; \mathbf{F}_p) = \mathcal{A}^*$, and the image consists of multiples of β, the Bockstein. In fact, $TAQ^*(H\mathbf{F}_p, S; H\mathbf{F}_p)$ can be described as follows (at least at $p = 2$): there is $y \in TAQ^1(H\mathbf{F}_p, S; H\mathbf{F}_p)$, and you get all the elements as $\mathrm{Sq}^{s_1}\mathrm{Sq}^{s_2}\cdots\mathrm{Sq}^{s_r}(y)$ where $s_r > 3$ and the sequence $(s_1, s_2, \cdots, s_r)$ is admissible.

TAQ carries power operations.

Let's now talk about the relationship to AQ cohomology. In general,

$$AQ_*(A, R; M) \not\simeq TAQ_*(HA, HR; HM).$$

A quick example: we know that

$$AQ_*(k[x], k; k) = \begin{cases} k & * = 0, \\ 0 & \text{else.} \end{cases}$$

But, using some Atiyah-Hirzebruch-type spectral sequence (see (9.1)), Richter showed:

$$TAQ_*(Hk[x], Hk; Hk) = Hk_*H\mathbb{Z}.$$

As mentioned, there is a sseq:

$$E^2_{p,q} = AQ_p(A, k; M) \otimes TAQ_q(Hk[x], Hk; HM) \Rightarrow TAQ_{p+q}(HA, Hk; HM). \tag{9.1}$$

This is like an Atiyah-Hirzebruch spectral sequence because $k[x]$ plays the role of a "basepoint". This is due to Richter.

9.4. "Calculating" TAQ. Everything that was true in the ordinary case is true here. In particular, flat (this means cofibrant) base change, additivity, transitivity LES, all hold here.

A sort of Hurewicz theorem also holds.

Theorem 9.9. *Let R and A be connective, and suppose $\varphi : R \to A$ is an n-equivalence for some $n \geq 1$, with cofiber $C\varphi$. Then $\Omega_R(A)$ is n-connected, and*

$$\pi_{n+1}\Omega_R(A) \simeq \pi_{n+1}C\varphi \simeq TAQ_{n+1}(A, R; H\pi_0 A).$$

What's the isomorphism? There is an identity map $\Omega_R(A) \xrightarrow{id} \Omega_R(A)$, which gives a map $A \to A \vee \Omega_R(A)$. I can compose with the projection to $\Omega_R(A)$, and this gives a map $\delta : A \to \Omega_R(A)$. This is the *universal derivation.*

Now, the isomorphism is described as follows. If I take $R \xrightarrow{\varphi} A \xrightarrow{\delta} \Omega_R(A)$, which is nullhomotopic, I get $\tau : C\varphi \to \Omega_R(A)$, which induces the isomorphism.

Let me finish off by talking about some spectral sequences. There is a universal coefficients spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{\pi_* A}_{s,t}(TAQ_*(A, R; A), \pi_* M) \Rightarrow TAQ_*(A, R; M)$$

and similarly for TAQ^*, where you replace Tor with Ext. For coefficients in $H\mathbf{F}_p$, Basterra constructs a Miller spectral sequence:

$$E_2^{s,t} = \mathrm{Hom}_{\mathbf{F}_p}(\mathcal{L}^{\mathbf{F}}_S(\mathbf{F}_p \otimes_{\mathcal{R}} Q^{alg}(-))(H_*(A; \mathbf{F}_p))_t, \mathbf{F}_p) \Rightarrow TAQ^*(A, S; H\mathbf{F}_p)$$

where:

- $\mathcal{R}$ is the Dyer-Lashof algebra.
- $\mathbf{F}$ is a comonad that's associated to the free algebra functor $\{$graded $\mathbf{F}_p$-vector spaces$\} \to \{$unstable algebras over $\mathcal{R}\}$.
- Q^{alg} is the algebra of indecomposables, and
- $\mathcal{L}^{\mathbf{F}}_S(-)$ is a comonad $\mathbf{F}$-left derived functor.

This is hard to compute!

Talk 10: Topological André-Quillen cohomology II (Yu Zhang)

Notation:

- R is a commutative S-algebra
- A is a commutative R-algebra
- NUCA is the category of nonunital commutative algebras
- $\mathcal{C}_{R/A}$ is CAlg_R / A
- $\mathcal{N}_A$ is the category of nonunital commutative A algebras
- $\mathcal{M}_A = \mathrm{Mod}_A$

Recall: we had three pairs of adjunctions:

$$\mathcal{C}_{R/A} \underset{\text{forget}}{\overset{-\wedge_R A}{\rightleftarrows}} \mathcal{C}_{A/A} \underset{I}{\overset{K}{\leftrightarrows}} \mathcal{N}_A \underset{Z}{\overset{Q}{\rightleftarrows}} \mathcal{M}_A$$

Now take the homotopy categories of everything in sight:

$$\mathrm{Ho}\,\mathcal{C}_{R/A} \underset{\text{forget}}{\overset{-\wedge^{\mathbf{L}}_R A}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{C}_{A/A} \underset{\mathbf{R}I}{\overset{K}{\leftrightarrows}} \mathrm{Ho}\,\mathcal{N}_A \underset{Z}{\overset{\mathbf{L}Q}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{M}_A$$

The middle pair forms a Quillen equivalence. The other two pairs form Quillen adjunctions. You have to derive the functors that don't preserve weak equivalences.

Define $\mathrm{Ab}^A_R : \mathrm{Ho}\,\mathcal{C}_{R/A} \to \mathrm{Ho}\,\mathcal{M}_A$ to be the composite $\mathbf{L}Q\ \mathbf{R}I(- \wedge^{\mathbf{L}}_R A)$. Let M be an A module, define

$$TAQ^*(A, R; M) := \pi_{-*}F_A(\mathrm{Ab}^A_R(A), M)$$

For this talk, we'll focus on $Q : \mathcal{N}_A \to \mathcal{M}_A$, which we defined via the pushout diagram

$$\begin{array}{ccc} N \wedge_A N & \longrightarrow & N \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q(N) \end{array}$$

This is called the indecomposables functor.

We also have the "zero multiplication" functor $Z : M_A \to N_A$, which is what you think it is. We also have an adjunction (Q, Z), that's moreover a Quillen adjunction since Z preserves weak equivalences and acyclic fibrations.

10.1. Indecomposables vs. stabilization. Let $X \in \mathrm{Top}_*$. We have a suspension map $\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X)$. We can go one step further to get $\pi_n(X) \to \pi_{n+2}(\Sigma^2 X)$. And we can go on. By homotopy excision, the sequence of suspension maps will eventually become all isomorphisms. In particular, the sequence of homotopy groups will stabilize. Define

$$\pi^s_n(X) = \mathrm{colim}_{k\to\infty}\, \pi_{n+k}(\Sigma^k X)$$

The colimit is actually achieved at some finite stage.

Here's another way to look at it, using the adjunction (Σ, Ω). The suspension map $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ is the same as $\pi_n(X) \to \pi_n(\Omega\Sigma X)$ induced by the unit map $X \to \Omega\Sigma X$. Similarly, $\pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X)$ is the same as $\Omega\Sigma X \to \Omega^2\Sigma^2 X$, again induced by the unit. We therefore have a sequence of maps $X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots$. Now we can define the stabilization of X to be

$$\underline{Q}(X) = \mathrm{colim}_k\, \Omega^k \Sigma^k X.$$

(Note that we're writing $\underline{Q}$ to distinguish from our indecomposables functor Q.) Then $\pi_n(\underline{Q}X) = \mathrm{colim}_k\, \pi_n(\Omega^k\Sigma^k X) = \pi^s_n(X)$.

Now, we'll work in $\mathcal{N}_A$, which is a simplicial model category. It is equipped with a suspension-loop adjunction (E, Ω), where $E := -\otimes S^1$ and $\Omega := \underline{hom}(S^1, -)$. We can define stabilization:

Definition 10.1. Let

$$\underline{Q}N = \mathrm{hocolim}_{k\to\infty}\, \Omega^k E^k N.$$

Note that E is different from Σ in $\mathcal{M}_A$, but Ω is the same. In particular, $\pi_n(X) = \pi_{n+1}(\Sigma X)$ for all n, but this is not always true for E. Similarly, $\pi_n(\underline{Q}N) = \operatorname{colim}_{k\to\infty} \pi_{n+k} E^k N$.

Let's now consider a special case: if $M \in \mathcal{M}_A$, we can let $\mathbf{N}M$ be $\bigvee_{k>0} M^{(k)}/\Sigma_k$. This is the free A-nonunital commutative algebra. Then $Q(\mathbf{N}M) = M$, and we have the following.

Theorem 10.2 (Basterra-Mandell '05). *If A and M are both cofibrant, the natural map $M \to \mathbf{N}M \to \underline{Q}\mathbf{N}M$ is a weak equivalence. In short, there is a weak equivalence:*

$$Q(\mathbf{N}M) \simeq \underline{Q}\mathbf{N}M$$

Here's some intuition from Maria Basterra about why you might expect indecomposables to agree with stabilization. Suppose X is n-connected; then it turns out that the map $X \to QX$ is $(2n+1)$-connected. So $\operatorname{colim} \Omega^n E^n X \to \operatorname{colim}_n \Omega^n Q E^n X$ is an equivalence. You can show that $QEX \simeq \Sigma QX$ and so $\operatorname{colim}_n \Omega^n Q E^n X \simeq \Omega^n \Sigma^n QX$, but that is just equivalent to QX because $QX \in \mathrm{Mod}_A$ and Mod_A is already a stable category.

How about the homotopy excision theorem?

Theorem 10.3 (Ching-Harper '16). *There is a theory of higher homotopy excision and Blakers-Massey theorems for structured ring spectra.*

10.2. Q for nonunital E_n-A-algebras.

Definition 10.4. Let $\mathcal{C}_n$ denote the little n-cubes operad. An E_n-A-*algebra* is an A-module M together with structure maps

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} (M \wedge_A \cdots \wedge_A M) \to M,\ m \geq 0$$

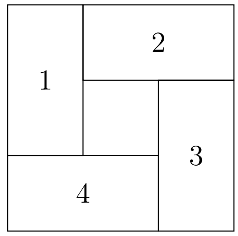
satisfying usual conditions. If we only require structure maps for $m > 0$, we get the notion of a *non-unital E_n-A-algebra.*

We still want a (Q, Z) adjunction. Let $\mathcal{NC}_n^A$ be the category of nonunital E_n-A-algebras. We define the zero multiplication functor $Z : \mathcal{M}_A \to \mathcal{NC}_n^A$. By giving M the structure map $\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} (M \wedge_A \cdots \wedge_A M) \to M$, defined to be trivial for $m > 1$, but for $m = 1$, we define $\mathcal{C}_n(1)_+ \wedge M \to *_+ \wedge M \simeq M$ (i.e., you send $\mathcal{C}_n(1)$ to the point). Now each $f \in \mathcal{C}_n(m)$ gives an m-ary multiplication function $M \wedge_A \cdots \wedge_A M \to M$.

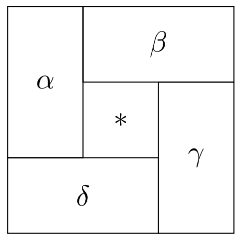
For nonunital commutative A-algebras, modding out by the decomposable part is the same as modding out by the binary product. However, for nonunital E_n-A-algebras, modding out by the binary product is not enough: we need to mod out all m-ary products.

Question: what is the difference here?

Example 10.5. Consider the little 2-cubes operad C_2, and the C_2-space $\Omega^2 X$. Let $f \in C_2(4)$ be



Let $\alpha, \beta, \gamma, \delta \in \Omega^2 X = \mathrm{Map}(S^2, X)$. Then $\alpha \otimes_f \beta \otimes_f \gamma \otimes_f \delta$ (i.e. multiplying these four elements using the multiplication f) is:



But since f can't be separated by any two cubes, $\alpha \otimes_f \beta \otimes_f \gamma \otimes_f \delta$ can't be written as a binary product.

Definition 10.6. For $N \in \mathcal{NC}_n^A$, define $Q(N)$ via the coequalizer:

$$\coprod_{m>0} \mathcal{C}_n(m)_+ \wedge_{\Sigma_m} N^{(m)} \underset{g}{\overset{f}{\rightrightarrows}} N \to Q(N)$$

where f is given by structure maps, and g is trivial for $m > 1$ and for $m = 1$, it's $\mathcal{C}_n(1)_+ \wedge N \to *_+ \wedge N \simeq N$.

Remark 10.7. For the $m = 1$ level, we're taking the $\mathcal{C}_n(1)$-orbit of N. Similarly, we have (Q, Z) which form a Quillen adjunction.

10.3. Bar construction for nonunital E_n-A-algebras. Recall that given a group G, the classifying space BG is $B(*, G, *)$, i.e., the realization of the bar construction:

$$* \times * \leftrightarrows * \times G \times * \leftrightarrows * \times G \times G \times * \qquad \cdots$$

To define the bar construction for $N \in \mathcal{NC}_1^A$, we face two difficulties:

- N has no identity element, so we don't have degeneracies. This is not really a problem though because we can freely generate a simplicial object.
- N has no associative multiplication structure. For instance, think of loop spaces, where loop concatenation is not associative, but it is associative up to homotopy. But we can make it associative. The question is how? Recall the Moore loop space. For $X \in \mathrm{Top}_*$, the Moore loop space $\Omega' X$ is the subspace of $X^{\mathbf{R}_{\geq 0}} \times \mathbf{R}_{\geq 0}$ consisting of pairs (ω, r) such that $\omega(t) = 0$ for $t \geq r$ and $t = 0$. For concatenation, define $(\omega_1, r_1) \cdot (\omega_2, r_2) = (\omega, r_1 + r_2)$,

where

$$\omega(t) = \begin{cases} \omega_1(t) & t \leq r_1, \\ \omega_2(t) & t > r_1. \end{cases}$$

In this case, loop concatenation is associative.

Using a similar idea, we can make $P_+ \wedge N$ associative, where $P = (0, \infty) \subseteq \mathbf{R}$. Let $\widetilde{N} = P_+ \wedge N$, and define

$$* \Leftarrow \widetilde{N} \Lleftarrow \widetilde{N} \wedge_A \widetilde{N} \qquad \cdots$$

Take

$$\widetilde{B}N = \coprod_{m \geq 0} (\widetilde{N})^{(m)} \wedge \Sigma^\infty_+ \Delta^m / \sim$$

to be the geometric realization.

Interesting things happen when we iterate the bar construction:

Theorem 10.8 (Basterra-Mandell '11). *Let N be cofibrant. There is a weak equivalence*

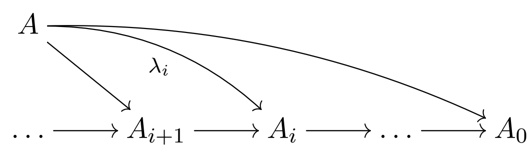
$$\widetilde{B}^n N \simeq \Sigma^n Q N.$$

We can use the iterated bar construction to compute the indecomposable parts.

Talk 11: Obstruction theory for connective spectra (Leo Herr)

This is going to involve some definitions that we'll get to in a bit.

Magic Fact 11.1 (Kriz). Let R be a connective commutative S-algebra, and let A be a connective R-algebra (not necessarily commutative). Then there exists a *Postnikov tower* of R-algebras A_i



such that:

(1) $\pi_k A \xrightarrow{\lambda_{i*}} \pi_k A_i$ is an isomorphism for $0 \leq k \leq i$

(2) $\pi_k A_i = 0$ for $k > i$

(3) The fiber sequence

$$\Sigma^{i+1} H\pi_{i+1} A \to A_{i+1} \to A_i$$

is an "extension" of R-algebras associated to a derivation. This gives rise to an element $k_{i+1} \in TDer^1_R(A_i, \Sigma^{i+1} H\pi_{i+1} A)$. If A is commutative, then $k_{i+1} \in TAQ^1(A_i, \Sigma^{i+1} H\pi_{i+1} A)$.

11.1. $\Omega^{\mathbf{assoc}}$, TAQ and $TDer$.

Notation:

- $C^{\text{ass}}_{R/A}$ is the category of associative R-algebras over A
- $\mathcal{M}_{A\text{-}B}$ is the category of (A, B)-bimodules

Recall we have functors $A \vee - : \mathcal{M}_{A\text{-}A} \to C^{\text{ass}}_{R/A}$ sending M to $A \vee M$. We have

$$(A \vee M)^{\wedge 2} = (A \wedge A) \vee (A \wedge M) \vee (M \wedge A) \vee (M \wedge M).$$

Take the map that sends the first piece to A, the second two pieces to M (all via the action or multiplication) and the last piece to zero. (Think about $\mathbb{C}[\varepsilon]/\varepsilon^2 \simeq \mathbb{C} \oplus \varepsilon\mathbb{C}$.) We will describe the (topological) Kähler differentials $\Omega^{\text{ass}}_A(B)$, which satisfies

$$hC^{\text{ass}}_{R/A}(B, A \vee M) \simeq h\mathcal{M}_{A\text{-}A}(\Omega^{\text{ass}}_A(B), M).$$

If B is a cofibrant object in $C^{\text{ass}}_{R/A}$, then define Ω^{ass}_B as a cofibrant replacement for the fiber in

$$\Omega^{ass}_B \to B \wedge B^{op} \xrightarrow{m} B$$

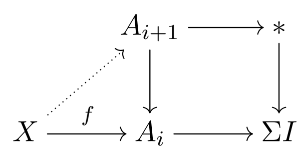
and define

$$\Omega^{\text{ass}}_A(B) = A \wedge A^{op} \wedge_{B \wedge B^{op}} \Omega^{\text{ass}}_B.$$

Define

$$TDer_R(A, M) = F_{A \wedge A^{op}}(\Omega^{\text{ass}}_A, M),$$
$$TAQ_R(A, M) = F_A(\Omega^{\text{comm}}_A, M).$$

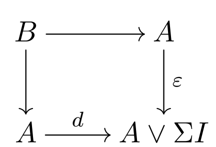
11.2. Obstructions. Suppose you have a lift from X to the i^{th} Postnikov stage A_i, and you want to find obstructions to getting a lift $X \to A_{i+1}$. If we were working in the normal world of topology, we would be looking for a lift



where $I = \Sigma^{i+1} H\pi_{i+1}A$ is the fiber of $A_{i+1} \to A_i$. You get a lift iff $X \to A_i \to \Sigma I$ is null; in topology, this corresponds to a class in $H^{i+2}(X; \pi_{i+1}A)$.

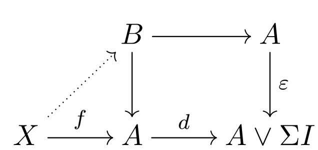
Now let's see what happens in the world of R-algebras. Let A be a cofibrant R-algebra, $I \in \mathcal{M}_{A\text{-}A}$, and $d : A \to A \vee \Sigma I$. The map d will play an analogous role to the map $A_i \to \Sigma I$ to the Eilenberg-MacLane spectrum. Let ε be the inclusion of a summand $A \to A \vee \Sigma I$.

Let B be the homotopy pullback in:



The homotopy fiber of the upper horizontal arrow $B \to A$ is I. The homotopy fiber sequence $I \to B \to A$ is called the (topological) (singular) "extension" associated to the derivation $d : A \to A \vee \Sigma I$.

Theorem 11.2. *Let X be a cofibrant R-algebra. Suppose we have a map $X \xrightarrow{f} A$. Then there is a lift $X \to B$ in*



iff the induced derivation

$$X \xrightarrow{f} A \xrightarrow{d} A \vee \Sigma I \quad \in TDer^1(X, I)$$

is homotopic to zero.

If *at least some* lift exists in the fibration $\mathrm{Map}_{C_R^{\mathrm{ass}}}(X, B) \to \mathrm{Map}_{C_R^{\mathrm{ass}}}(X, A)$, the homotopy fiber over f (i.e. using f as a basepoint) is weakly equivalent to $\Omega^\infty TDer_R(X, I)$.

Slogan: "In one degree you should have obstructions, in the degree below that you have 'how many are there', and in the degree below that there are automorphisms."

11.3. Magic fact proof sketch. So how do you actually construct the A_i's? "Gluing cells" gives a map $A \to A_0 := H\pi_0 A$ which is the identity on π_0. (Idea: kill all the higher homotopy groups as you do in the normal Postnikov tower, but you need the small object argument.) (Note it's really important that it's connective – otherwise where do you start?) Dugger and Shipley show that you can proceed similarly, but you don't get the extensions perspective.

Inductive step: given A_i by Dan's TAQ I talk,

$$\pi_k \Omega^{\mathrm{ass}}_{A_i/A} = \begin{cases} 0 & k \leq i \\ \pi_{i+1} A & k = i+1 \\ ? & \text{else.} \end{cases}$$

Now glue cells to Ω to get a map $k_{i+1} : \Omega^{\mathrm{ass}}_{A_i/A} \to \Sigma^{i+2} H\pi_{i+1} A$. Now k_{i+1} is the required derivation in $TDer^1(A_i, \Sigma^{i+1} H\pi_{i+1} A)$. (I use a relative version of the Kähler differentials; basically you do the exact same thing but with $\Omega_{A/R'} \to A \wedge_{R'} A \to A$ instead.)

11.4. Applications: E_n genera, moduli of A_∞-structures. I'm borrowing heavily from a talk by Mandell.

What is a genus? Answer: an assignment $j : M \mapsto j(M) \in R$ from manifolds with extra structure that is "cobordism-invariant": $j(M) = 0$ if $M = \partial W$.

What is R? A group, a ring, a graded group or ring,... but it could be π_* of an (ideally multiplicative) cohomology theory.

What do I mean by "extra structure"? Let me describe two things which are miraculously cohomology theories. Let $MSO_*(X)$ denote oriented manifolds over X and $*$ is just the dimension of the manifold, modulo ∂W. (It's a little tricky once you start amping up the extra structure to specify what structure you need on ∂W, but let's not worry about that). There's also $MU_*(X)$, "stably almost complex" manifolds M over X, again modulo ∂W, again graded by the dimension of the manifold.

M always embeds into $\mathbb{R}^N$ for $N \gg 0$ and we consider the normal bundle of this embedding. For various large N, take the colimit of these normal bundles over the diagram given by the maps $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N'}$, $N' > N$. Putting an almost complex structure on the colimit bundle is precisely a "stably almost complex" structure on M, constituting an element of MU_*, or $MU_*(X)$ if M has a fixed map to X.

A (cohomological) genus j is a map of multiplicative cohomology theories $MSO \to R$ or $MU \to R$, where R is some multiplicative cohomology theory.

Observations: MSO and MU are E_∞-rings (alternatively, commutative S-algebras).

Which genera are maps of E_∞-rings? $E_1 = A_\infty$-rings? E_n-rings?

Theorem 11.3 (Mandell, Chadwick)**.** *Let R be an E_2-ring and suppose $\pi_n R = 0$ for odd n (we sometimes call this "even").* Any *map of E_1-ring spectra $MU \to R$ lifts to one of E_2-ring spectra. If $\frac{1}{2} \in \pi_0 R$, then $MSO \to R$ also all lift to E_2.*

Corollary 11.4. *The Quillen idempotent $MU \to MU$ is (can be lifted to) an E_2-ring map. So BP is an E_2-ring (although more is known…).*

Proof sketch. You can use obstruction theory to get to E_1; assume our map is already E_1.

Consider the forgetful map from E_2-ring maps to E_1-maps. Take π_*, in particular π_0; we're going to show it's surjective. Recall the Pontryagin-Thom setup:

- $BU(n)$ classifies $\mathbb{C}^n$-vector bundles
- $PU(n)$ is the universal $U(n)$-space (which is contractible)
- $EU(n)$ is the universal $\mathbb{C}^n$-vector bundle
- $TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$
- $MU = \mathrm{colim}_n \, \Sigma^{\infty - 2n} TU(n)$

Define the Thom diagonal on the spaces $TU(n) \to TU(n) \wedge BU(n)_+$; I can take the suspension and deloop to get the map $MU \to MU \wedge_R BU_+$. Precomposition with this Thom diagonal yields a map $R^*MU \otimes R^*\Sigma^\infty_+ BU \to R^*MU$.

In other words, the data $f : MU \to R$, $g : \Sigma^\infty_+ BU \to R$ give rise to $MU \to MU \wedge BU_+ \overset{f\wedge g}{\to} R \wedge R \overset{m}{\to} R$. I'm mostly interested in fixing some $f \in R^*MU$; then you get an induced map $R^*\Sigma^\infty_+ BU \to R^*MU$. The Thom isomorphism theorem says that this is an isomorphism.

If R is E_{n+1}, the multiplication map $R \wedge R \overset{m}{\to} R$ is E_n. Suppose the fixed map $f \in R^*MU$ is E_n. The map $MU \to MU \wedge BU_+$ is always going to be E_∞. If g is E_n, then $MU \wedge BU_+ \to R \wedge R$ is E_n. Let $\mathcal{E}_n$ denote E_n-ring maps. We have an action of $\mathcal{E}_n(\Sigma^\infty_+ BU, R)$ on $\mathcal{E}_n(MU, R)$. An E_n-refined Thom isomorphism yields that, when we may fix $f \in \mathcal{E}_n(MU, R)$, the induced map $\mathcal{E}_n(\Sigma^\infty_+ BU, R) \to \mathcal{E}_n(MU, R)$ is an isomorphism.

Consider R connective (you can reduce to that case). Look at the Postnikov tower; I claim you will get a spectral sequence by trying to lift

$$\begin{array}{ccc} & & R_n \\ & \nearrow & \downarrow \\ A & \longrightarrow & R_{n-1} \end{array}$$

that computes $\pi_*\mathcal{E}_n(A, R)$ for arbitrary A. The spectral sequence is

$$h\mathcal{E}_{n/H\pi_0 R}(A, H\pi_0 R \vee \Sigma^p H\pi_q R) \implies \pi_{p-q}\mathcal{E}_{n/H\pi_0 R}(A, R).$$

The E_n-refined Thom isomorphism gives an E_n $H\pi_0 R$-algebra isomorphism

$$H\pi_0 R \wedge MU \simeq H\pi_0 R \wedge \Sigma^\infty_+ BU.$$

Substituting both sides into the spectral sequence above gives an isomorphism on E^2-terms and of

$$\pi_*\mathcal{E}_n(\Sigma^\infty_+ BU, R) \simeq \pi_*\mathcal{E}_n(MU, R).$$

A computation for BU shows that the map on π_0 from E_2-maps to R to E_1-maps is surjective. Then so too for MU. Hidden in our assumptions was that R was E_{n+1}, i.e., E_3, but a careful argument reduces to R being E_2. □

TALK 12: APPLICATION OF TAQ: BP IS E_4 (Guchuan Li)

Unlike the previous material in this document, which is comprised of scribe notes from talks, the notes for Talk 12 are written by Guchuan.

The content is mainly based on the first five sections in Basterra and Mandell's paper "The Multiplication on BP". Some notations are different from the ones (those in parentheses) in the paper to keep consistent with previous talks. This is a talk based on previous ones but I tried to make it more self-contained.

12.1. Outline. The goal for the talk is to prove the following theorem.

Theorem 12.1 (Basterra-Mandell)**.** *BP is an E_4-ring spectrum; the E_4-ring structure is unique up to automorphism in the category of E_4-ring spectra.*

Corollary 12.2 (Basterra-Mandell)**.** *The derived category of BP modules has a symmetric monoidal smash product $\wedge_{BP}$.*

Remark 12.3. In Pax's talk, we have seen that BP admits a $(2p^2 + 2p - 2)$-stage structure. The relationship of this result to Theorem 12.1 is not understood.

The strategy is to lift E_4 structure along the Postnikov tower by checking certain properties of obstructions lying in TAQ.

12.2. TAQ & Postnikov towers for E_n ring spectra. We have seen TAQ in various settings in previous talks and Postnikov towers in R-modules in the last talk; here we will work in E_n R-algebras and develop Postnikov towers in E_n R-algebras compatible with the R-module ones with k-invariants in (E_n) TAQ. This will be the main machinery for the proof of Theorem 12.1.

12.2.1. *Setting.* Let R be a connective commutative S-algebra, A be a connective commutative R-algebra and M be a A-module. Here we ask for connectivity because we will need a place to start the Postnikov tower. In the application, we will take $R = S_{(p)}$, $A = H\mathbb{Z}_{(p)}$ and $M = H\mathbb{F}_p$. We want to work in the category of E_n R-algebras lying over A.

Definition 12.4. An E_n *R-algebra* X is an R-module with R-module maps

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} (X \wedge_R \cdots \wedge_R X) \to X$$

where $\mathcal{C}_n$ is the Boardman-Vogt little n-cubes operad.

Let's put E_n in front of A-algebra everywhere and adjust the setting from previous talks to the E_n A-algebra one. Let $\mathscr{C}^n_{R/A}$ ($\mathfrak{UC}_n^{R/A}$) be the category of E_n R-algebras lying over A. An object is an E_n R-algebra X together with a map of E_n R-algebras $\varepsilon\colon X \to A$; a morphism is a map of E_n R-algebras over A. Let $\mathscr{N}^n_A$ ($\mathfrak{NC}^A_n$) be the category of E_n "non-unital" A-algebras, the same as the category of E_n A-algebras except not asking for unital maps. Let $\mathscr{M}_A$ be the category of A-modules as before.

From Yu's talk, we have seen that these categories are related by Quillen adjunctions

$$\mathscr{C}^n_{R/A} \underset{F}{\overset{A\wedge_R -}{\rightleftarrows}} \mathscr{C}^n_{A/A} \underset{I}{\overset{K}{\leftrightarrows}} \mathscr{N}^n_A \underset{Z}{\overset{Q}{\rightleftarrows}} \mathscr{M}_A \tag{12.1}$$

where F is the forgetful functor, K attaches a unit ($K(-) = A \vee -$), I takes the (point-set) fiber of the augmentation $- \to A$, Z turns an A-module into a non-unital E_n A-algebra by giving trivial multiplications and trivial E_n structure maps, and Q is the indecomposables functor, defined by the coequalizer

$$\bigvee_{m>0} \mathcal{C}_n(m)_+ \wedge_{\Sigma_m} N^m \rightrightarrows N \to QN.$$

(Here $N^m = N \wedge_R \cdots \wedge_R N$ with m copies of N. One map is the action map for N and the other map is the trivial map on the factors for $m > 1$ and the map $\mathcal{C}_n(1)_+ \wedge N \overset{\simeq}{\to} N$).

Remark 12.5. The setting is similar to the previous one for R-algebras, but here we need to encode the E_n structures: the indecomposables functor Q needs to remember higher structures; $KZ(M) = A \vee M \in \mathscr{C}^n_{R/A}$ is the "square-zero extension" for an A-module M, in addition to the multiplication structure, we also need to put the "square-zero" E_n structure maps

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} (A \vee M)^m \to (A \vee M)^m/\Sigma_m \to A \vee M$$

where the map is induced by the multiplication on A and the A-module structure on M on the summands with one or fewer factors of M and the trivial map on the summands with two or more factors of M. The same happens for $\mathscr{N}^n_A$, and we need to modify the E_n operad to $\tilde{\mathcal{C}}_n$ for the non-unital case where $\tilde{\mathcal{C}}_n(0) = \emptyset$ and $\tilde{\mathcal{C}}_n(k) = \mathcal{C}_n(k)$ for $k \geqslant 1$.

12.2.2. *TAQ.* Because of the Quillen adjunctions in (12.1), we can make our definitions of topological Quillen (co)homology in one of homotopy categories of $\mathscr{C}^n_{R/A}$, $\mathscr{N}^n_A$ or $\mathscr{M}_A$. Here are notations. We denote E_n topological André-Quillen (co)homology as $H^{\mathcal{C}_n}_*(H^*_{\mathcal{C}_n})$ instead of using TAQ. Also for $X \in \mathscr{C}^n_{R/A}$ cofibrant, let $N \in \mathscr{N}^n_A$ cofibrant with a weak equivalence $KN \to A \wedge_R X$ in $\mathscr{C}^n_{A/A}$ (ignoring the technical condition of cofibrancy, N roughly is $\mathbb{R}I(A \wedge^{\mathbb{L}}_R X)$). Recall that the Kähler differentials $\Omega_R(X)$ is $\mathbb{L}Q\mathbb{R}I(X \wedge^{\mathbb{L}}_R X)$. Let $\Omega^A_R(X) = \Omega_R(X) \wedge_X A$.

	$\mathscr{C}^n_{R/A}$	$\mathscr{N}^n_A$	$\mathscr{M}_A$
$H^*_{\mathcal{C}_n}(X; M)$	$ho\mathscr{C}^n_{R/A}(X, A \vee \Sigma^* M)$	$\mathrm{Ext}^*_A(\mathbb{L}QN, M)$	$\mathrm{Ext}^*_A(\Omega^A_R(X), M)$
$H^{\mathcal{C}_n}_*(X; M)$		$\mathrm{Tor}^A_*(\mathbb{L}QN, M)$	$\mathrm{Tor}^A_*(\Omega^A_R(X), M)$

One can check that $H^*_{\mathcal{C}_n}$ is a cohomology theory and $H^{\mathcal{C}_n}_*$ is a homology theory with relative versions, and we have universal coefficient spectral sequences to relate them.

Remark 12.6. The equivalence of the above definitions follows directly from Quillen adjunctions, and K, I form a Quillen equivalence, so it does not matter that K goes the other way in 12.1.

The definition in $\mathscr{M}_A$ presents a good analogue to the algebraic Quillen (co)homology; the definition in $\mathscr{C}^n_{R/A}$ is more homotopic and the place where obstructions and invariants naturally live; the definition in $\mathscr{N}^n_A$ is where all computation happens (see the following Theorem 12.7).

Let's recall the following theorem from Yu's talk. I want to highlight this key fact that makes all computations possible in the application. We will need it for the Postnikov towers and the computation of the obstruction.

Theorem 12.7 (Basterra-Mandell)**.** *Let $N \in \mathscr{N}^n_A$ be cofibrant, then $\tilde{B}N$ has E_{n-1} structure and $\tilde{B}^n N \simeq \Sigma^n QN$ where $\tilde{B}$ is the reduced bar construction.*

We need to recall one more fact about TAQ from Daniel's talk – that there are natural maps from $H^R_*(X; M)$ to $H^{\mathcal{C}_n}_*(X; M)$ and $H^*_{\mathcal{C}_n}(X; M)$ to $H^*_R(X; M)$. By definition, we have $H^R_*(X; M) = \pi_*(X \wedge_R M)$ and $H^{\mathcal{C}_n}_*(X; M) = \pi_*(QN \wedge_R M)$ where $A \vee N = KN \simeq X \wedge_R A$; the natural map is induced by $X \wedge_R M = X \wedge_R A \wedge_A M \simeq KN \wedge_A M = (A \vee N) \wedge_A M \to N \wedge_A M \to ZQN \wedge_A M \simeq QN \wedge_A M$.

12.3. Postnikov towers.

12.3.1. *Postnikov towers of R-algebras.* In Leo's talk, we have seen the Postnikov tower of an R-algebra (due to I. Kriz). We will review it and then discuss the Postnikov tower of an E_n R-algebra with compatible k-invariants.

Let X be a connective cofibrant R-algebra. Then there is a unique (up to weak equivalence as towers) Postnikov tower in R-algebras lying over $A = H\pi_0 X$.

$$\begin{array}{ccccc} & & \vdots & & \\ & \nearrow & \downarrow & & \\ & & X[1] & \xrightarrow{k_2} & \Sigma^3 H\pi_2 X \\ & \nearrow & \downarrow & & \\ X & \longrightarrow & X[0] & \xrightarrow{k_1} & \Sigma^2 H\pi_1 X \end{array}$$

where $\pi_* X[n] = \pi_* X$ for $* \leqslant n$ and $\pi_* X[n] = 0$ for $* > n$ and $X[n] \to X[n-1] \xrightarrow{k_n} \Sigma^{n+1} H\pi_n X$ is a fiber sequence and k_n is also unique up to weak equivalence.

From Leo's talk, we see that this can be used to give a filtration of X that helps the computation of $\mathrm{Map}_{E_n}(MU, X)$ and shows that any map of ring spectra from MU to an even E_2-ring spectrum X lifts to a map of E_2-ring spectra map $MU \to X$ (this is due to Chadwick and Mandell). In particular, the Quillen idempotent $MU_{(p)} \to MU_{(p)}$ is E_2 and BP is E_2. In this talk, we will set up the Postnikov tower for E_n R-algebras and lift the E_4 structure on $H\mathbb{Z}_p = BP[0]$ along the Postnikov tower to an E_4 structure on BP.

12.3.2. *E_n Postnikov towers.* We will work in $\mathscr{C}^n_{R/A}$.

Theorem 12.8 (Basterra-Mandell)**.** *Let X be a connective cofibrant E_n R-algebra with $H\pi_0 X = A$; then X is in $\mathscr{C}^n_{R/A}$ and there exists a Postnikov tower for X in $\mathscr{C}^n_{R/A}$ with $k^n_q \in H^{q+1}_{\mathcal{C}_n}(X[q-1]; H\pi_q X)$ such that*

(1) The following diagrams are homotopy pullbacks.

$$\begin{array}{ccc} X[n+1] & \longrightarrow & A \\ \downarrow & & \downarrow \\ X[n] & \longrightarrow & \Sigma^{n+2} H\pi_{n+1} X \vee A \end{array}$$

(2) Forgetting the E_n structure, $\{X[q]\}$ is a Postnikov tower of R-modules. The natural map $H^{q+1}_{\mathcal{C}_n}(X[q-1]; H\pi_q X) \to H^{q+1}(X[q-1]; H\pi_q X)$ takes k^n_q to the R-module k-invariants k_q (which we will denote as k^0_q).

(3) The data $\{X[q], k^n_q\}$ is unique up to weak equivalence, i.e. if there is a weak equivalence between towers $\{X[q]\}$ and $\{X'[q]\}$, then the weak equivalence sends k^n_q to k'^n_q.

Remark 12.9. The above theorem allows us to talk about $\{X[q], k^n_q\}$ for a given connective E_n-R-algebra X.

This is the main machinery we use to identify obstructions for lifting E_n structures from $X[q]$ to $X[q+1]$ and lifting a ring map to an E_n map. Before going into the application, we first discuss the proof.

The existence and "uniqueness" proof of $\{X[q]\}$ is formal and "cellular arguments" work. However, the compatibility of k-invariants needs some effort. Here is the key fact.

Theorem 12.10 (Basterra-Mandell ("Hurewicz theorem")). *Let M be a connective A-module, and $X \to Y$ a q-connected map in $\mathscr{C}^n_{R/A}$. Then*

$$H^{\mathcal{C}_n}_*(Y, X; M) = 0 \text{ for } * \leqslant q,$$
$$H_{q+1}(Y, X; M) \xrightarrow{\cong} H^{\mathcal{C}_n}_{q+1}(Y, X; M),$$

where the map is the natural map from H_* to $H^{\mathcal{C}_n}_*$.

We will not go into the details of the proof but emphasize the proof needs Theorem 12.7 and the observation that $\tilde{B}^n X \to \tilde{B}^n Y$ is $(q+1)$-connected.

Sketch idea of the proof of Theorem 12.8. The construction is similar to the R-algebra case. The remaining part is to deal with the k-invariants. By the construction, $X \to X[q-1]$ is q-connected. By the Hurewicz Theorem, there is a natural isomorphism between $H^{\mathcal{C}_n}_{q+1}(X[q-1], X; A)$ and $H_{q+1}(X[q-1], X; A)$. By universal coefficients spectral sequences, we will have a correspondence between $H^{q+1}_{\mathcal{C}_n}(X[q-1], X; A)$ and $H^{q+1}(X[q-1], X; A)$, which allows us to choose $\tilde{k}^n_q \in H^{q+1}_{\mathcal{C}_n}(X[q-1], X; A)$. Let k^n_q be the image of $\tilde{k}^n_q$ under $H^{q+1}_{\mathcal{C}_n}(X[q-1], X; A) \to H^{q+1}_{\mathcal{C}_n}(X[q-1]; A)$. Then one can check, by construction, that k^n_q hits k_q under the natural map.

To be a little more precise, by the Hurewicz Theorem and universal coefficients spectral sequences, there is a canonical isomorphism

$$H^{q+1}_{\mathcal{C}_n}(X[q-1], X; A) \cong Hom_{\pi_0 R}(\pi_q A, \pi_q A)$$

and $\tilde{k}^n_q \in H^{q+1}_{\mathcal{C}_n}(X[q-1], X; A)$ corresponds to $id \in Hom_{\pi_0 R}(\pi_q A, \pi_q A)$. □

12.3.3. *Obstruction theories.* We have the following corollaries of Theorem 12.8.

Corollary 12.11 (Basterra-Mandell). *Start with an R-module X with a fixed commutative $\pi_0 R$-algebra structure on $\pi_0 X$ and a $\pi_0 X$-module structure on $\pi_* X$. An E_n R-algebra over an $H\pi_0 X$ structure on $X[q-1]$ extends to a compatible structure on $X[q]$ of an E_n R-algebra over $H\pi_0 X$ if and only if $k^0_q \in H^{q+1}(X[q-1]; H\pi_q X)$ lifts to $k^n_q \in H^{q+1}_{\mathcal{C}_n}(X[q-1]; H\pi_q X)$.*

This gives a way to inductively extend the E_n structure along the tower. Because $X[q]$ is the fiber of $X[q-1] \to \Sigma^{q+1} H\pi_q X \vee A$, if we can show this map is an E_n map, i.e. k^0_q lifts to k^n_q, then the fiber has an E_n structure.

Corollary 12.12 (Basterra-Mandell). *Let X and Y be E_n R-algebras lying over $H\pi_0 X$. Then a E_n map $Y \to X[q-1]$ lifts to a E_n map $Y \to X[q]$ if and only if $f^*_{q-1} k^n_q \in H^{q+1}_{\mathcal{C}_n}(Y; H\pi_q X)$ vanishes.*

12.4. BP is E_4. Now we are ready to prove that BP is E_4. We will take $R = S_{(p)}$ and $A = H\mathbb{Z}_{(p)}$.

12.4.1. *Outline of the proof.* For the existence, we will lift the E_4 structure on $BP[1] = H\mathbb{Z}_{(p)}$ to the structure on BP inductively along the Postnikov tower of BP by lifting k_q^0 to k_q^4. We construct $k_q^4(BP[q-1])$ by constructing an E_4 map $f\colon BP[q-1] \to MU[q-1]$ and use the fact MU is E_4, hence there is $k_q^4(MU[q-1])$, and let $k_q^4(BP[q-1])$ be the composition

$$BP[q-1] \xrightarrow{f} MU[q-1] \to \Sigma^{q+1} H\pi_q MU \vee \mathbb{Z}_{(p)} \to \Sigma^{q+1} H\pi_q BP \vee \mathbb{Z}_{(p)}$$

where the last map is induced by the complex orientation of BP. To construct such an f, we compute $H^*_{\mathcal{C}_4}(BP[q-1]; H\mathbb{Z}_{(p)})$ in a certain range and check that obstructions to lifting $BP[q-1] \to H\mathbb{Z}_{(p)}$ to $f\colon BP[q-1] \to MU[q-1]$ all vanish. Hence, by Corollary 12.12, we have the desired f and complete the proof. The vanishing of the obstructions is based on the main lemma below.

Lemma 12.13 (Main Lemma)**.** *Assume there is an E_4 structure on $X = BP[2q] = BP[2q+1]$. Then for degrees $\leq 2q+1$, $H_*^{\mathcal{C}_4}(X; H\mathbb{Z}_{(p)})$ is torsion free and evenly concentrated.*

Remark 12.14. We use homology because it is easier to deal with “tensors". From the universal coefficients spectral sequences, we immediately have that $H^*_{\mathcal{C}_4}(X; H\mathbb{Z}_{(p)})$ is also evenly concentrated.

The uniqueness also depends on Lemma 12.13, which shows that there is no obstruction to lifting $BP \to H\mathbb{Z}_{(p)}$ to $BP \to BP'$ where BP' is BP with a different E_4 structure. Such a lift is necessarily an isomorphism on $H^0(-; \mathbb{F}_p)$ and compatible with Steenrod operations, so it induces an isomorphism on cohomology and so it is a weak equivalence.

12.4.2. *Inductive step.* To illustrate the inductive proof we just outlined, let's do the case of lifting E_4 structure on $BP[1]$ to $BP[2]$. This may be trivial since $BP[1] = BP[2]$ at primes $p > 2$, but we will do the general approach that works for all $BP[q]$. We start with the E_4 S-algebra $BP[1]$ and an E_4 S-algebra map $f_1\colon BP[1] \to MU[1]$, and try to push to the $q = 2$ case. We first show that $BP[2]$ is E_4. We have the following diagram.

$$\begin{array}{ccccccc}
BP[2] & & MU[2] & & & & \\
\downarrow{\scriptstyle g_1(BP)} & & \downarrow{\scriptstyle g_1(MU)} & & & & \\
BP[1] & \xrightarrow{f_1} & MU[1] & \xrightarrow{k_2^4(MU)} & \Sigma^3 H\pi_2 MU \vee H\mathbb{Z}_{(p)} & \xrightarrow{h} & \Sigma^3 H\pi_2 BP \vee H\mathbb{Z}_{(p)}
\end{array}$$

Let $k_2^4(BP) = h \circ k_2^4(MU) \circ f_1$. Then it is an E_4 map and a lift of $k_2^0(BP)$, thus by Corollary 12.11, $BP[2]$ is E_4. In fact, $BP[2]$ is the fiber of $k_2^4(BP)$.

Next we need to construct an E_4 map $f_2\colon BP[2] \to MU[2]$ to finish this inductive step. We have an E_4 map $f_1 \circ g_1(BP)\colon BP[2] \to MU[1]$. By Corollary 12.12, the obstruction to lift this to an E_4 map $f_2\colon BP[2] \to MU[2]$ lies in $H^3_{\mathcal{C}_4}(BP[2], H\pi_2 MU)$, which by Lemma 12.13 vanishes. Note that in general the obstruction lies in $H^{q+1}_{\mathcal{C}_4}(BP[q], H\pi_q MU)$, so when q is odd, the coefficient $H\pi_q MU$ is trivial, and hence the whole group is trivial; when q is even, the cohomology degree $q+1$ is odd, and by Lemma 12.13, the cohomology groups are evenly concentrated, so the group vanishes.

12.4.3. *Main Lemma.* Now the problem reduces to proving Lemma 12.13. The key ingredient is Theorem 12.7. First, we can reduce to the case with coefficients $H\mathbb{F}_p$, where it is evenly concentrated instead of $H\mathbb{Z}_{(p)}$. This is because if $H_*^{\mathcal{C}_4}(BP[q]; H\mathbb{F}_p)$ is evenly concentrated, then the Bockstein spectral sequence from $H_*^{\mathcal{C}_4}(BP[q]; H\mathbb{F}_p)$ to $H_*^{\mathcal{C}_4}(BP[q]; H\mathbb{Z}_{(p)})$ collapses for dimension reasons, so $H_*^{\mathcal{C}_4}(BP[q]; H\mathbb{Z}_{(p)})$ is evenly concentrated.

Lemma 12.15 (Main Lemma$'$)**.** *Assume there is an E_4 structure on $BP[2q+1]$. Then in degrees $\leq 2q+1$, $H_*^{\mathcal{C}_4}(BP[2q+1]; H\mathbb{F}_p)$ is torsion free and evenly concentrated.*

We use Theorem 12.7 to set up the following spectral sequence to compute $H_*^{\mathcal{C}_n}(BP[q-1]; H\mathbb{F}_p)$ inductively on n. When $n=0$ it is the ordinary cohomology, and $n=4$ is what we want.

Lemma 12.16. *Let X be an E_n S-algebra lying over $H\mathbb{Z}_{(p)}$. The quotient map $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ gives the E_∞ map $H\mathbb{Z}_{(p)} \to H\mathbb{F}_p$. Then for $0 \leqslant j < n$, there is a spectral sequence*

$$E_2^{s,t} = \mathrm{Tor}^{\mathbb{F}_p \oplus H_{*-j}^{\mathcal{C}_j}(X;\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathbb{F}_p \oplus H_{s+t-(j+1)}^{\mathcal{C}_{j+1}}(X; \mathbb{F}_p)$$

with $d_r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

We will come back to the construction of the spectral sequence later. Let's see how to use this spectral sequence to prove Lemma 12.13 and why this approach stops at E_4.

Recall that $H^*_{\mathcal{C}_0}(BP, \mathbb{F}_p) = H^*(BP, \mathbb{F}_p)$ at an odd prime p (the $p=2$ case is similar but with a slightly different formula, so we will focus on the odd prime case to describe the method). Here are two easy homological algebra computations:

$$\mathrm{Tor}_*^{\mathbb{F}_p[x_1,\cdots,x_n]}(\mathbb{F}_p, \mathbb{F}_p) = E(y_1, \cdots, y_n)$$

where $E(-)$ is an exterior algebra and $|y_i| = 1$;

$$\mathrm{Tor}_*^{E(x_1,\cdots,x_n)}(\mathbb{F}_p, \mathbb{F}_p) = \bigotimes_{0\leqslant i\leqslant n} \Gamma(y_i)$$

where $\Gamma(y_i) = \bigotimes_{k\geqslant 0} \mathbb{F}_p[r_k(y_i)]/(r_k(y_i))^p$ is a divided power algebra and $|r_k(y_i)| = p^k$.

Denote $H^*_{\mathcal{C}_n}(BP, \mathbb{F}_p) = B_n$ and compute them from $n=1$ to $n=4$ inductively as follows:

	E^2	E^∞	degree
B_0		$\mathbb{F}_p[\xi_1, \xi_2, \cdots]$	$\lvert\xi_i\rvert = 2p^i - 2$
B_1	$E(\sigma\xi_1, \sigma\xi_2, \cdots)$	collapse	$\lvert\sigma\xi_i\rvert = 2p^i - 1$
B_2	$\mathbb{F}_p[r_0\sigma^2 x_1, r_1\sigma^2 x_1, \cdots]$	collapse	$\lvert r_i\sigma^2 x_1\rvert = 2p^{i+1}$
B_3	$E(\sigma r_0\sigma^2 x_1, \sigma r_1\sigma^2 x_1, \cdots)$	collapse	$\lvert\sigma r_i\sigma^2 x_1\rvert = 2p^{i+1} + 1$
B_4	Divided power algebra	evenly concentrated	even

Most of this is direct computation and collapse follows from the multiplicative structure and dimension reasons – except that the E^2 of B_2, as Tor of an exterior algebra, should be a divided power algebra, but here is the place where the Dyer Lashof operations help to solve extension problems on the E^2 page and it turns out to be a polynomial algebra after suitably changing the base. The same trick does not work on B_4 since the E^2 page is more complicated and it is harder to compute Dyer Lashof operation actions there, and that is the reason why this method stops with an E_4 structure.

Remark 12.17. The σ appearing in the name indicates the suspension and the shift of degree by 1.

The collapse needs multiplicative structure of the spectral sequence and the Dyer Lashof operations need to be modified to act on each E^r page. Some work is needed and that is sections 6 and 7 in the paper.

Let me wrap up with a sketch of the Dyer Lashof operation arguments and the construction of the spectral sequence in Lemma 12.16.

By direct computation, the E^2 page of B_2 is

$$\bigotimes_{i\geqslant 1}\Big(\bigotimes_{k\geqslant 0}\mathbb{F}_p[r_k\sigma^2\xi_i]/(r_k\sigma^2\xi_i)^p\Big).$$

Note that ξ_i comes from $H^*(H\mathbb{F}_p, \mathbb{F}_p)$, and we know the Dyer Lashof operation actions there. In particular, $Q^{p^i}\xi_i = \xi_{i+1} + \textit{decomposables}$. Recall that Q^{p^i} is a p^{th} power on the right degree, so we have

$$(r_j\sigma^2\xi_i)^p = Q^{p^{i+j}} r_j\sigma^2\xi_i = r_j\sigma^2\xi_{i+1} + \textit{decomposables}.$$

Up to suitable change of basis, $x_i \sim \xi_i$, so we can absorb the decomposables part and have

$$(r_j\sigma^2 x_i)^p = r_j\sigma^2 x_{i+1},$$

which gives the extension that changes the divided power algebra into the polynomials.

Finally, we go back to the construction of the spectral sequence in Lemma 12.16. It arises from the filtration given by the bar construction and in our example, we work over the field $\mathbb{F}_p$, so the flatness over a field allows us to compute the E_1 page with differentials from the bar complex (it happens to be a resolution of $\mathbb{F}_p$) inside the Tor and get a simple E^2 page as in Lemma 12.16.

Part IV: Goerss-Hopkins obstruction theory

Interlude: Informal introduction to Goerss-Hopkins obstruction theory
(Dylan Wilson)

This is Dylan's response to a request to provide more intuition behind Goerss-Hopkins obstruction theory, delivered during an informal questions seminar.

Suppose you're trying to build spaces. You could use a cell structure, which is reflected in the homology of X. You could use a Postnikov tower, which starts with the homotopy groups. Alternatively, you could being with the input of $H^*(X, \mathbb{F}_p)$ together with the action of the Steenrod algebra. Now what do you get?

In the cellular story, the easy objects were D^n or Δ^n; in the Postnikov story, they were $K(\pi, n)$'s; in this new story (the Adams story), they're products $\prod_i K(\mathbb{F}_p, m_i)$. We can try to resolve X by these.

Step 1: Build a map $X \xrightarrow{\rho} \prod K(\mathbb{F}_p, n_i)$ such that ρ^* is surjective on cohomology.

Step 2: Take the cofiber and repeat.

Step 3: We get a spectral sequence for maps, and an obstruction theory for objects. Suppose I want to build a map $Y \to X$, and already have in mind a map $\varphi : H^*X \to H^*Y$. The obstructions to lifting live in

$$D^{t+2}_{\substack{\text{unst.} \\ \text{alg}/\mathcal{A}}}(H^*X, \Omega^t H^*Y).$$

This is some sort of unstable Adams spectral sequence. This is used in the proof of the Sullivan conjecture (by Miller); a good reference is the book by Lannes. There's also a bit about this in Goerss-Jardine (under 'Obstruction theory').

Remark. Why does Quillen homology always seem to arise? Say you're computing maps $S^t \to \mathrm{Map}_{E_\infty}(X, Y; \varphi)$. Some adjunctions eventually lead to considering collections of commutative diagrams of algebra maps

$$\begin{array}{ccc} X & \longrightarrow & Y \oplus \Omega^k Y \\ & \searrow^{\varphi} & \downarrow \\ & & Y \end{array}$$

But that's the definition of a derivation. (I learned this observation from some older Talbot notes from a talk by Hopkins.)

In Goerss-Hopkins obstruction theory, the easy objects are $T_n(DE_\alpha)$. Let me explain what I mean by that a bit more.

Slogan: algebra is "easy"; topology is hard. Wouldn't it be nice if some pieces of topology were the same as some piece of algebra? In Goerss-Hopkins obstruction theory, the plan is:

Step 1: Find E_∞-algebras which are controlled by algebra that you understand.

Step 2: Resolve everything by these.

Step 3: Profit.

This is sort of the outline of all of algebraic topology.

My algebraic approximation to E_∞-ring maps is E_*E-comodule algebra maps. If I have a map $X \to Y$ of E_∞-rings, then I certainly get a map $E_*X \to E_*Y$ of E_*E-comodules. This is our piece of algebra that we purport to understand. Since our starting information only sees stuff that E sees, we'll eventually have to E-complete.

As a warmup, what if we just have spectrum maps instead of E_∞ maps? We have a map $\pi_0 \operatorname{Map}_{\mathrm{Sp}}(X, Y) \to \operatorname{Hom}_{E_*E}(E_*X, E_*Y)$. The Adams condition produces/requires the existence of spectra E_α such that $[DE_\alpha, E \wedge Y] \simeq \operatorname{Hom}_{E_*E\text{-comod}}(E_*DE_\alpha, E_*E \otimes_{E_*} Y)$. As my test objects, take free E_∞ algebras on DE_α. Now you have to go simplicial, because a map between two of these is not really reflected in E_*E-comodule maps.

What do I mean by that? At the very least, I should be able to understand E_∞ maps $\operatorname{Free}_{E_\infty}(DE_\alpha) \to \operatorname{Free}_{E_\infty}(DE_\alpha)$. The claim is that maps $DE_\alpha \to \operatorname{Free}_{E_\infty}(DE_\beta)$ is the same as $\operatorname{Hom}_{E_*}(E_*DE_\alpha, E_* \operatorname{Free}_{E_\infty}(DE_\alpha))$. If we insist on computing E_∞ maps in terms of E_*E-comodule algebra maps, then we would ask that

$$\operatorname{Hom}_{E_\infty}(\operatorname{Free}_{E_\infty}(DE_\alpha), \operatorname{Free}_{E_\infty}(DE_\beta)) = \operatorname{Hom}_{E_*E\text{-Alg}}(E_*(DE_\alpha), E_*(DE_\beta)).$$

But that's not true.

[Aside: the free associative thing is really easy: it's $E_*(\bigvee_{n\geq 0} DE_\alpha^{\wedge n})$. By projectivity, this is $\bigoplus (E_*DE_\alpha)^{\otimes n}$ by Künneth. So, in the associative case, there's no need to 'go simplicial'.]

But that's not the case with the free E_∞ thing. The free commutative object is $\bigvee E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n}$. Part of the problem is that this is not $\bigvee B\Sigma_n \wedge X^{\wedge n}$. Instead, you have to resolve the E_∞ operad by a simplicial operad, whose pieces have $\pi_0 \simeq \Sigma_n$ so that algebra and topology coincide for our test objects. But it's still not quite true that I've got E_*E-comodule algebra maps. Instead, I get "E_*(resolution)"-algebra maps. But things are spread out enough that this is as close as you're going to get, and this is good enough.

At the end of the day, we have a way to study E_∞-rings via simplicial E_*(resolution)-algebras in simplicial E_*E-comodules.

Moral: don't compute until you have reduced to something that is maximally easy to compute.

Talk 13: Goerss-Hopkins obstruction theory I (Dominic Culver)

First I'm going to tell you what Goerss-Hopkins obstruction theory is; then I'm going to develop the machinery needed for the next talk.

Throughout the talk, E is a homotopy commutative ring spectrum. We will also make the assumption that E_*E is flat over E_*. The main question that Goerss-Hopkins obstruction theory tries to answer is:

Question 13.1. Let A be an algebra in the category of E_*E-comodules. When is there an E_∞-ring spectrum R such that $E_*R \cong A$ in the category of E_*E-comodule algebras?

Goerss-Hopkins take a moduli approach. Let $\mathcal{E}(A)$ be the category with objects E_∞-rings R with $E_*R \cong A$ as E_*E-comodule algebras, and morphisms E_∞-ring maps which are E_*-isomorphisms. Let $TM(A) := B\mathcal{E}(A)$ be the classifying space; they try to compute this. What they really want to know is whether this space is nonempty.

They study this by relating it to another space. The define the *space of potential* ∞*-stages* $TM_\infty(A) = B(\text{potential } \infty\text{-stages})$. Paul will define this rigorously later, but the idea is as follows: potential ∞-stages are $X_\bullet \in s\,\mathrm{Alg}_{E_\infty}$ such that

$$\pi_k E_* X_\bullet \cong \begin{cases} A & k = 0 \\ 0 & k > 0. \end{cases}$$

Here $E_*X_\bullet$ means taking E-homology levelwise; and $\pi_k E_* X_\bullet$ is the $k$thhomotopy group of this simplicial abelian group. We will get a spectral sequence

$$\pi_s E_t X_\bullet \implies E_{s+t}|X_\bullet|.$$

If $X_\bullet$ is E_∞, then so is its realization $|X_\bullet| \in \mathrm{Alg}_{E_\infty}$.

Theorem 13.2 (Goerss-Hopkins)**.** *Geometric realization gives a weak equivalence*

$$|-| : TM_\infty(A) \xrightarrow{\simeq} TM(A).$$

So now we have to show that $TM_\infty(A)$ is nonempty. Goerss-Hopkins define intermediary objects $TM_n(A)$, the classifying spaces of potential n-stages $X_\bullet$. We will have $X_\bullet \in s\,\mathrm{Alg}_{E_\infty}$, and ask that

$$\pi_i E_* X_\bullet \cong \begin{cases} A & i = 0 \\ 0 & 1 \le i \le n+1 \\ ? & i \ge n+2. \end{cases}$$

You should think of these as giving a resolution of A up to a stage. There will be a functor $P_m : TM_n(A) \to TM_m(A)$ where $0 \le m \le n \le \infty$.

Theorem 13.3 (Goerss-Hopkins)**.** *There is a waek equivalence* $TM_\infty(A) \xrightarrow{\simeq} \mathrm{holim}_n\, TM_n(A)$.

So we've got a tower

$$\begin{array}{ccc} & & TM_\infty(A) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & TM_n(A) \\ & & \downarrow \\ & & TM_{n-1}(A) \\ & & \downarrow \\ & & \vdots \\ & \nearrow & \downarrow \\ & & TM_1(A) \\ & \nearrow & \downarrow \\ * & \longrightarrow & TM_0(A) \end{array} \tag{13.1}$$

They start with a basepoint in $TM_0(A)$ and try to lift that up the tower. If you get a basepoint in $TM_\infty(A)$ then you win.

They'll show that there is a space $\widehat{\mathcal{H}}^{n+2}(A, \Omega^n A)$ and a homotopy pullback square

$$\begin{array}{ccc} TM_n(A) & \longrightarrow & B\operatorname{Aut}(A, \Omega^n A) \\ \downarrow & & \downarrow \\ TM_{n-1} & \longrightarrow & \widehat{\mathcal{H}}^{n+2}(A, \Omega^n A) \end{array}$$

and the fiber of the right-hand column (and hence the left-hand column) is $\widehat{\mathcal{H}}^{n+1}(A, \Omega^n A)$. Here $\widehat{\mathcal{H}}^{n+2}(A, \Omega^n A)$ is a space whose π_0 is $D^{n+2}_{E_T/E_*}(A, \Omega^n)$ (here D is Quillen cohomology, and I'll define the rest of the stuff later).

Corollary 13.4 (Goerss-Hopkins)**.** *There are obstructions to existence living* $D^{n+2}_{E_*T/E_*E}(A, \Omega^n A)$*, and the obstructions to uniqueness lie in* $D^{n+1}_{E_*T/E_*E}(A, \Omega^n A)$*.*

One of the most important examples of this is $A = E_*E$, for example for the Lubin-Tate theories E_n.

Here's another perspective to the realization problem that you might come up with (following Behrens, c.f. appendix to the construction of *tmf*). The naïve idea is the following: suppose you want to realize A. You want to build a simplicial free resolution $A \leftarrow W_\bullet$ and a space $X_\bullet \in s\operatorname{Alg}_{E_\infty}$ simultaneously such that $E_*X \cong W_\bullet$. Then you could just realize X_0. The issue is that you can't do that.

I can write $W_n = \overline{W}_n \otimes L_n W$ and $X_n = \overline{X}_n \wedge L_n X$ where $L_n W$ etc. are latching objects and $\overline{X}_n$ is a free E_∞-ring on something, i.e. $\overline{X}_n = \mathbb{P} Y_n = \bigvee_{k \geq 0} E\Sigma_{k+} \wedge_{\Sigma_k} Y_n^{\wedge k}$. I'd also like $\overline{W}_n = E_* X_n = E_* \mathbb{P} Y_n$. I'd like to be able to write down a triple $\mathbb{T}$ such that this is $\overline{W}_n = \mathbb{T} E_* Y_n$. But to write this $\mathbb{T}$, you'd have to compute $E_*(E\Sigma_{k+} \wedge_{\Sigma_k} Y_n^{\wedge k})$, but that would involve computing the DL operations. That would be tantamount to already knowing that E is E_∞; but the whole point is that we didn't know that E is E_∞. But Goerss-Hopkins resolve the operad such that you can write something like this down. This will come up in Paul's talk.

Let $X_\bullet$ be a simplicial spectrum and let T be a simplicial operad. The free T-algebra is

$$T(X) = \operatorname{diag}\{T_p(X_q)\}_{p,q \geq 0}.$$

Declare that a simplicial T-algebra is an algebra over T. Recall that if $X_\bullet \in s\operatorname{Alg}_T$ then $|X_\bullet| \in \operatorname{Alg}_{|T|}$.

So Goerss and Hopkins have an operad C and want to resolve it with a simplicial operad $T \to C$ such that:

(1) $|T| \simeq C$

(2) $\pi_0 T_n(q)$ is Σ_q-free
(Why? Given $X_\bullet \in s\operatorname{Alg}_T$ you want to understand $E_* T(X)$, which involves understanding $T_n(q)_+ \wedge_{\Sigma_q} X^{\wedge q}$, but if $\pi_0 T_n(q)$ is Σ_q-free, then this is just $T_n(q)/\Sigma_{q+} \wedge X^{\wedge q}$.)

Theorem 13.5. *Let C be an E_∞-operad. Then there exists an augmented simplicial operad $T \to C$ such that:*

(1) it is Reedy cofibrant (to deal with the fact that T is not unique);

(2) for all n, q, $\pi_0 T_n(q)$ is Σ_q-free;

(3) $|T| \simeq C$;

(4) if $E_*(C(q))$ is projective over E_*, then $E_* T$ will be cofibrant, and $E_* T \to E_* C$ induces a weak equivalence in simplicial operads over E_*-modules.
(This allows control over cofibrant objects in $\operatorname{Alg}_{E_* T}$.)

Recall we had $T_n(q)_+ \wedge_{\Sigma_q} X^\wedge q \simeq T_n(q)/\Sigma_{q+} \wedge X^{\wedge q}$. Resolving the simplicial operad made the $T_n(q)/\Sigma_{q+}$ easier, but you still have $X^{\wedge q}$. You want a Künneth spectral sequence. I will introduce a model category that gives this, plus something about resolving spectra with "projective spectra".

Suppose $E_* E$ is flat over E_*. They need to impose the *Adams condition*: that you can write $E = \operatorname{holim} E_\alpha$ where E_α is a finite cellular spectrum, and:

(1) $E_*(DE_\alpha)$ is projective over E_* (here D means Spanier-Whitehead dual)

(2) If M is an E-module spectrum, then $[DE_\alpha, M] \to \operatorname{Hom}_{E_*}(E_*(DE_\alpha), M)$ is an isomorphism.

For example, S^0, $H\mathbb{F}_p$, MO, MU, and any Landweber-exact theory (e.g. the Lubin-Tate theories) satisfy this condition.

Adams needs this to construct a resolution-like thing that gives rise to the universal coefficient spectral sequence.

The point is that Goerss and Hopkins use the DE_α to build topological resolutions where you have control over the algebras.

Definition 13.6. Let $\mathscr{P}$ be the minimal collection of spectra such that:

(1) $S^0 \in \mathscr{P}$;

(2) $DE_\alpha \in \mathscr{P}$;

(3) $\mathscr{P}$ is closed under Σ and Σ^{-1};

(4) $\mathscr{P}$ is closed under $\vee$;

(5) For all $P \in \mathscr{P}$ and E-module spectra M, the Künneth map $[P, M] \to \mathrm{Hom}_{E_*}(E_*P, M_*)$ is an isomorphism.

You use these to build up cofibrant resolutions and cofibrant replacements. You want to use the elements of $\mathscr{P}$ to detect weak equivalences, and then define a model structure that actually does that.

Definition 13.7. Suppose $f : X_\bullet \to Y_\bullet$ is a map of simplicial spectra.

(1) If $\pi_*[P, X_\bullet] \to \pi_*[P, Y_\bullet]$ is an isomorphism for all P, then say f is a $\mathscr{P}$-weak equivalence.

(2) If $f_\bullet : [P, X_\bullet] \to [P, Y_\bullet]$ is a fibration of simplicial abelian groups, then say f is a $\mathscr{P}$-fibration.

Theorem 13.8. *This determines a simplicial model structure on simplicial spectra.*

There are two notions of homotopy groups:

- ("type I") $\pi_{s,t}(X_\bullet; P) = \pi_s[\Sigma^t P, X_\bullet]$
- ("type II") $\pi^\natural_{s,t}(X_\bullet; P) = [\Sigma^t P \wedge \Delta^s/\partial\Delta^s, X_\bullet]$, where the term in the RHS is defined as the pushout:

$$\begin{array}{ccc} P \otimes \Delta^0 & \longrightarrow & P \otimes \Delta^s/\partial\Delta^s \\ \downarrow & & \downarrow \\ * & \longrightarrow & P \wedge \Delta^s/\partial\Delta^s \end{array}$$

When you geometrically realize, you're getting an actual homotopy class of maps – you land in $[\Sigma^{s+t}P, |X_\bullet|]$. It is clear that the groups $\pi_{s,t}$ (type I) detect $\mathscr{P}$-equivalences. But it turns out that this is also true for $\pi^\natural_{s,t}$.

The *spiral exact sequence* is an exact sequence of the following form:

$$\cdots \to \pi^\natural_{s-1,t+1}(X_\bullet; P) \to \pi^\natural_{s,t}(X_\bullet; P) \to \pi_{s,t}(X_\bullet; P) \to \pi^\natural_{s-2,t+1}(X_\bullet; P) \to$$
$$\cdots \to \pi^\natural_{0,t+1}(X_\bullet; P) \to \pi^\natural_{1,t}(X; P) \to \pi_{1,t}(X_\bullet; P) \to 0.$$

You can assemble an exact couple to get a spectral sequence

$$\pi_{s,t}(X_\bullet; P) \implies [\Sigma^{s+t}P, |X_\bullet|].$$

Let $E = \varinjlim E_\alpha$. Then $\pi_*\Sigma_t X_\bullet \cong \varinjlim \pi_*(E_\alpha)_* X_\bullet \cong \varinjlim [DE_\alpha, X_\bullet]$. So you recover the spectral sequence $\pi_s E_* X_\bullet \implies E_{s+t}|X_\bullet|$.

If $X_\bullet \to Y_\bullet$ is a $\mathscr{P}$-equivalence, then there is a map of spectral sequences

$$\begin{array}{ccc} \pi_{s,t}(X_\bullet; P) & \Longrightarrow & [\Sigma^{s+t}P, |X_\bullet|] \\ \downarrow & & \downarrow \\ \pi_{s,t}(Y_\bullet; P) & \Longrightarrow & [\Sigma^{s+t}P, |Y_\bullet|] \end{array}$$

Being a weak equivalence in the model structure is the same as inducing an isomorphism on the E_2 pages. In simplicial spectra the usual model structure is the Reedy model structure, which corresponds to weak equivalences inducing an isomorphism on E_1-pages. I don't care about simplicial spectra; I care about simplicial T-algebras.

Theorem 13.9 (Goerss-Hopkins)**.** *This model structure can be lifted to the category* $s\,\mathrm{Alg}_T$*.*

Let $X \in \mathrm{Alg}_C$. Recall T was augmented over C. I can turn X into a constant simplicial spectrum. Paul will need to take a cofibrant replacement $P_T(X) \to X$; being cofibrant means it's built out of things in $\mathscr{P}$. A result of these spectral sequences is that $\pi_* E_* P_T(X) \cong E_* X$.

$P_T(X)$ has an "underlying degeneracy diagram" of the form $T(Z)$, for $Z_\bullet$ a simplicial spectrum where Z_n is a wedge of objects in $\mathscr{P}$. You can go through the definitions and show that $E_* P_T(X) \cong (E_* T)(E_* Z)$. This is the thing you should think of as a free resolution.

When Goerss and Hopkins wrote down moduli spaces, they only cared about things up to E_*-isomorphisms.

Theorem 13.10. *You can Bousfield localize* $s\,\mathrm{Alg}_T$ *such that*

- $f : X_\bullet \to Y_\bullet$ *is a weak equivalence iff* $\pi_* E_* X_\bullet \to \pi_* E_* Y_\bullet$ *is an isomorphism.*
- $f : X_\bullet \to Y_\bullet$ *is an* E_**-fibration if it is a* $\mathscr{P}$ *fibration.*

It turns out that this is a good model structure.

Talk 14: Goerss-Hopkins obstruction theory II (Paul VanKoughnett)

Unlike most of the other material in this document, which is comprised of scribe notes from talks, the notes for Talk 14 are written by Paul.

14.1. The spectral sequence for algebra maps. Sources for this are [GH04], [GH], and the appendix to [Beh14].

Let $\mathcal{O}$ be an operad in spectra. Say we have two $\mathcal{O}$-algebras X and Y, we want to say something about $\mathrm{Maps}_{\mathcal{O}}(X, Y)$, and we know something about $E_* X$ and $E_* Y$ for some homology theory E which is an $\mathcal{O}$-algebra. Let's also make two assumptions about E, and one about $\mathcal{O}$:

- $E_* E$ is flat over E_* (the **Adams condition**);

- E is a filtered homotopy colimit of finite cellular spectra E_α such that E_*DE_α is a projective E_*-module, and for any E-module M there's a universal coefficient isomorphism $[DE_\alpha, M] \cong \mathrm{Hom}_{E_*\,\mathrm{Mod}}(E_*DE_\alpha, M_*)$ (the **other Adams condition**);
- there exists a monad C on E_*E-comodules such that E_* of an $\mathcal{O}$-algebra is naturally a C-algebra, and such that if X is cofibrant and E_*X is a projective E_*-module, then $E_*\mathcal{O}(X) \cong C(E_*X)$ (the homology theory is **adapted** to the operad).

The Adams condition is used to guarantee a good theory of E_*E-comodules. The other Adams condition is used to construct the resolution model structure that Dominic mentioned, essentially giving us a supply of 'projective objects' in spectra, such that resolutions by these objects also give nice resolutions by E_*E-comodules. Notice that the first Adams condition guarantees that E_*DE_α has the analogous properties as an E_*E-comodule. Finally, the adaptation of E to $\mathcal{O}$ will be used to construct our resolutions: we will simultaneously be resolving X as an $\mathcal{O}$-algebra and E_*X as a C-algebra.

Let's point out that the adaptedness condition, in general the hardest of these to prove, is satisfied in two convenient cases. First, $E_*\mathcal{A}_\infty$ is the associative operad in E_*E-comodules. More generally, if $\mathcal{O}$ is an operad over $\mathcal{A}_\infty$, so that $\pi_0\mathcal{O}(q)$ is a free Σ_q-set, then $E_*\mathcal{O}$ is naturally an operad in E_*E-comodules. Second, one can show that the p-complete K-theory of an E_∞-algebra is naturally a θ-algebra in comodules, and the free θ-algebra monad works in this case. This one also generalizes by [**Rez09**] and [**BarF15**]: they showed that the Morava E_n-theory of a $K(n)$-local E_∞-algebra is naturally an algebra over a monad $\mathbb{T}$ defined in terms of subgroups of formal groups (and which has been computed at low primes and $n = 2$).

So, let's be in such a case and consider E_*X as a C-algebra in E_*E-comodules. This has a set of generators $\{x_i\}$, which we can represent by comodule maps $x_i : S \to E \wedge X$. Each of these factors through some $S \to E_\alpha \wedge X$, meaning that they're in the image of some comodule map $E_*DE_\alpha \to E_*X$. Let $Z_0 = \bigvee_{x_i} DE_\alpha$; then E_*Z_0 is a projective comodule, we have an $\mathcal{O}$-algebra map $P_0 = \mathcal{O}(Z_0) \to X$, and

$$C(E_*Z_0) = E_*\mathcal{O}(Z_0) \to E_*X$$

is surjective. Likewise, we can find generators of the kernel of this map, represent them via maps $E_\alpha \to P_0$, and combine these into a map of free $\mathcal{O}$-algebras $P_1 \to P_0$, where E_*P_1 is a free C-algebra on a projective comodule. Continuing this way (and omitting some details), we get a simplicial resolution $P_\bullet \to X$, such that $P_\bullet$ is a simplicial $\mathcal{O}$-algebra that's levelwise free, $E_*P_\bullet$ is a simplicial C-algebra on a levelwise projective simplicial comodule, and $|P_\bullet| \simeq X$ as an $\mathcal{O}$-algebra, $|E_*P_\bullet| \cong E_*X$ as a C-algebra. This can be interpreted as a cofibrant resolution in a certain model category of simplicial $\mathcal{O}$-algebras, the **resolution model category**.

Now consider the double cosimplicial space

$$\mathrm{Maps}_{\mathcal{O}}(P_\bullet, E^{\bullet+1} \wedge Y).$$

If we want, we can think of this as a cosimplicial space by taking the diagonal.

Proposition 14.1. *We have*

$$\mathrm{Tot}\,\mathrm{Maps}_{\mathcal{O}}(P_\bullet, E^{\bullet+1} \wedge Y) \simeq \mathrm{Maps}_{\mathcal{O}}(X, Y_E^\wedge),$$

where $Y_E^\wedge = \mathrm{Tot}\, E^{\bullet+1} \wedge Y$ *is the* E*-completion.*

Corollary 14.2. *Given a map of $\mathcal{O}$-algebras $\varphi : X \to Y$, there is a Bousfield-Kan spectral sequence*

$$E_2^{s,t} = \pi^s \pi_t(\mathrm{Maps}_{\mathcal{O}}(P_\bullet, E^{\bullet+1} \wedge Y), \varphi) \Rightarrow \pi_{t-s}(\mathrm{Tot}\, \mathrm{Maps}_{\mathcal{O}}(P_\bullet, E^{\bullet+1} \wedge Y), \varphi).$$

Note that the basepoint φ on the E_2 page is the map $P_T(X) \to X \xrightarrow{\varphi} Y \to E^{\bullet+1} \wedge Y$.

It remains to identify the E_2 page. Under the Dold-Kan correspondence, this is the cohomology of a cochain complex whose terms are homotopy groups of mapping spaces

$$\mathrm{Maps}_{\mathcal{O}}(P_q, E^{q+1} \wedge Y).$$

By the freeness hypothesis on $P_T(X)$, this is equivalent to

$$\mathrm{Maps}_{\mathcal{O}}(\mathcal{O}(Z_q), E^{q+1} \wedge Y) \simeq \mathrm{Maps}_{\mathrm{Sp}}(Z_q, E^{q+1} \wedge Y).$$

Z_q is supposed to satisfy the universal coefficient theorem for E-homology, so the π_0 of this (which is really just a set) is

$$\pi_0 \mathrm{Maps}_{\mathrm{Sp}}(Z_q, E^{q+1} \wedge Y) \cong \mathrm{Hom}_{E_*E}(E_*Z_q, E_*(E^{q+1} \wedge Y)) \cong \mathrm{Hom}_{C/E_*E}(C(E_*Z_q), E_*(E^{q+1} \wedge Y)).$$

This is a mapping space of C-algebras in E_*E-comodules. Now, $E_*(E^{\bullet+1} \wedge Y)$ contracts to E_*Y, while $C(E_*Z_\bullet)$ is a resolution of the C-algebra E_*X, so that by a similar argument as before, the associated cosimplicial object has cohomotopy concentrated in π^0, and equal to

$$\pi^0 \pi_0(\mathrm{Maps}_{\mathcal{O}}(P_\bullet, E^{\bullet+1} \wedge Y)) \cong \mathrm{Maps}_{C/E_*E}(E_*X, E_*Y).$$

In higher degrees, we likewise get

$$\pi_t(\mathrm{Maps}_{\mathrm{Sp}}(Z_q, E^{q+1} \wedge Y), \varphi) \cong \mathrm{Maps}_{C/E_*E}(C(E_*Z_q), E_*((E^{q+1} \wedge Y)^{S^t}))_{/E_*\varphi}.$$

Now, $E_*(M^{S^t})$, just as a comodule algebra, is a square-zero extension of E_*M by $\Omega^t E_*M$ (Ω^t being a shift in comodules). We've fixed where our maps are supposed to go in E_*M – they're all $E_*\varphi$ – so we really only care about maps into the degree t part. These are a sort of derivations of C-algebras:

$$\mathrm{Der}_{C/E_*E}(C(E_*Z_q), \Omega^t E_*(E^{q+1} \wedge Y))_{/E_*\varphi}.$$

Again, $C(E_*Z_q)$ is a resolution of E_*X by a cofibrant simplicial C-algebra, and $E_*(E^{q+1} \wedge Y)$ contracts to E_*Y. So the cohomology of this complex calculates left derived functors of derivations of C-algebras,

$$\mathbb{L}^* \mathrm{Der}_{C/E_*E}(E_*X, \Omega^t E_*Y) = D^*_{C/E_*E}(E_*X, \Omega^t E_*Y).$$

Thus, we have obtained the following theorem.

Theorem 14.3. *Given an $\mathcal{O}$-algebra map $\varphi : X \to Y$, here is a second quadrant spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}(\mathrm{Maps}_{\mathcal{O}}(X, Y_E^\wedge), \varphi)$$

with

$$E_2^{0,0} = \mathrm{Maps}_{C/E_*E}(E_*X, E_*Y),$$
$$E_2^{s,t} = D^s_{C/E_*E}(E_*X, \Omega^t E_*Y)_{/E_*\varphi} \text{ for } s > 0 \text{ and } t \geq 0,$$

and $E_2^{s,t} = 0$ otherwise.

Bousfield's definition of this sort of spectral sequence – computing the homotopy groups of the totalization of a cosimplicial space from the homotopy groups of the levels – actually gives us slightly more. Bousfield was also interested in the question of whether the totalization is nonempty, given that the levels are. Unlike the analogous question for a simplicial space, this question is nontrivial because a totalization involves a limit. Given a point x in the 0th level, C^0, of the cosimplicial space $C^\bullet$, we first need to know that it's homotopic to something that goes to the same place in C^1 under the coface maps d^0 and d^1, which is the same as asking that it lives in π^0. We next need to know that, possibly after moving x again, we can get the coface maps to C^2 to agree, which turns out to be asking something about $\pi^2\pi_1 C$. The general result, as applied in this case, is:

Theorem 14.4. *Let X and Y be $\mathcal{O}$-algebras, and let $f : E_*X \to E_*Y$ be a C-algebra map. Then there are successively defined obstructions to realizing f as E_* of a map of $\mathcal{O}$-algebras living in*

$$D^{t+1}_{C/E_*E}(E_*X, \Omega^t E_*Y),\ s \geq 1,$$

and obstructions to the uniqueness, up to homotopy, of this lift, living in

$$D^{t}_{C/E_*E}(E_*X, \Omega^t E_*Y),\ s \geq 1.$$

Note that, although the obstruction groups only use the C-algebra structure on E_*X and E_*Y, the cosimplicial object they're coming from is defined in terms of $\mathcal{O}$-algebra structures on X and Y. Thus, you can't use this obstruction theory to construct $\mathcal{O}$-algebra structures on spectra. There's also a method that does this, which I'll discuss in section 4; for now, let me mention that it gives very similar obstruction groups: the obstructions to realizing a C-algebra in E_*E-comodules live in

$$D^{t+\varepsilon}_{C/E_*E}(A_*, \Omega^t A_*),\ t \geq 1,$$

where ε is now 2 for existence and 1 for uniqueness.

Here's a variation. If Y is an E-module, then the cosimplicial resolution $E^{\bullet+1} \wedge Y$ already retracts to Y as a spectrum; we can look at maps in E_*-modules instead of E_*E-comodules; and we don't need to resolve Y like this to apply the universal coefficient theorem. So we can get rid of the resolution of Y from the argument entirely, and in particular, we don't have to assume that E is an $\mathcal{O}$-algebra! Thus, in this case, we get

Theorem 14.5. *Given an $\mathcal{O}$-algebra map $\varphi : X \to Y$, there is a second quadrant spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}(\mathrm{Maps}_{\mathcal{O}}(X, Y), \varphi)$$

with

$$E_2^{0,0} = \mathrm{Maps}_{C/E_*}(E_*X, Y_*),$$
$$E_2^{s,t} = D^s_{C/E_*}(E_*X, \Omega^t Y_*)_{/E_*\varphi} \text{ for } s > 0 \text{ and } t \geq 0,$$

and $E_2^{s,t} = 0$ otherwise.

Given $\mathcal{O}$-algebras X and Y and a C-algebra map $f : E_*X \to Y_*$, there are successively defined obstructions to realizing f as E_* of a map of $\mathcal{O}$-algebras living in

$$D^{t+1}_{C/E_*}(E_*X, \Omega^t E_*Y),\ t \geq 1,$$

and obstructions to the uniqueness, up to homotopy, of this lift, living in

$$D^{t}_{C/E_*E}(E_*X, \Omega^s E_*Y),\ t \geq 1.$$

14.2. Resolving the operad. In practice, the Dyer-Lashof operations monad C may be hard to find, it may not even exist, and when it does exist, it may be hard to compute André-Quillen cohomology over it. A key example is in the construction of an E_∞ structure on Morava E-theory: Goerss and Hopkins wanted to use E-theory as the homology theory and the X and Y, and couldn't assume that it had E_∞ Dyer-Lashof operations, much less that they were computable. Their solution was to resolve the operad by a simplicial operad $T_\bullet \to \mathcal{O}$, and simultaneously resolve X by a simplicial algebra over this simplicial operad, that still had the same nice properties with respect to E.

The important technical fact, discussed in more detail by Dominic, is that there exists a resolution $T_\bullet \to \mathcal{O}$ in the category $s\mathsf{Op}(\mathrm{Sp})$ of simplicial operads in spectra, such that

- $T_\bullet$ is Reedy cofibrant as a simplicial operad,
- $|T_\bullet| \to \mathcal{O}$ is a weak equivalence in Op,
- for each n and $q \geq 0$, $\pi_0 T_n(q)$ is a free Σ_q-set,
- $E_* T_\bullet$ is Reedy cofibrant as a simplicial operad in E_*-modules,
- and $|E_* T_\bullet| \to E_*\mathcal{O}$ is a weak equivalence in $\mathsf{Op}(E_* \,\mathrm{Mod})$.

Both X and Y are simplicial T-algebras concentrated in simplicial degree zero. Thus, as a second important technical fact, there exists a resolution $P_T(X)_\bullet \to X$ such that

- $P_T(X)_\bullet$ is cofibrant in the E_2 resolution model category structure on $s\,\mathrm{Alg}_T$,
- for each n, $P_T(X)_n = T(Z_n)$, where Z_n has a Künneth isomorphism
$$[Z_n, M] \cong \mathrm{Hom}_{E_* \,\mathrm{Mod}}(E_* Z_n, M_*)$$
for any E-module M,
- and the underlying degeneracy diagram of $P_T(X)_\bullet$ is T of a free degeneracy diagram in spectra.

We now have the following theorem:

Theorem 14.6. *If E is an $\mathcal{O}$-algebra and $\varphi : X \to Y$ an $\mathcal{O}$-algebra map, there is a spectral sequence*
$$E_2^{s,t} \Rightarrow \pi_{t-s} \,\mathrm{Tot}\, \mathrm{Maps}_{T_\bullet}(P_T(X)_\bullet, E^{\bullet+1} \wedge Y) \cong \pi_{t-s} \,\mathrm{Maps}_{\mathcal{O}}(X, Y_E^\wedge),$$
with
$$E_2^{0,0} = \mathrm{Maps}_{E_*\mathcal{O}/E_*E}(E_*X, E_*Y),$$
$$E_2^{s,t} = D^s_{E_*T/E_*E}(E_*X, \Omega^t E_*Y) \text{ for } s > 0 \text{ and } t \geq 0$$
and $E_2^{s,t} = 0$ otherwise. If $f : E_*X \to E_*Y$ is an $E_*\mathcal{O}$-algebra map, there are successively defined obstructions to the existence and uniqueness of a lift of f to an $\mathcal{O}$-algebra map $X \to Y$, living in
$$D^{t+\varepsilon}_{E_*T/E_*E}(E_*X, \Omega^t E_*Y),\ t \geq 1,$$
with ε is 1 for existence and 0 for uniqueness. There are similar simplifications when Y is an E-module and E isn't necessarily an $\mathcal{O}$-algebra.

14.3. Examples.

Example 14.7. Any homology is adapted to the $\mathcal{A}_\infty$ operad: in fact, $E_*\mathcal{A}_\infty$ is the associative operad in E_*E-comodules. Thus, the groups in the spectral sequence take the form

$$D^s_{\mathsf{Ass}/E_*E}(E_*X, \Omega^t E_*Y).$$

If Y is an E-module and E_* is a commutative ring, this reduces to

$$D^s_{\mathsf{Ass}/E_*}(E_*X, \Omega^t Y_*) = HH^{s+1}_{E_*}(E_*X, \Omega^t Y_*).$$

In particular, the obstruction groups for realizing E_*X as the E-homology of an $\mathcal{A}_\infty$-algebra are

$$D^{t+\varepsilon}_{\mathsf{Ass}/E_*}(E_*X, \Omega^t X_*) = HH^{t+\varepsilon+1}_{E_*}(E_*X, \Omega^t X_*),$$

where ε is 2 for existence and 1 for uniqueness. Taking $E = X$, we get the same obstruction *groups* as in Robinson's theory. I have no idea if the obstruction classes are the same or not; it would be a little strange if they were, as these obstruction classes don't seem to be saying anything about $\mathcal{A}_n$ structures.

Example 14.8. As I mentioned, p-complete K-theory is adapted to the E_∞-operad, with the monad C being the free graded θ-algebra functor. Briefly, a **θ-algebra** has a pth power Adams operation ψ^p satisfying the Frobenius congruence $\psi^p(x) \cong x^p \pmod{p}$ and a witness θ to the congruence (so that $\psi^p(x) - x^p = p\theta(x)$), such that θ satisfies the relations required to make ψ^p a ring homomorphism in the universal case. The free θ-algebra is fairly simple – for example, the free θ-algebra on a set of generators $\{x_i\}$ is a polynomial ring generated by $\{\theta^j x_i\}$ – and so one can actually compute obstruction groups in this case. One prominent example is the assembly of the $K(1)$- and $K(2)$-local parts of TMF – see [**Beh14**].

Example 14.9. Mod p homology $H\mathbb{F}_p$ is also adapted to the E_∞-operad, with the monad C being the free unstable algebra over the Dyer-Lashof algebra. The obstruction groups can be computed as Ext of unstable *modules* over the Dyer-Lashof algebra from a Dyer-Lashof 'cotangent complex'.

Example 14.10. If $\mathcal{O}$ is the *trivial* operad, an $\mathcal{O}$-algebra is just a *spectrum*, and a module over an $\mathcal{O}$-algebra is just *another spectrum*. Any homology theory E is adapted to $\mathcal{O}$. The groups in the mapping space spectral sequence take the form

$$D^s_{\mathrm{triv}/E_*E}(E_*X, \Omega^t E_*Y) = \mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*Y).$$

Of course, this is just the E-based Adams spectral sequence. Notice that we can interpret the obstruction classes to realizing a map, in

$$\mathrm{Ext}^{s+1,s}_{E_*E}(E_*X, E_*Y),$$

as the classes in the -1 column of the Adams spectral sequence receiving differentials from the given class in bidegree $(0, 0)$.

Example 14.11. The Goerss-Hopkins project took some of its inspiration from the paper [**BDG04**] on realizing Π-algebras as homotopy groups of spaces. Here, a Π-algebra is roughly a $\mathbb{Z}_{\geq 0}$-graded set equipped with the algebraic structure accruing to the homotopy groups of a space – a group structure on π_1 and abelian group structures on $\pi_{\geq 2}$, composition and Whitehead products, and so on. The corresponding stable concept is a π_*S-module. The André-Quillen cohomology here is about as hard as you can imagine, but manageable in simple cases, and [**BauF15**] have been able to describe the obstructions to realization when there are only two nonzero homotopy groups. A modern update to [BDG04] is given by [Pst17].

14.4. The moduli space of realizations. If we start with just a C-module E_*X, we can try to lift it to an $\mathcal{O}$-algebra X by climbing a sort of Postnikov tower.

Recall that we had a category of 'projective spectra' $\mathcal{P}$ which was generated by the finite complexes DE_α coming from E. If $X = X_\bullet$ is a simplicial spectrum, it has two kinds of homotopy groups in the resolution model category:

$$\pi_i(X;P) = \pi_i([n] \mapsto [P, X_n]),$$

the homotopy groups of a simplicial abelian group defined by mapping $P \in \mathcal{P}$ in levelwise, and

$$\pi_n^\natural(X;P) = \pi_n \operatorname{Maps}_{\mathcal{P}-\mathsf{Res}}(P,X) = [P \wedge \Delta^n/\partial\Delta^n, X],$$

a group of homotopy classes of maps, in the resolution model category, between simplicial spectra. These fit into a **spiral exact sequence**:

$$\cdots \to \pi_{n-1}^\natural(X;\Sigma P) \to \pi_n^\natural(X;P) \to \pi_n(X;P) \to \pi_{n-2}^\natural(X;\Sigma P) \to \cdots$$

Either the natural or the levelwise homotopy groups (with coefficients in all generators P) detect $\mathcal{P}$-equivalences. If X is a simplicial spectrum, then E_*X is a simplicial E_*E-comodule, and

$$\pi_p E_q X \cong \operatorname{colim} \pi_p(E_\alpha)_q X \cong \operatorname{colim} \pi_p[\Sigma^q DE_\alpha, X] = \operatorname{colim} \pi_p(X; \Sigma^q DE_\alpha).$$

Likewise, we can define

$$\pi_n^\natural E_* X = \operatorname{colim}_\alpha \pi_n^\natural(X; DE_\alpha).$$

Finally, note that $\pi_0 = \pi_0^\natural$.

I'll now start describing the moduli space of realizations in terms of approximations to realizations in the E_2-model structure. I'll be in the setting of the previous section, when E is adapted to a simplicial operad $T_\bullet$; if this gets too confusing, you can think about the case of the first section, when E was adapted to an ordinary operad and we considered simplicial algebras over that.

Definition 14.12. Let A be a (discrete) $E_*\mathcal{O}$-algebra. For $0 \le n \le \infty$, a simplicial $T_\bullet$-algebra X is a **potential n-stage** for A if:

- $\pi_0 E_* X \cong A$ as an $E_*\mathcal{O}$-algebra,
- $\pi_k E_* X = 0$ for $1 \le k \le n+1$,
- $\pi_k^\natural(X;P) = 0$ for $k > n$ and all $P \in P$.

There is a moduli space $\mathcal{TM}_n(A)$ of potential n-stages, the nerve of the category of potential n-stages and simplicial E_*-equivalences.

If X is a potential n-stage for A, then the spiral exact sequence breaks up into isomorphisms $\pi_{k-1}^\natural E_*\Omega X \cong \pi_k^\natural E_* X$ for $k \le n$, giving $\pi_k^\natural E_* X \cong \Omega^k A$ for these k. Likewise, $\pi_{n+2} E_* X \cong \pi_n^\natural E_* \Omega X \cong \Omega^{n+1} A$. All other natural and levelwise homotopy groups are zero. There's a spectral sequence

$$\pi_p E_q X \Rightarrow E_{p+q}|X|,$$

and one can show that the only possible differential here is an isomorphism, giving $|X| \simeq 0$.

On the other hand, a potential ∞-stage has $E_*X \simeq A$ as simplicial $T_\bullet$-algebras, concentrated in simplicial degree zero. So $|X|$ is an $\mathcal{O}$-algebra and the spectral sequence computing $E_*|X|$ from E_*X_n collapses to give $E_*|X| \cong A$. That is, X is a realization of A. In fact, we have:

Proposition 14.13. *The moduli space of realizations of A (and E_*-equivalences) is equivalent to*

$$\mathcal{TM}_\infty(A) \simeq \operatorname{holim}_{n<\infty} \mathcal{TM}_n(A).$$

Now let's think about the potential 0-stages.

Definition 14.14. Write B_A for a simplicial $T_\bullet$-algebra with the properties that

- $\pi_0 E_* B_A \cong A$ as an $E_*T_\bullet$-algebra, and
- for all simplicial $T_\bullet$-algebras Y, the natural map
$$[Y, B_A] \to \operatorname{Hom}_{E_*\mathcal{O}}(\pi_0 E_* Y, A),$$
from homotopy classes of maps in the resolution model category to maps of $E_*\mathcal{O}$-algebras, is an isomorphism.

All such objects are E_*-equivalent, and we have $\pi_0^\natural(B_A; P) = \operatorname{Hom}_{E_*E}(E_*P, A)$, $\pi_0(E_*B_A) = A$, $\pi_2(E_*B_A) = \Omega A$, and all other homotopy groups are zero. In particular, B_A is a potential 0-stage for A. Conversely, let X be a potential 0-stage for A. Then the isomorphism $\pi_0 E_* X \xrightarrow{\cong} A$ defines a map $X \to B_A$ that is an isomorphism on natural homotopy groups, and thus an E_*-equivalence. Thus, we have

Proposition 14.15.

$$\mathcal{TM}_0(A) \simeq B\operatorname{Aut}(B_A) \simeq B\operatorname{Aut}(A).$$

Why do these even exist, though? Let's go further and prove that Eilenberg-Mac Lane objects exist in all degrees. First we have to define them.

Definition 14.16. Let M be an A-module. Write $B_A(M, n)$ for a simplicial $T_\bullet$-algebra, equipped with a map from B_A, with the properties that

- $\pi_k E_* B_A \to \pi_k E_* B_A(M, n)$ is an isomorphism for $k < n$,
- $\pi_n E_* B_A(M, n) \cong M$ as an A-module,
- $\pi_k^\natural(B_A(M, n); P) = 0$ if $k > n$.

Okay, so why do these exist? It's easy to get something with the right π_0 by using generators and relations. You then attach cells – free $T_\bullet$-algebras on projectives in each simplicial degree between 1 and n – to get the right π_1 through π_n. Finally, you apply the nth Postnikov truncation (in the 'natural' direction).

One again finds, using the spiral exact sequence, that $\pi_* E_* B_A(M, n)$ is generated over $\pi_* E_* B_A$ by an M in degree n and a ΩM in degree $n+2$. However, $\pi_* E_* B_A$ has an A in degree 0 and a ΩA in degree 2. We'd like something that looks like a square-zero extension of A in degree 0 and an M in degree n. This we can do algebraically by forming the pushout in $E_*T_\bullet$-algebras of

$$E_*B_A(M,n) \leftarrow E_*B_A \to A,$$

which kills the ΩA factors, and taking the $(n+1)$th Postnikov truncation, which kills the ΩM. Call this $E_*T_\bullet$-algebra $K_A(M,n)$. Again, one shows that

Proposition 14.17. *Let X be a simplicial $T_\bullet$-algebra equipped with a map $X \to B_A$ (which is equivalent to a map $E_*X \to E_*B_A$ of $E_*T_\bullet$-algebras). Then the map $E_*B_A(M,n) \to K_A(M,n)$ induces*

$$\mathrm{Maps}_{T_\bullet/B_A}(X, B_A(M,n)) \xrightarrow{\cong} \mathrm{Maps}_{E_*T_\bullet/A}(E_*X, K_A(M,n)),$$

where the source mapping space is, as usual, taken in the resolution model category.

Note that

$$\pi_0 \,\mathrm{Maps}_{E_*T_\bullet/A}(E_*X, K_A(M,n)) = D^n_{E_*T_\bullet/A}(E_*X, M),$$

and since $\Omega_{E_*T_\bullet/A}(K_A(M,n)) = K_A(M,n-1)$,

$$\pi_k \,\mathrm{Maps}_{E_*T_\bullet/A}(E_*X, K_A(M,n)) = D^{n-k}_{E_*T_\bullet/A}(E_*X, M).$$

So, given a potential n-stage X for A, we get a map

$$E_*X \to K_A(\Omega^{n+1}A, n+2)$$

over A by obstruction theory (these are precisely the first two homotopy groups of E_*X). By the proposition, this lifts to

$$X \to B_A(\Omega^{n+1}A, n+2)$$

over B_A.

Meanwhile, the $(n-1)$th Postnikov section of X is a potential $(n-1)$-stage: $E_*P_{n-1}X$ has the right natural homotopy groups, and its levelwise homotopy groups are A in degree 0 and $\Omega^n A$ in degree $n+1$ by the spiral exact sequence. Thus, we get an equivalence

$$E_*P_{n-1}X \simeq K_A(\Omega^{n+1}A, n),$$

which corresponds to a map

$$P_{n-1}X \to B_A(\Omega^{n+1}A, n)$$

inducing an isomorphism on $\pi_{n+1}E_*$. This fits into a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & B_A \\ \big\downarrow & \lrcorner & \big\downarrow \\ P_{n-1}X & \longrightarrow & B_A(\Omega^n A, n+1) \end{array}$$

in simplicial $T_\bullet$-algebras over B_A.

Conversely, given such a square

$$\begin{array}{ccc} X & \longrightarrow & B_A \\ \big\downarrow & \lrcorner & \big\downarrow \\ Z & \longrightarrow & B_A(\Omega^n A, n+1). \end{array}$$

in which the bottom left corner Z is a potential $(n-1)$-stage, then X is a potential n-stage if and only if the bottom map is an equivalence on $\pi_{n+1}E_*$, or in other words if it induces an equivalence $E_*Z \to K_A(\Omega^n A, n+1)$. But the two nontrivial homotopy groups of E_*Z give a homotopy pullback square

$$\begin{array}{ccc} E_*Z & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \xrightarrow{\theta} & K_A(\Omega^n A, n+2). \end{array}$$

Thus $E_*Z \to K_A(\Omega^n A, n+1)$ is an equivalence if and only if this diagram is a loop space diagram, i. e., if and only if θ is nullhomotopic. Thus, we get an obstruction to lifting a potential $(n-1)$-stage to a potential n-stage living in

$$\pi_0 \operatorname{Maps}_{E_*T_\bullet/A}(A, K_A(\Omega^n A, n+2)) \cong D^{n+2}_{E_*T_\bullet/A}(A, \Omega^n A).$$

Let $\mathcal{H}^{n+2}(A, \Omega^n A)$ be the *space* $\operatorname{Maps}_{E_*T_\bullet/A}(A, K_A(\Omega^n A, n+2))$, and let $\widehat{\mathcal{H}}^{n+2}(A, \Omega^n A)$ the homotopy quotient by the action of $\operatorname{Aut}(A, \Omega^n A)$. Since θ was only determined up to this action, we get a map

$$\mathcal{TM}_{n-1}(A) \to \widehat{\mathcal{H}}^{n+2}(A, \Omega^n A)$$

and a homotopy pullback square

$$\begin{array}{ccc} \mathcal{TM}_n(A) & \longrightarrow & B\operatorname{Aut}(A, \Omega^n A) \\ \downarrow & & \downarrow \\ \mathcal{TM}_{n-1}(A) & \longrightarrow & \widehat{\mathcal{H}}^{n+2}(A, \Omega^n A). \end{array}$$

This is the desired decomposition of the moduli space. Note that obstructions to uniqueness of the lift from a potential $(n-1)$-stage Z to a potential n-stage X live in π_0 of the fiber of the vertical maps, which is $D^{n+1}_{E_*T_\bullet/A}(A, \Omega^n A)$.

Theorem 14.18. *If A is an $E_*\mathcal{O}$-algebra, there are successively defined obstructions to realizing A as E_*X, for X an $\mathcal{O}$-algebra, living in*

$$D^{t+\varepsilon}_{E_*T_\bullet/E_*E}(A, \Omega^t A),$$

where ε is 2 for existence and 1 for uniqueness.

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Talk 15: Applications of Goerss-Hopkins obstruction theory (Haldun Özgür Bayindir)

I'll start with Hopkins-Miller and Goerss-Hopkins, and then I'll say a little about the result of Tyler Lawson and Niko Naumann that $BP\langle 2\rangle$ for $p=2$ is E_∞.

The Hopkins-Miller theorem constructs a (contravariant) functor from the category of formal group laws (of finite height over a perfect field) to A_∞-ring spectra. (Hopkins and Miller never wrote up their result, so the main reference for this is Rezk's notes.) Goerss-Hopkins constructed a functor from formal group laws to E_∞-ring spectra.

15.1. Review of formal group laws. Let E be a complex-oriented spectrum. Let L_1, L_2 be line bundles over a space X. Given a line bundle L there is a characteristic class $c_E(L)\in E^2X$. In ordinary homology, $c_{H\mathbb{Z}}(L_1\otimes L_2) = c_{H\mathbb{Z}}(L_1) + c_{H\mathbb{Z}}(L_2)$. But in general, $c_E(L_1\otimes L_2) = F(c_E(L_1), c_E(L_2))$ where F is a group law over E_*. What is this?

Definition 15.1. $F(x,y)\in R[[x,y]]$ is a formal group law over R if:

(1) (unit) $F(x,0)=x=F(0,x)$

(2) (commutative) $F(x,y)=F(y,x)$

(3) (associative) $F(F(x,y),z)=F(x,F(y,z))$

Example 15.2. The formal group law of $H\mathbb{Z}$ is $F(x,y)=x+y$. The formal group law of KU is the *multiplicative formal group law* $F(x,y)=x+y-xy$.

A morphism $f: F\to G$ of formal group laws is $f(x) = R[[x]]$ such that $f(0)=0$ and $f(F(x,y)) = G(f(x), f(y))$.

Let FG be the category whose objects are pairs (k,Γ) where k is a perfect field and Γ is a finite-height formal group law over k, and a morphism $(k_1,\Gamma_1)\to(k_2,\Gamma_2)$ is a map $i: k_1\to k_2$ together with $f:\Gamma_1\to i^*\Gamma_2$.

The Hopkins-Miller theorem produces a functor $FG^{op}\to A_\infty$-ring spectra sending (k,Γ) to $E_{k,\Gamma}$, where $E_{k,\Gamma}$ is complex oriented. It comes from the Landweber exact functor theorem.

It was already known (using the Landweber exact functor theorem) that given a formal group law satisfying certain properties, you can produce a spectrum; the advance here is the extra structure, which is gotten using obstruction theory. You also have to show that morphisms of formal groups go to A_∞ maps.

Corollary 15.3. *The Morava stabilizer group acts on Morava E-theory E_n corresponding to the Honda p-typical formal group law as an A_∞ spectrum.*

Using this, you can construct higher real K-theories constructed as homotopy fixed point spectra E_n^{hG}, where G is a finite subgroup of the Morava stabilizer group (often taken to be the maximal finite subgroup). One outcome is the following: the functor, which is not fully faithful, is sort of fully faithful in the following sense: there is a weak equivalence

$$FG((k_1,\Gamma_1),(k_2,\Gamma_2)) \simeq \mathrm{Map}_{A_\infty}(E_{k_2,\Gamma_2}, E_{k_1,\Gamma_1})$$

which can be transferred to E_∞-ring spectra. Here the LHS has the discrete topology. To prove this weak equivalence, we need to show that the RHS has contractible connected components which are in bijective correspondence with the points on the LHS.

15.2. Applying Goerss-Hopkins obstruction theory. Let $E = E_{k_1,\Gamma_1}$ and $F = E_{k_2,\Gamma_2}$. We've seen that obstructions to A_∞-structures lie in the André-Quillen cohomology groups

$$\mathrm{Der}^{n+2}_{E_*}(E_*E, E_{*+n})$$

for $n \geq 1$. Obstructions to uniqueness lie in $\mathrm{Der}^{n+1}_{E_n}(E_nE, E_{*+n})$ for $n \geq 1$. Here Der^n denotes André-Quillen cohomology for associative rings. (You have to resolve your ring with free associative rings.) Recall

$$\mathrm{Der}^s(R,M) \cong HH^{s+1}(R,M)$$

for $s > 0$. (I.e. these obstruction groups coincide with Robinson's obstruction groups.)

The spectral sequence in Goerss-Hopkins obstruction theory specializes to the following in our case:

$$E_2^{s,t} = \begin{cases} \mathrm{Hom}_{E_*}(E_*F, E_*) & s=t=0 \\ \mathrm{Der}^s_{E_*}(E_*F, E_{*+t}) & t-s \geq -1, t>0 \end{cases} \qquad \Longrightarrow \qquad \pi_{t-s}\,\mathrm{Map}_{A_\infty}(F,E,\varphi)$$

where $\mathrm{Maps}_{A_\infty}(F,E,\varphi)$ denotes that space of A_∞ maps is based at $\varphi: F \to E$. Obstructions to lifting a map in $E_2^{0,0}$ to a map of E_∞-ring spectra lie on the line $s-t=1$ for $s \geq 2$. Obstructions to homotopy uniqueness of the lift lie on the line $s=t$ for $s \geq 1$.

We're going to show that all the relevant terms are zero, so any morphism you pick in $E_2^{0,0}$ lifts. We need to show $E_2^{s,t} = 0$ for $(s,t) \neq (0,0)$. Since $E_2^{0,0} \cong FG((k_1,\Gamma_1),(k_2,\Gamma_2))$, this implies the weak equivalence of spaces

$$FG((k_1,\Gamma_1),(k_2,\Gamma_2)) \simeq \mathrm{Map}_{A_\infty}(E_{k_2,\Gamma_2}, E_{k_1,\Gamma_1}).$$

By an argument very similar to the last proof in Talk 6 (using flat base change and SES's), we can reduce to showing the following:

Goal 15.4. $\mathrm{Der}^s_{E_0/\mathfrak{m}}(E_0F/\mathfrak{m}, E_t/\mathfrak{m}) = 0$ where $\mathfrak{m}$ is the maximal ideal in E_0 and Der^* is AQ cohomology for associative rings.

15.3. A_∞ case. In the previous talks, we've used the fact that étaleness implies the obstructions vanish, but this fact has been black-boxed. I'm going to show you how this works in this case. Let σ be the Frobenius endomorphism $E_0F/\mathfrak{m} \to E_0F/\mathfrak{m}$.[6] The fact we're going to use is that σ is an automorphism of $E_0F/\mathfrak{m}$. (See §21.4 in Rezk's notes; this is one place where you use the assumption that the fields k_1 and k_2 are perfect.)

[6]Note that we're skipping over something. We need Frobenius to be a $E_0/\mathfrak{m}$-linear map but it's not, so you actually use something called the relative Frobenius.

Step 1: Show that commutative AQ-homology is trivial in all degrees. Compute commutative AQ-homology by resolving your ring with a cofibrant simplicial resolution $Q_\bullet \to E_0F/\mathfrak{m}$ in simplicial commutative $E_0/\mathfrak{m}$-algebras. Then compute the cotangent complex

$$L^{\text{comm}}_{E_0F/\mathfrak{m}\big/E_0/\mathfrak{m}} = \Omega_{Q_\bullet/E_0/\mathfrak{m}} \otimes_{Q_\bullet} E_0F/\mathfrak{m}.$$

(This is just classical AQ-homology.) What does σ do to differentials? We have

$$d\sigma(x) = dx^p = px^{p-1}dx$$

and that's zero because we're working in characteristic p. So σ induces both an isomorphism (see above) and the zero map. This shows that the cotangent complex in the commutative case is contractible. Unfortunately, this doesn't show that our E_∞ obstruction groups vanish, because there is a difference between AQ cohomology for commutative algebras and derivations with respect to an E_∞ operad, which is the kind of derivations we need to vanish. For the A_∞ case, we will use a spectral sequence that compares this to associative algebra AQ homology (in step 2); for the E_∞ case, we'll have to use two spectral sequences to get from vanishing of commutative AQ cohomology to vanishing of the right operad derivations.

Step 2: I claim there is a spectral sequence

$$H_s(\Lambda^{t+1}_{E_0F/\mathfrak{m}} L^{\text{comm}}_{E_0F/\mathfrak{m}\big/E_0/\mathfrak{m}}) \implies H_{s-1} L^{\text{assoc}}_{E_0F/\mathfrak{m}\big/E_t/\mathfrak{m}}$$

(where Λ means exterior powers) that comes from applying $P \mapsto \Omega_{P\big/E_0/\mathfrak{m}} \otimes_P E_0F/\mathfrak{m}$ to a bisimplicial resolution

$$\begin{array}{ccccc} \cdots & P_{2,\bullet} & \longrightarrow & P_{1,\bullet} & \\ & \downarrow & & \downarrow & \\ \cdots & Q_2 & \rightrightarrows & Q_1 & \longrightarrow E_0F/\mathfrak{m} \end{array}$$

where $Q_\bullet \to E_0F/\mathfrak{m}$ is a resolution of commutative algebras, and $P_{p,\bullet} \to Q_p$ is a resolution of associative algebras. We showed L^{comm} was contractible, and the RHS is AQ homology for associative rings therefore L^{assoc} is also contractible. So

$$\text{Der}^s_{E_0/\mathfrak{m}}(E_0F/\mathfrak{m}, E_t/\mathfrak{m}) = H^s\,\text{Der}_{E_0/\mathfrak{m}}(L^{\text{assoc}}_{E_0F/\mathfrak{m}\big/E_0/\mathfrak{m}}, E_t/\mathfrak{m}) = 0.$$

15.4. E_∞ case. At this point, we're done with the A_∞ version of our goal. But we also want to do E_∞ obstruction theory. This requires showing the vanishing of the module of derivations associated to an E_∞ operad. More precisely, suppose $\mathcal{E}$ is a simplicial E_∞-operad in k-modules and S is an étale ring. It is sufficient to show $D^*_{\mathcal{E}}(S, M) = 0$. *I'm confused what S and M actually are in this case... (Relevant sections of the paper are 6.8, 7.4, 7.5, and 7.6.) –Eva*

Start with a cofibrant resolution $A \to S$ in $\mathcal{E}$-algebras. Then π_*A has Dyer-Lashof operations. To see this, one uses the normalization functor from simplicial k-modules to chain complexes over k. Our simplicial operad is a simplicial operad in k-modules, but if you fix an operadic degree, you get a simplicial module. You can see $\mathcal{E}$ as an operad in simplicial k-modules. If you do that and apply the normalization functor at each operadic degree, then $N\mathcal{E}$ is an E_∞-operad in chain complexes. Because of this, we obtain that NA is an E_∞-dg algebra (an algebra over this operad in chain complexes). Over a field of characteristic p, you end up having Dyer-Lashof operations in homology.

Let $\mathcal{R}$-alg be the category of commutative rings with Dyer-Lashof operations. We want to calculate $\mathrm{Der}^s_{\mathcal{R}}(\pi_* A, M)$, the AQ cohomology in $\mathcal{R}$-algebras.

Fact 15.5. *Free $\mathcal{R}$-algebras are also free commutative algebras. This implies that a free resolution in simplicial $\mathcal{R}$-algebras is a free resolution in commutative rings.*

It is not hard to show that AQ-homology for $\mathcal{R}$-algebras and commutative rings are the same in this case. In general, you have a spectral sequence

$$E_2 = \mathrm{Ext}^p_{\mathcal{R}}(D^{\mathcal{R}}_q(\pi_* A), M) \implies \mathrm{Der}^{p+q}_{\mathcal{R}}(\pi_* A, M).$$

We have that $D^{\mathcal{R}}_q(\pi_* A) = 0$, and so the RHS $= 0$. There is another spectral sequence

$$\mathrm{Der}^p_{\mathcal{R}}(\pi_* A, M)^q \implies D^{p+q}_{\mathcal{E}}(A, M)$$

and the LHS being zero implies the RHS being zero. This shows $D^*_{\mathcal{E}}(S, M) = 0$.

15.5. Result of Lawson and Naumann about $BP\langle 2\rangle$. Lawson and Naumann has shown that $BP\langle 2\rangle$ for $p = 2$ is E_∞. Here are the main steps in the proof:

(1) Start with $L_{K(2)}BP\langle 2\rangle$, which they show is the fixed point spectrum of a Lubin-Tate spectrum. We know that Lubin-Tate spectra are E_∞, and therefore so is this.

(2) Consider the map $L_{K(1)}BP\langle 2\rangle \to L_{K(1)}L_{K(2)}BP\langle 2\rangle$. Show that the image of the homotopy groups are invariant under a power operation θ iff $L_{K(1)}BP\langle 2\rangle$ is E_∞.

(3) Show by using rational homotopy theory that $L_{K(0)}BP\langle 2\rangle$ is E_∞.

(4) $BP\langle 2\rangle$ is the connective cover of $L_{K(0)\vee K(1)\vee K(2)}BP\langle 2\rangle$, which is E_∞ by using chromatic pullback squares, and therefore so is $BP\langle 2\rangle$.

Part V: Comparison of obstruction theories

Talk 16: Comparison results I (Arpon Raskit)

There are all these cohomology theories floating around; everyone is asking themselves whether they are all the same. Our goal is to understand how various cohomology theories are related, and how to organize the mind.

16.1. Abelianization and stabilization. Recall we had:

- classical André-Quillen homology constructed via "derived abelianization":
  $$\mathbb{L}\,\mathrm{Ab} : \mathrm{ho}(s\,\mathrm{CAlg}_{R/A}) \to \mathrm{ho}(\mathrm{Ab}(-)) \simeq \mathrm{ho}(s\,\mathrm{Mod}_A)$$
  Here R and A are ordinary discrete rings.
- Topological André-Quillen homology, constructed in Yu's talk in terms of stabilization. We saw that $\mathbb{L}Q = \mathbb{L}\Omega^\infty\Sigma^\infty : \mathrm{ho}(E_\infty\,\mathrm{Alg}^{nu}_A) \to \mathrm{ho}(\mathrm{Mod}_A)$ where Q was the indecomposables functor (I/I^2) and $\Omega^\infty\Sigma^\infty = \operatorname{colim}\Omega^n\Sigma^n$.

Definition 16.1. Let C be a nice setting for homotopy theory (e.g. a simplicial model category) and suppose C is pointed (i.e. has a zero object). (For example, you might think of pointed spaces or pointed simplicial sets, but also $s\,\mathrm{CAlg}_{A/A}$ or $E_\infty\,\mathrm{Alg}_{A/A}$.)

In such a setting there is a notion of suspension $\Sigma : C \to C$ and loops $\Omega : C \to C$. E.g., a pointed simplicial model category is tensored and cotensored over based spaces, so we can think of "$\Sigma X = S^1 \wedge X$" and "$\Omega X = F_*(S^1, X)$".

Given these, we can talk about spectra in C: there's another homotopy theory, which I'll denote $\mathrm{Sp}(C)$, whose objects are sequences $\{X_n\}$ in C equipped with maps $\Sigma X_n \to X_{n+1}$. You can set up an adjunction $\Sigma^\infty : C \rightleftarrows \mathrm{Sp}(C) : \Omega^\infty$. It all works basically the way it does in spaces. This generality was considered by Schwede, Basterra-Mandell, Rezk (complete Segal spaces), Lurie, ...

There's also a notion of abelian group objects in C, and these assemble into another homotopy theory $\mathrm{Ab}(C)$. There's an adjunction $\mathrm{Ab} : C \rightleftarrows \mathrm{Ab}(C) : U$. You should think of these things as two forms of linearizing your objects in C; one form is taking the free abelian group on something (linearizing over the integers), and the other form is stabilization (linearizing over the sphere spectrum).

Example 16.2. If $C = \mathrm{Spaces}_*$ then you could stabilize by forming the category of spectra $\mathrm{Sp}(C)$, so $\pi_*\Sigma^\infty X = \pi_*^s X$ ("linearized over $\mathbb{S}$"). You could also "abelianize" by forming $\mathrm{Ab}(C)$, the category of abelian group spaces, which is equivalent to the category of non-negatively graded chain complexes of abelian groups. Then $\pi_*\,\mathrm{Ab}(X) = \widetilde{H}_*(X;\mathbb{Z})$. Think of this as "linearization over $\mathbb{Z}$".

Remark 16.3. Suppose C just has a final object. Then you can form C_*, the category of pointed objects $* \to X$. Then we have an adjunction $(\ )_+ : C \overset{*\sqcup -}{\rightleftarrows} C_* : U$. We obtain adjunctions $\Sigma^\infty_+ : C \rightleftarrows \mathrm{Sp}(C_*) : \Omega^\infty$ and $\mathrm{Ab}_+ : C \rightleftarrows \mathrm{Ab}(C_*) : U$. (We don't normally notate this $\mathrm{Ab}_+ \dots$)

Let me talk more about how André-Quillen homology fits into this picture. For example, take $C = s\,\mathrm{CAlg}_{R/A}$ where R is a simplicial commutative ring and A is a simplicial commutative R-algebra. This has a final object A, and the coproduct is $\otimes_R$. Then the pointed objects are objects with a map from A, i.e. $C_* \simeq s\,\mathrm{CAlg}_{A/A}$, and $B_+ \simeq A \otimes_R B$. We have an equivalence of categories

$$s\,\mathrm{Mod}_A \underset{I}{\overset{A\oplus -}{\rightleftarrows}} \mathrm{Ab}(s\,\mathrm{CAlg}_{A/A}).$$

André-Quillen homology is $\mathrm{Ab}_+ : s\,\mathrm{CAlg}_{R/A} \to s\,\mathrm{Mod}_A$. In particular, the cotangent complex in this setting, which we'll denote $L^{\mathrm{alg}}_{A/R}$, is defined to be $\mathrm{Ab}_+(A)$.

Now we do the analogous thing for E_∞ algebras. For example, take $C = E_\infty \mathrm{Alg}_{R/A}$, where R is an E_∞ ring spectrum and A is an E_∞ R-algebra. This has final object A, coproduct is $\wedge_R$, and $B_+ \simeq A \wedge_R B$. Pointed objects are $E_\infty \mathrm{Alg}_{A/A}$. In this setting we have stabilization; we have an equivalence, due to Basterra and Mandell:

$$\mathrm{Mod}_A \underset{I}{\overset{A\vee -}{\rightleftarrows}} \mathrm{Sp}(E_\infty \mathrm{Alg}_{A/A}).$$

This setup is what gives rise to TAQ, which is given by stabilization $\Sigma^\infty_+ : E_\infty \mathrm{Alg}_{R/A} \to \mathrm{Mod}_A$. Here the cotangent complex is $L^{\mathrm{top}}_{A/R} := \Sigma^\infty_+ A$.

These do not agree.

Example 16.4. Let k be a field. We get $L^{\mathrm{alg}}_{k[x]/k} \simeq k[x]$ because $k[x]$ is a free object in simplicial commutative algebras. On the other hand, $L^{\mathrm{top}}_{Hk[x]/Hk} \simeq Hk[x] \wedge_{\mathbb{S}} H\mathbb{Z}$. So they're not the same.

So I did abelianization in the simplicial commutative setting and stabilization in the topological setting. What happens when I do stabilization in the simplicial commutative setting?

Here's the setup: let R be a simplicial commutative ring, A a simplicial commutative R-algebra, and we want to understand the stabilization functor $\Sigma^\infty_+ : s\,\mathrm{CAlg}_{R/A} \to \mathrm{Sp}(s\,\mathrm{CAlg}_{R/A})$. Unfortunately the target is *not* Mod_A or $s\,\mathrm{Mod}_A$. The distinction will be discussed in the next talk tomorrow – Schwede knows what this is.

Idea: compare to the E_∞ setting. Any simplicial commutative ring can be viewed as an E_∞-ring space, and any such can be delooped into a connective E_∞ ring spectrum. So we have a functor $\theta : s\,\mathrm{CAlg}_{\mathbb{Z}} \to E_\infty \mathrm{Alg}^{\mathrm{connective}}_{H\mathbb{Z}}$. (This is forgetting the strict commutativity.)

Claim 16.5. *θ preserves homotopy limits and colimits.*

Proof. Both simplicial commutative rings and E_∞-ring spaces are spaces equipped with more structure; homotopy limits are computed on the underlying spaces. Since the underlying spaces aren't changed by this functor, homotopy limits are preserved. The same argument applies for filtered colimits.

For finite colimits, it suffices to check that the initial object is preserved ($\mathbb{Z}$ gets sent to $H\mathbb{Z}$ so that's fine), and that pushouts are preserved (pushouts are relative tensor and smash products, and these are the same thing). □

Consider the following diagram

$$\begin{array}{ccc} s\,\mathrm{CAlg}_{R/A} & \xrightarrow{\theta} & E_\infty\,\mathrm{Alg}_{R/A} \\ \downarrow{\scriptstyle \Sigma^\infty_{\mathrm{alg},+}} & & \downarrow{\scriptstyle \Sigma^\infty_{\mathrm{top},+}} \\ \mathrm{Sp}(s\,\mathrm{CAlg}_{R/A}) & & \mathrm{Sp}(E_\infty\,\mathrm{Alg}_{R/A}) \\ \downarrow{\scriptstyle \Omega^\infty_{\mathrm{alg}}} & & \downarrow{\scriptstyle \Omega^\infty_{\mathrm{top}}} \\ s\,\mathrm{CAlg}_{A/A} & \xrightarrow{\theta} & E_\infty\,\mathrm{Alg}_{A/A} \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\ s\,\mathrm{Mod}_A & \longrightarrow & \mathrm{Mod}_A \end{array} \tag{16.1}$$

I claim this diagram commutes. The top rectangle is, on objects,

$$\begin{array}{ccc} B & \longmapsto & \theta(B) \\ \downarrow & & \downarrow \\ \mathrm{colim}_n\,\Omega^n\Sigma^n B_+ & \longmapsto & \mathrm{colim}_n\,\Omega^n\Sigma^n\theta(B_+) \end{array}$$

I can think of suspension and loops as specific homotopy limits and colimits, so the previous claim implies that this commutes. The bottom square in (16.1) obviously commutes.

Conclusion: Let $B \in s\,\mathrm{CAlg}_{R/A}$. Then

$$I(\Omega^\infty_{\mathrm{alg}}\Sigma^\infty_{\mathrm{alg},+}B) \simeq I(\Omega^\infty_{\mathrm{top}}\Sigma^\infty_{\mathrm{top},+}\theta(B))$$
$$I(\Omega^\infty_{\mathrm{alg}}\Sigma^\infty_{\mathrm{alg},+}A) \simeq L^{\mathrm{top}}_{\theta(A)/\theta(R)}$$

So TAQ and the topological cotangent complex can also be computed as stabilization in the simplicial commutative setting. The moral is that, in general,

$$\mathrm{Ab} \neq \mathrm{Sp}\,.$$

Why are these the same in characteristic 0? Sp and Ab *are* the same in characteristic zero. This boils down to the fact that $H_*(B\Sigma_n;\mathbb{Q}) = 0$.

16.2. Comments by Maria on abelianization vs. stabilization. For abelianization to agree with stabilization we need some sort of equivalence between the category of abelian objects in the category C and the stable category $Sp(C)$. For example, in the category of augmented commutative algebras we may have that $AQ(k) \neq TAQ(Hk)$; more precisely, the category of spectra in $s\,\mathrm{CAlg}_k$ is equivalent to Mod_{Dk} where $Dk = Hk \overset{\mathbb{L}}{\wedge} H\mathbb{Z}$ (see Claim

17.11) whereas the category of abelian objects in $s\,\mathrm{CAlg}_k$ is $s\,\mathrm{Mod}_k$, and these don't agree in general. However, when working rationally they do and so $AQ(R) = TAQ(HR)$ in this case.

For a fixed commutative algebra k and an associative k algebra B, we can consider the category of associative algebras over B. Here the homotopy category of abelian group objects and the homotopy category of connective spectra are both equivalent to the homotopy category of B-bimodules which explains why there isn't an analogous problem with HH vs. THH. As another example, look at the category of augmented commutative S-algebras in your favorite stable category. The category of spectra in this category is Mod_S which agrees to the category of abelian objects.

(What causes these to agree in the associative case and in the case of simplicial commutative rings over a $\mathbb{Q}$-algebra B? For the latter case, see Schwede, "Spectra in model categories" 3.2.2 and 3.2.3. The gist is that this boils down to a connectivity argument involving a free object, which is simpler in these cases. For the rational commutative case, rationality of B causes $A^{\otimes_B n}/\Sigma_n$ to split as a summand of $A^{\otimes_B n}$, allowing you to measure connectivity of $\mathrm{Sym}(A)$ in terms of the connectivity of A. In the associative case, the free associative algebra on A is easy to write down and measure connectivity of.)

"But," you protest, "TAQ was constructed using an exactly analogous procedure to AQ. How did one of them end up being abelianization and the other, stabilization?" One heuristic is that the indecomposables functor Q is doing different things in the case of discrete algebra vs. ring spectra – in the spectra case, Q is quotienting out not just by the multiplication, but also by things like Dyer-Lashof operations.

16.3. Comparison with Γ-homology.

Theorem 16.6 (Basterra-McCarthy)**.** *Let R be an ordinary commutative ring and A an ordinary commutative R-algebra. Then*

$$H\Gamma(A|R;A) \simeq I(\Omega^\infty_{alg}\Sigma^\infty_{alg,+}) \simeq L^{top}_{HA/HR}.$$

In other words, in this ordinary setting, Γ-homology agrees with TAQ.

Proof ingredients. There is a complex $\widetilde{C\Gamma}(B/A)$ (for B an augmented simplicial commutative A-algebra) which computes Γ-homology, in the sense that

$$H\Gamma(A|R;A) \simeq \widetilde{C\Gamma}(A \overset{\mathbb{L}}{\otimes}_R A/A).$$

And for $I = \ker(B \to A)$ there is a map $\varphi : I \to \widetilde{C\Gamma}(B/A)$ with the property:

(∗) if I is n-connected then the map φ is $(2n+1)$-connected.

This connectivity statement is the key ingredient. I'll just show how it implies that Γ-homology is the same as stabilization. We have

$$I(\Omega^\infty\Sigma^\infty(A \overset{\mathbb{L}}{\otimes}_R A)) \simeq I(\operatorname{colim}\Omega^n\Sigma^n(A \overset{\mathbb{L}}{\otimes}_R A)).$$

I commutes with the filtered colimits and Ω^n because these are both computed at the level of modules, so this is $\operatorname{colim}\Omega^n I(\Sigma^n(A\overset{\mathbb{L}}{\otimes}_R A))$. There is a map from here to $\operatorname{colim}\Omega^n\widetilde{C\Gamma}(\Sigma^n(A\overset{\mathbb{L}}{\otimes}_R A))$ and the point is that this is an equivalence: each suspension increases the connectivity, so in the colimit this becomes an equivalence by (∗). One has to show that Γ-homology preserves coproducts in this setting, so there's an equivalence to $\operatorname{colim} \Omega^n \Sigma^n \widetilde{C\Gamma}(A \overset{\mathbb{L}}{\otimes}_R A)$. In simplicial modules, $\Omega^n\Sigma^n$ doesn't do anything – it's just a shift out and back. So this is just $\widetilde{C\Gamma}(A \overset{\mathbb{L}}{\otimes}_R A)$. □

16.4. Goerss-Hopkins. One reason to phrase abelianization and stabilization in such generality is so that you can do it anywhere you want. For example, in Goerss-Hopkins obstruction theory you want to form André-Quillen homology for simplicial E_∞-algebras.

If k is an ordinary commutative ring, the Dold-Kan correspondence says $s\operatorname{Mod}_k \simeq ch_{\geq 0}(\operatorname{Mod}_k)$; there's also "stable Dold-Kan" $\operatorname{Mod}_{Hk} \simeq ch(\operatorname{Mod}_k)$. On the point-set level, these equivalences don't respect the monoidal structure. However, they do respect derived tensor products.

Mandell proved that I can take E_∞-algebras in these different settings, and get the same homotopy theories.

Theorem 16.7 (Mandell)**.** *We have*

$$E_\infty s\operatorname{Alg}_k := E_\infty(s\operatorname{Mod}_k) \simeq E_\infty(ch_{\geq 0}\operatorname{Mod}_k) =: E_\infty dg_{\geq 0}\operatorname{Alg}_k$$

and similarly

$$E_\infty \operatorname{Alg}_{Hk} \simeq E_\infty dg \operatorname{Alg}_k .$$

Forming the cotangent complex in any of these settings amounts to the same thing.

Remark 16.8 (Basterra-Richter)**.** For Goerss-Hopkins obstruction theory applied to putting an E_∞-ring structure on homotopy commutative ring spectrum E (e.g. Lubin-Tate theory) we were faced with (the possibility of) obstructions in $AQ^*_{sE_\infty}(A|k;k)$ where $k = E_*$ and $A = E_*E$. It turns out that this is equivalent to $H\Gamma^*(A|k;k)$. Compare this to TAQ in the usual E_∞-algebra setting, and use the Basterra-Mandell comparison of TAQ and $H\Gamma$. (Issue: there are some (flatness) restrictions.)

Talk 17: Comparison Results II (Aron Heleodoro)

In the first part of the talk, I'll talk about algebraic theories, the general setting used by Schwede. In the second part, I'll talk about comparing stabilization to abelianization in this setting. In the third part, I'll relate that back to Γ-homology.

17.1. Algebraic theories. Following Schwede's paper "Stable homotopy of algebraic theories", we'll be working in the general setting of algebraic theories and T-algebras.

Definition 17.1. Let Γ_* be the opposite of the category of all pointed finite sets. Then a (simplicial) *algebraic theory* is the data of a limit preserving functor $F : \Gamma_* \to T_*$ where the category T_* has the same objects as Γ_* and is enriched over simplicial sets. Note this implies that $F(n^+) \simeq F(1^+)^n$.

A *model* of T (a.k.a. T-*algebra*) in $\mathcal{C}$ (a category with finite products) is a product-preserving functor $X : T \to \mathcal{C}$. The collection of all T-algebras forms $T\text{-alg}(\mathcal{C})$.

Example 17.2. Let F_n denote the free group on n letters. There is an algebraic theory called the theory of groups where T is the full subcategory of Grp^{op} with objects F_n (i.e. $n^+ \in T$ should be thought of as $F_n \in \mathrm{Grp}$, and $\mathrm{Hom}(n^+, k^+) = \mathrm{Hom}_{\mathrm{Grp}}(F_k, F_n)$). Then T-algebras X in a category $\mathcal{C}$ are the same as group objects in $\mathcal{C}$: the underlying object is $X(1^+)$, and the multiplication is gotten from the structure maps.

Nonexample 17.3. Recall that [special] Γ-spaces are functors $X : \Gamma^{op} \to \mathrm{Spaces}$ such that $X(n^+) \simeq X(1^+)^n$; Segal showed that, when $\pi_0 X(1^+)$ is a group, these are models of infinite loop spaces (equivalently, connective spectra), where you're supposed to think of $X(1^+)$ as the underlying space and $X(n^+)$ as its n-fold Cartesian product. But note that these are functors out of Γ^{op}, not Γ, and hence not T-algebras in any obvious sense.

Exercise 17.4. Show that there is an equivalence between algebraic theories T, and monads on Set_* preserving filtered colimits. The category of T-algebras is equivalent to the category of algebras over the corresponding monad.

For any simplicial theory T, there is a forgetful functor from T-algebras to pointed simplicial sets (i.e. taking a T-algebra X to $X(1^+)$), which has a left adjoint (free T-algebra functor). The claim is that the monad arising from the free-forgetful adjunction between T-algebras and Sets preserves filtered colimits, because it is just given by Kan extending the original functor $F : \Gamma_* \to T_*$.

Example 17.5. Consider $\mathcal{E} : \mathrm{Spaces}_* \to \mathrm{Spaces}_*$ sending $X \mapsto \bigvee_{n \geq 0} E\Sigma_{n,+} \wedge_{\Sigma_n} X^{\wedge n}$. Schwede calls this the Barratt-Eccles theory of infinite loop spaces. The idea is that $\mathcal{E}(X)$ is a model for $\Omega^\infty \Sigma^\infty(X)$. The claim is that $\mathcal{E}$ can be regarded as the free-forgetful monad $s\,\mathrm{Set} \to T\text{-alg} \to s\,\mathrm{Set}$ for some simplicial theory T. (Concretely, $\mathrm{Hom}_T(k^+, 1^+) = \mathcal{E}(k^+)$, and that determines T.) Algebras X over this theory such that $\pi_0 X$ is a group, are infinite loop spaces.

Given a Γ-space $\gamma : \Gamma^{op} \to \mathrm{Spaces}_*$, we can use left Kan extension Lan_i along the Yoneda embedding $i : \Gamma^{op} \to \mathrm{Spaces}$ to form $\widetilde{\gamma} = \mathrm{Lan}_i(Y) : \mathrm{Spaces}_* \to \mathrm{Spaces}_*$. In addition to the monoidal structure given by $\wedge$, now we have another monoidal structure $\circ$ given by composition. It is easy to associate a spectrum $S(\widetilde{\gamma})$ enriched over simplicial sets: declare the n^{th} space to be $\widetilde{\gamma}(S^n)$.

Lemma 17.6. *There is an equivalence between simplicial algebraic theories ($F : \Gamma \to T$ where T is simplicially enriched) and monoids in* (Γ*-spaces*, $\circ$).

Remark 17.7. Algebraic theories have a symmetric monoidal structure, and there is an equivalence

$$(T_1 \otimes T_2)\text{-algebras} \simeq T_1\text{-alg}(T_2\text{-alg}(\mathcal{C})) \simeq T_2\text{-alg}(T_1\text{-alg}(\mathcal{C})).$$

17.2. Stability and abelianization. Given an algebraic theory T, there is a category $\mathrm{Sp}(T)$ of *spectra in T-algebras*, defined as the category of T-algebras valued in spectra. Equivalently, you can mimic the Boardman category of spectra in the setting of T-algebras (a spectrum is a collection of T-algebras X_n, etc.).

Let the free T-algebra functor F^T be the left adjoint to the forgetful functor $U : T\text{-alg} \to \text{Spaces}$, and let T^s be the restriction of $U \circ F^T$ to Γ^{op}. So T^s is a functor $\Gamma^{op} \to \text{Spaces}$, i.e. a Γ-space.

It turns out that "stabilization is representable", in the following sense:

Theorem 17.8 (Schwede)**.** *There is an equivalence*

$$T\text{-}alg(\Gamma\text{-}Spaces) \simeq T^s\text{-}mod.$$

Idea of proof. Consider the assembly map in Γ-spaces. Suppose I have a Γ-space, i.e. a simplicial functor $\widetilde{X} : \text{Spaces}_* \to \text{Spaces}_*$. Then for $K, L \in \text{Spaces}_*$ I get a map $\widetilde{X}(K) \wedge L \to \widetilde{X}(K \wedge L)$. If $X, Y \in \Gamma$-spaces $X(n^+) \wedge Y(m^+) \to X(n^+ \wedge Y(m^+)) \to X(Y(n^+ \wedge m^+))$. This gives me a natural transformation $\psi_\gamma : X \wedge Y \to X \circ Y$ between the two monoidal structures. A consequence of the exercise is that

$$(T^s, \circ)\text{-alg}(\Gamma\text{-Spaces}) \simeq T\text{-alg}(\Gamma\text{-Spaces}).$$

ψ then induces a functor

$$\psi^* : (T^s, \circ)\text{-alg}(\Gamma\text{-Spaces}) \to (T^s, \wedge)\text{-alg}(\Gamma\text{-Spaces}).$$

An algebra over the RHS is just a T^s-module. □

An alternative characterization of T^s is as the endomorphism Γ-ring of the free T-algebra on one generator. There's a related and analogous story for abelianization. Let T^{ab} denote the endomorphism ring of the free abelian group object on one generator. Then we have

$$\mathrm{Ab}(T) \cong T^{ab}\text{-mod}$$

where $\mathrm{Ab}(T)$ is the category of abelian group objects in T-algebras (equivalently, T-algebras valued in abelian groups).

Example 17.9. If T is the theory of pointed sets (i.e. $T = \Gamma$ and the map is the identity), then $T^s = S$ and $T^{ab} = \mathbb{Z}$; the natural map $T^s \to HT^{ab}$ is the Hurewicz map.

How are these things related? We have the following adjunctions:

$$\Gamma\text{-Spaces} \underset{H}{\overset{L}{\rightleftarrows}} \mathrm{Ab} \underset{(-)_{ab}}{\overset{U}{\rightleftarrows}} \mathrm{Set}$$

where H is the Eilenberg-Maclane functor and L is its left adjoint. Schwede checked that these lift to when you take T-algebras on all these categories:

$$T\text{-alg}(\Gamma\text{-Spaces}) \underset{H}{\overset{L}{\rightleftarrows}} T\text{-alg}(\mathrm{Ab}) \underset{(-)_{ab}}{\overset{U}{\rightleftarrows}} T\text{-alg}(\mathrm{Set})$$

The total backwards composite is $X \mapsto \widetilde{\Sigma}^\infty X$ where $\widetilde{\Sigma}^\infty(X)(n^+) = \coprod_{i=1}^n X$ in T-alg.

Theorem 17.10.

(1) $L(T^s) \cong T^{ab}$, *and if* T *is a discrete theory (i.e. valued in sets as opposed to in simplicial sets) then* $T^{ab} \cong \pi_0 T^s$.

(2) The map $T^s \to HT^{ab}$ *is an isomorphism on* π_0 *and an epimorphism on* π_1. *(You should think of this as a Hurewicz theorem.)*

(3) $L(\widetilde{\Sigma}^\infty X) = X_{ab}$.

Let k be a commutative ring and $T = \mathrm{Comm}_k^{\mathrm{aug}}$.

Claim 17.11. *There is a (noncanonical) stable equivalence* $Dk := T^s \cong Hk \overset{L}{\wedge} H\mathbb{Z}$.

Finally, I'll say how this relates to other theories that we have.

17.3. Stable homotopy of T-algebras and stable homotopy of Γ-modules. Recall: let $A \in \mathrm{Comm}_k^{\mathrm{aug}}$; then we defined the Loday functor $\mathcal{L}(A|k;k)(n^+) = k \otimes A^{\otimes n}$. Given this we can get a Γ-space, and from that a spectrum. Say that $\pi_*^s(\mathcal{L}(A|k;k))$ is the stable homotopy of Γ-modules. Define $\pi_*^T(A) := \pi_*^s(\widetilde{\Sigma}^\infty A)$.

Theorem 17.12 (Basterra-Richter)**.** $\pi_*^s(\widetilde{\Sigma}^\infty A) \cong \pi_*^s(\mathcal{L}(A|k;k))$

Basterra claims that this is mostly done by Schwede.

There exists a spectral sequence for stable homotopy theories of algebraic theories, where

$$E^2_{p,q} = AQ_p(A|k; \pi_q^{st}(Dk)) \implies \pi_{p+q}^{st}(\widetilde{\Sigma}^\infty A).$$

By one of the earlier talks on Γ-homology, there also exists a spectral sequence for Γ-homology

$$E^2_{p,q} = AQ_p(A|k; \pi_q^s(\mathcal{L}(k[x]|k;k))) \implies \pi_{p+q}^s(\mathcal{L}(A|k;k)).$$

We know $\pi_*^s(\mathcal{L}(k[x]|k;k)) \cong Hk_* H\mathbb{Z}$, hence these two spectral sequences are the same.

Theorem 17.13. *We have*

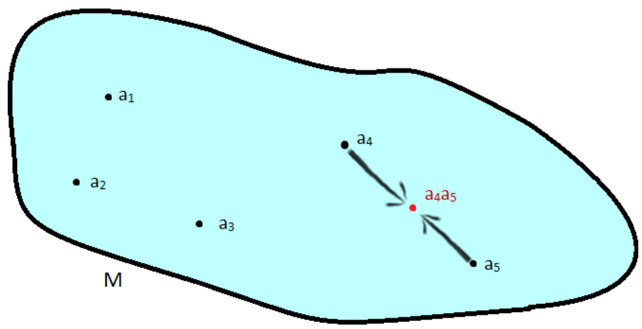
$$H\Gamma_*(A|k;M) \cong \pi_*^{st}(\mathcal{L}(A|k;M)).$$

When A *is flat over* k,

$$H\Gamma_*(A|k;k) \cong TAQ_*(HA|Hk;Hk).$$

BONUS TALK: Factorization homology (Inbar Klang)

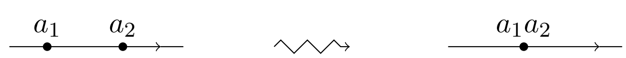
Let's start with ordinary homology. Let M be a space and A a topological abelian group. Define $A[M]$ to be the space of configurations of points in M labeled by A, such that when points collide, their labels multiply:



If A is a discrete abelian group, then the Dold-Thom theorem says that $H_*(M;A) \cong \pi_* A[M]$.

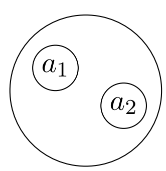
Factorization homology arises from the question: "what if A is not abelian?" (I learned about this point of view on factorization homology from Jeremy Miller.)

What is the problem? If a_1 and a_2 collide, it's unclear whether to label the new point a_1a_2 or a_2a_1. But if A is associative and M is an oriented 1 -manifold, then this is fine: the orientation on the manifold tells you which way to multiply.



More generally, if A is an E_n-algebra and M is a "parallelized" (more commonly known as "framed") n-manifold (i.e. it has a chosen isomorphism $TM \cong M \times \mathbb{R}^n$), this is also fine:

Since A is an E_n-algebra, there are an S^{n-1}'s worth of ways to multiply two elements; for example, if $n = 2$, the space of ways to multiply a_1 and a_2



is the space of ways the disks above can move around each other. If M is a parallelized manifold, then there are "S^{n-1} ways" two points can collide; multiply their labels using the operation corresponding to the direction of collision. (This is a hand-wavy way to define this, and it looks like it involves choices; in reality, the rigorous construction, using a two-sided monadic bar construction, does not involve choices.)

So far A has been a space, but you can also do this for chain complexes or spectra.

Example. Suppose $n = 1$, $M = S^1$, and A is an associative algebra. The factorization homology is defined to be $A[S^1] =: \int_{S^1} A$. These are configurations of points on S^1 labeled by A such that when points collide, their labels multiply. Labeling points around the circle by $a_0, ..., a_p$, we see that this is the cyclic bar construction. This is a simplicial object whose

p-simplices are $A^{\otimes p+1}$ and the face maps (which correspond to collision of points) are multiplication of adjacent elements $a_i a_{i+1}$ (or $a_p a_0$). So factorization homology for $M = S^1$ gives (T)HH; that is,

$$\pi_* \int_{S^1} A \cong (T)HH_*(A).$$

In general, factorization homology has input a parallelized (i.e. framed) manifold M^n and an E_n-algebra A (space, spectrum, chain complex, etc.). The output $\int_M A$ is whatever kind of thing A was (a space, spectrum, chain complex, etc.). You can take π_* to get a more algebraic invariant. $\int_{S^1} A \simeq HH(A)$ if A is a differential graded algebra, and $\int_{S^1} A \simeq THH(A)$ if A is a ring spectrum.

Factorization homology is functorial in M and A (see also below on axioms).

Comparison with other definition. Let $\mathrm{Disk}_n^{\mathrm{fr}}$ be a (topological, or infinity) category whose objects are finite disjoint unions of D^n's and morphisms are embeddings preserving the framing:

$$\mathrm{Map}(\bigsqcup_k D^n, \bigsqcup_\ell D^n) = \mathrm{Emb}^{\mathrm{fr}}(\bigsqcup_k D^n, \bigsqcup_\ell D^n).$$

Ayala and Francis define a $\mathrm{Disk}_n^{\mathrm{fr}}$-algebra to be a symmetric monoidal functor $A : \mathrm{Disk}_n^{\mathrm{fr}} \to \mathrm{Spectra}$ (or spaces, or chain complexes...). They define factorization homology $\int_M A$ as a homotopy colimit of the functor A over disks in M.

There is a correspondence between E_n algebras and $\mathrm{Disk}_n^{\mathrm{fr}}$-algebras: a symmetric monoidal functor $A : \mathrm{Disk}_n^{\mathrm{fr}} \to \mathrm{Spectra}$ gives an E_n algebra $A(D^n)$. We obtain an operation $A(D^n)^{\wedge k} = A(\bigsqcup_k D^n) \to A(D^n)$ for each embedding $\bigsqcup_k D^n \hookrightarrow D^n$, because A is symmetric monoidal and a functor from $\mathrm{Disk}_n^{\mathrm{fr}}$.

Factorization homology can also be defined for manifolds which are not framed; in this case, the E_n algebra A needs to have more structure (this is often confusingly called a framed E_n-algebra, or an algebra over the framed little disks operad). In terms of functors, we need $A : \mathrm{Disk}_n \to \mathrm{Spectra}$, where Disk_n has the same objects as Disk_n, but morphisms are all embeddings, not just ones preserving the framing.

Homology axioms. Why is this a homology theory?

(1) There is a "dimension axiom" $\int_{\mathbb{R}^n} A \simeq A$.

(2) This is functorial in M and A (the functoriality in M is not for all maps of spaces, just open embeddings of manifolds– or ones preserving the framing, for factorization homology of framed manifolds).

(3) There is an excision axiom: suppose $M = M' \cup_{M_0 \times \mathbb{R}} M''$ (where $\dim M_0 = n-1$ and all the rest are n-dimensional). Then

$$\int_M A \simeq \int_{M'} A \overset{\mathbb{L}}{\otimes}_{\int_{M_0 \times \mathbb{R}} A} \int_{M''}.$$

(Because of the $\mathbb{R}$, $\int_{M_0 \times \mathbb{R}} A$ is an (E_1-) algebra; given an embedding $\bigsqcup_k \mathbb{R} \hookrightarrow \mathbb{R}$, we get an embedding $\bigsqcup_k M_0 \times \mathbb{R} \hookrightarrow M_0 \times \mathbb{R}$ and therefore a map $(\int_{M_0 \times \mathbb{R}} A)^{\otimes k} \to \int_{M_0 \times \mathbb{R}} A)$.

Example. Consider $M = S^1$ covered by the open, overlapping top half and bottom half of the circle (thought of as two copies of $\mathbb{R}$), with overlap $S^0 \times \mathbb{R}$. This says

$$\int_{S^1} A \simeq \int_{\mathbb{R}} A \overset{\mathbb{L}}{\otimes}_{\int_{S^0 \times \mathbb{R}} A} \int_{\mathbb{R}} A \simeq A \overset{\mathbb{L}}{\otimes}_{A \overset{\mathbb{L}}{\otimes} A^{op}} A.$$

(The A^{op} is because of the orientation on the circle: the two pieces of $S^0 \times \mathbb{R}$ have opposite orientations.) This is Hochschild homology!

$\int_{S^n} A$ gives higher Hochschild homology, at least if A is E_∞.

Relationship to TAQ? Let $A \to k$ be an augmented E_n-algebra; think of $\widetilde{B}^n$ as an n-fold "delooping" or "suspension" (we used this to stabilize things and get TAQ). It turns out that $\widetilde{B}^n A$ is "reduced" factorization homology $\int_{(S^n,*)} A$. The intuition is as follows: say everything in S^n is labeled by A except for the north pole, which is labeled by k. Then use the augmentation $A \to k$ to multiply things at the north pole. (This is analogous to reduced homology.)

We know from Talk 10 that $TAQ \simeq \operatorname{colim} \Omega^n \widetilde{B}^n A$ is "shifts of $\widetilde{B}^n A$". We can apply $\widetilde{B}$ to A infinitely many times if and only if A is E_∞ (and can apply it n times, $\widetilde{B}^n$, if A is E_n).

Part VI: Negative results and future directions

Talk 18: $MU_{(p)} \to BP$ is not an E_∞ map (Gabe Angelini-Knoll)

We saw that MU is really important for doing chromatic homotopy theory; it is the Thom spectrum of $BU \to BF$, an infinite loop map. Consequently, MU is an E_∞ ring spectrum. If you p-localize, you get a splitting

$$MU_{(p)} \simeq \bigvee_{i \neq p^k - 1} \Sigma^{2i} BP$$

which is given in terms of specific maps $BP \xrightarrow{s} MU_{(p)} \xrightarrow{r} BP$. Here r is the map giving BP^* the universal p-typical formal group law.

Question 18.1. Is BP an E_∞ ring spectrum?

Question 18.2. Are s and r E_∞ ring maps?

I'll answer the second one; the first one will be answered in the next talk.

Theorem 18.3 (Hu-Kriz-May)**.** *There are no E_∞ ring maps $BP \to MU_{(p)}$.*

Theorem 18.4 (Johnson-Noel)**.** *The map r is not an E_∞ ring map for $p \leq 13$.*

This relies on computer calculations; they might have done a few more primes but they won't get all of them this way.

I'll focus on the proof of the Johnson-Noel result. There are two main ingredients: formal group laws and power operations. My definition of formal group law will be slightly different from the one we've seen before, but I'll show it's equivalent.

18.1. Formal group laws.

Definition 18.5. A formal group law over k is a connective, bicommutative, associative, topological Hopf algebra A with a choice of isomorphism $A \cong k[[x]]$.

To show this is the same as the definition in Özgür's talk, note that the Hopf algebra structure provides a coproduct map $\Delta : \underbrace{A}_{\cong k[[x]]} \to \underbrace{A \widehat{\otimes} A}_{\cong k[[x,y]]}$; we can produce a formal group law in the previous sense by taking the image of x under this map, thought of as a map $k[[x]] \to k[[x,y]]$.

Example 18.6. $MU^*(\mathbb{CP}^\infty) \cong MU^*[[x]]$, and the map $\underbrace{MU^*(\mathbb{CP}^\infty)}_{MU^*[[x]]} \to \underbrace{MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)}_{MU^*[[x,y]]}$ sends $x \mapsto x +_{MU} y$.

The Lazard ring L is the representing object for formal group laws, i.e. $\mathrm{Hom}_{\mathrm{Ring}}(L, K) \cong \mathrm{FGL}/K$.

Theorem 18.7 (Quillen). *The map $L \to MU^*$ associated to the formal group law $x +_{MU} y$ is an isomorphism.*

Given a map of ring spectra $MU \to E$, we can produce a formal group law $x +_E y$ using the induced map $MU^* \to E^*$ and the universal property of MU^*.

Definition 18.8. E is complex-oriented if $E^*(\mathbb{CP}^\infty) \cong E^*[[x]]$ and $E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ sends the Thom class x to the unit.

Proposition 18.9. *Ring maps $MU \to E$ are in bijection with complex orientations on E.*

Notation 18.10. $[i]_E(x) = \underbrace{x +_E x +_E \cdots +_E x}_{i}$

If E is complex oriented, then $E^*(BC_p) \cong E^*[[\xi]]/[p]_E\xi$. This will play an important role in Johnson-Noel's proof that $r : MU \to BP$ is not and E_∞ map.

Definition 18.11. A formal group law is *p-typical* if for all primes $q \neq p$, $\sum^F_{\zeta^q=1} \zeta x = 0$ where ζ is a primitive p^{th} root of unity.

18.2. Power operations. Let A be an E_∞ ring spectrum. We have free-forgetful adjunctions

$$E_n\text{-}A\text{-alg} \underset{U}{\overset{A\wedge -}{\rightleftarrows}} E_n\text{-alg} \underset{U}{\overset{\mathbb{P}^{E_n}}{\rightleftarrows}} \mathrm{Sp}$$

Let $\mathbb{P}^{E_n}_A(-) = A \wedge \mathbb{P}^{E_n}(-)$ be the right-to-left composition. Also abbreviate $\mathbb{P} := \mathbb{P}^{E_\infty}$. Say that R is an H_∞-algebra if it is a $\mathbb{P}$-algebra in $\mathrm{Ho}\,\mathrm{Sp}$.

Let X be a pointed space. For π a subgroup of Σ_k, define $D_\pi(X) := E\pi_+ \wedge_\pi X^{\wedge k}$. Recall the free E_∞-algebra on a spectrum R is $\mathbb{P}(R) = \bigvee_{i\geq 0} D_{\Sigma_i}(R)$. If R were an E_∞ ring spectrum, then there would be a factorization[7]

$$\begin{array}{ccc} R^{\wedge k} & \xrightarrow{\mu} & R \\ \downarrow & \nearrow & \\ D_{\Sigma_k}(R) & & \end{array}$$

[7]Compare this to a strictly commutative ring spectrum R, where the multiplication map $R^{\wedge k} \to R$ factors through $R^{\wedge k}/\Sigma_k$.

and, equivalently, a counit map $\mathbb{P}(R) \to R$ for the adjunction. In particular, using the Barr-Beck theorem for adjunctions we can show that there are equivalences of categories

$$E_n\text{-}A\text{-alg} \cong \mathbb{P}_A^{E_n}\text{-alg}$$
$$E_n\text{-alg} \cong \mathbb{P}^{E_n}\text{-alg}$$

where $\mathbb{P}_A^{E_n}$ and $\mathbb{P}^{E_n}$ are the monads associated to the free-forgetful adjunctions (note that the forgetful functor is hidden from the notation).

Given this data, we can define the power operation $\mathscr{P}_{\pi,R} : R^0(X) \to R^0(D_\pi(X))$ via the assignment

$$\mathscr{P}_{\pi,R} : \ (f : X \to R) \quad \mapsto \quad \big(D_\pi(X) \xrightarrow{D_n(f)} D_\pi(R) \xrightarrow{\varepsilon} \mathbb{P}(R) \to R)\big)$$

and the power operation

$$P_{\pi,R} : R^0(X) \xrightarrow{\mathscr{P}_{\pi,R}} R^0(D_\pi X) \xrightarrow{\partial} R^0(B\pi_+ \wedge X)$$

where ∂ is induced by the map $B\pi_+ \wedge X \simeq E\pi_+ \wedge_\pi X \to E\pi_+ \wedge_\pi X^{\wedge k} = D_\pi(X)$.

There's a weaker assumption we could have made on R, however: say that R is H_∞ if it has maps $D_{\Sigma_k}(R) \to R$ satisfying appropriate commutative diagrams. Equivalently, R is an algebra over the monad $\mathbb{P}$ *in the homotopy category*. Given an H_∞ ring we can still define $\mathscr{P}_{\pi,R}$ and $P_{\pi,R}$ as above since they only depended on maps up to homotopy.

Definition 18.12 (H_∞^d-structures)**.** Say that R is H_∞^d if there are compatible maps $D_{\Sigma_k}(\Sigma^{di}R) \to \Sigma^{dik}R$ for all $i \in \mathbb{Z}$.

Remark 18.13.

- $H_\infty^d \implies H_\infty$ for any d (just set $i = 0$ in the above definition).
- R is H_∞^d iff $\bigvee_{i\in\mathbb{Z}} \Sigma^{di}R$ is H_∞.

Given an H_∞^d ring spectrum R, we can produce power operations

$$\mathscr{P}_{\pi,R} : R^{di}(X) \to R^{dik}(D_\pi(X))$$
$$P_{\pi,R} : R^{di}(X) \to R^{dik}(B\pi_+ \wedge X)$$

defined in the same way as before.

Remark 18.14. Note that the notion of an E_∞ spectrum and an H_∞ spectrum are not equivalent; cf. a paper of Justin Noel titled "$H_\infty \neq E_\infty$". An E_∞ spectrum is an H_∞ spectrum by considering as an object in the homotopy category (in particular the monad $\mathbb{P}$ is compatible with the functor to the homotopy category). Hence, if a spectrum is not H_∞ then it is not E_∞ and, similarly, we can show that a map is not a E_∞ map by showing that it is not H_∞.

18.3. The map $r : MU \to BP$ is not E_∞. By Remark 18.14 it suffices to prove that $r : MU \to BP$ is not H_∞. The first step will be to reduce to showing that $r : MU \to BP$ is not H_∞^2.

Proposition 18.15. *An H_∞-complex orientation $MU \to E$ is equivalent to an H^2_∞-complex orientation.*

PROOF. One direction is easy: $H^2_\infty \implies H_\infty$.

Now assume $MU \to E$ is H_∞. From Lewis-May-Steinberger we have

$$D_{\Sigma_k}(S^{2i}) \simeq B\Sigma_k^{V\otimes\mathbb{C}^i}$$

where V is the standard representation of Σ_k, and so the Thom isomorphism gives

$$E^*(D_{\Sigma_k}(S^{2i})) \cong E^*(B\Sigma_k^{V\otimes\mathbb{C}^i}) \cong E^*(\Sigma^{2ik}B\Sigma_k).$$

As E is complex oriented, in $E^{2ik}(D_{\Sigma_k}(S^{2i})) \to E^{2ik}(\Sigma^{2ik}B\Sigma_k)$ the unit in the RHS pulls back to a class $\mu_{i,k}$; this is the Thom class. To prove the proposition we need to show the diagram

$$\begin{array}{ccccccc} D_{\Sigma_k}(\Sigma^{2i}MU) & \longrightarrow & D_{\Sigma_k}(S^{2i}) \wedge D_{\Sigma_k}(MU) & \xrightarrow{(*)} & \Sigma^{2ik}MU \wedge MU & \longrightarrow & \Sigma^{2ik}MU \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D_{\Sigma_k}(\Sigma^{2i}E) & \longrightarrow & D_{\Sigma_k}(S^{2i}) \wedge D_{\Sigma_k}(E) & \longrightarrow & \Sigma^{2ik}E \wedge E & \longrightarrow & \Sigma^{2ik}E \end{array}$$

commutes, where $(*)$ is $\mu_{ik} \wedge H_\infty$-structure map. The left square always commutes by naturality of the natural transformation $D_{\Sigma_k}(- \wedge -) \to D_{\Sigma_k}(-) \wedge D_{\Sigma_k}(-)$, as does the right square (since $MU \to E$ is a ring map by assumption). It remains to prove why the middle square commutes. It is a smash of two diagrams

$$\begin{array}{ccc} D_{\Sigma_k}(S^{2i}) & \longrightarrow & \Sigma^{2ik}MU \\ \| & & \downarrow \\ D_{\Sigma_k}(S^{2i}) & \longrightarrow & \Sigma^{2ik}E \end{array}$$

$$\begin{array}{ccc} D_{\Sigma_k}(MU) & \longrightarrow & MU \\ \downarrow & & \downarrow \\ D_{\Sigma_k}(E) & \longrightarrow & E \end{array}$$

The second one commutes by the H_∞ structure. The first commutes because $MU \to E$ is a complex orientation and consequently Thom classes map to Thom classes. □

Corollary 18.16. *If $MU \to BP$ is an H_∞ map, then there exists a power operation $P_{C_p,BP}$ compatible with the map, i.e. the following diagram commutes:*

$$\begin{array}{ccc} MU^{2i} & \xrightarrow{P_{C_p,MU}} & MU^{2pi}(BC_p) \\ \downarrow & & \downarrow \\ BP^{2i} & \xrightarrow{P_{C_p,BP}} & BP^{2ip}(BC_p) \end{array}$$

If the map were E_∞ it would respect all the power operations. By Proposition 18.15, a complex orientation is H_∞ if and only if it is compatible with all H^2_∞ power operations.

Strategy 18.17. If there exists $x \in MU^{2*}$ such that $r_*(x) = 0$ (this is easy to check) but $r_* P_{C_p,MU}(x) \neq 0$ (this is hard to find) then the map $MU \xrightarrow{r} BP$ is not H_∞.

The class $r_* P_{C_p,MU}(x)$ that we want to be nonzero is in the bottom right corner of

$$\begin{array}{ccc} MU^{2*} & \xrightarrow{P_{C_p,MU}} & MU^{2p*}(BC_p) \cong MU^{2p*}[[\xi]]/[p]_{MU}\xi \\ \downarrow r_* & & \downarrow r_* \\ BP^{2*} & \xrightarrow{P_{C_p,BP}} & BP^{2p*}(BC_p) \cong BP^{2p*}[[\xi]]/[p]_{BP}\xi \end{array}$$

We have $r_* P_{C_p,MU}(x) = a_0 + a_1 x + a_2 x^2 + \dots$ where a_i are power series in ξ. Let $\langle p \rangle_E \xi = [p]_E(\xi)/\xi$. You can extend the previous diagram to:

$$\begin{array}{ccccc} MU^{2*} & \xrightarrow{P_{C_p,MU}} & MU^{2p*}(BC_p) \cong MU^{2p*}[[\xi]]/[p]_{MU}\xi & \xrightarrow{\varphi_*\chi^{2n}} & MU^{2p*+4n(p-1)}[[\xi]]/\langle p \rangle_{MU}\xi \\ \downarrow r_* & & \downarrow r_* & & \downarrow \\ BP^{2*} & \xrightarrow{P_{C_p,BP}} & BP^{2p*}(BC_p) \cong BP^{2p*}[[\xi]]/[p]_{BP}\xi & \longrightarrow & BP^{2p*+4n(p-1)}[[\xi]]/\langle p \rangle_{BP}\xi \end{array}$$

where $\varphi_* : MU^{2p*+4n(p-1)}[[\xi]]/[p]_{MU}\xi \to MU^{2p*+4n(p-1)}[[\xi]]/\langle p \rangle_{MU}$ is the reduction map and $\chi = \prod_{i=1}^{p-1}[i]_{MU}\xi \in MU^{2(p-1)}[[\xi]]/[p]_{MU}\xi$.

The claim (below) is that it suffices to check if $\varphi_*\chi^{2n} r_* P_{C_p,MU}$ is nonzero on the class $[\mathbb{C}\mathrm{P}^n] \in MU^*$ for some $n \neq p^n - 1$, where $[\mathbb{C}\mathrm{P}^n]$ is a lift of one of Quillen's choice of generators for $MU^* \otimes \mathbb{Q}$. Recall that $MU^* \cong \mathbb{Z}[[\mathbb{C}\mathrm{P}^1], [\mathbb{C}\mathrm{P}^2], \dots]$ for $|[\mathbb{C}\mathrm{P}^i]| = 2i$ and $BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $r_* : MU^* \to BP^*$ sends $[\mathbb{C}\mathrm{P}^n]$ to 0 when $n \neq p^i - 1$.

Theorem 18.18 (Bruner-May-McClure-Steinberger)**.** *If such a $P_{C_p,BP}$ exists, it must be $r_* P_{C_p,MU_{(p)}} s_*$. Furthermore, the map $MU \xrightarrow{r} BP$ is H_∞ only if*

$$MC_n(\xi) := \varphi_* \chi^{2n} r_* P_{C_p,MU}([\mathbb{C}\mathrm{P}^n]) = 0$$

in $BP^*[[\xi]]/\langle p \rangle_{BP}\xi$ for $n \neq p^i - 1$.

Theorem 18.19 (Johnson-Noel)**.** *$MC_n(\xi) \neq 0$ for some $n \neq p^k - 1$ for $p \leq 13$.*

In particular, the thing that's important for the next talk is, at $p = 2$, we have

$$MC_2(\xi) = \xi^6(v_1^6 + v_2^2) + \xi^7(v_1^7 + v_3) + \text{higher degree terms.}$$

In particular, v_3 is indecomposable.

Lawson reproves Johnson-Noel's result by a different means in his appendix and his answer has different power of ξ in particular

$$r_* P_{C_p,MU}(\mathbb{C}\mathrm{P}^2) = \xi^2(v_1^6 + v_2^2) + \xi^3(v_1^7 + v_3) + \text{higher degree terms.}$$

This can be accounted for by the fact that $\chi = \xi$ modulo (ξ^8) and $\langle p \rangle_{MU}\xi$ at $p = 2$ and therefore $\chi^4 = \xi^4$ modulo higher degree terms.

Here's an explicit formula you can supposedly compute with yourself:

$$r_* P_{C_p,MU}(x) = \prod_{i=0}^{p-1} [i]_{BP}\xi +_{BP} x.$$

At $p = 2$, this is:

$$\begin{aligned}\prod_{i=0}^{1} [i]_{BP}\xi +_{BP} x &= x(\xi +_{BP} x) \\ &= x(\exp_{BP}(\log_{BP}(\xi) + \log_{BP} x))\end{aligned}$$

which is computable by hand and this computation is done explicitly at the end of the Johnson-Noel paper.

18.4. About the Hu-Kriz-May paper on $BP \to MU$. We have a map $BP \to MU$. If this were an E_∞ ring map, it would have to respect Dyer-Lashof operations. This map has some nice properties, e.g. there's an inclusion $H_*(BP) \hookrightarrow H_*(MU)$ sending $\xi_1^2 \mapsto a_1$. There's a Dyer-Lashof operation $Q^8(a_1) = a_5$, which is indecomposable. It turns out that there is no indecomposable in degree 10 in $H_*(BP)$ that could possibly hit it.

Talk 19: BP is not E_{12} at $p = 2$ (Andy Senger)

This talk is based on the recent preprint "Secondary power operations and the Brown-Peterson spectrum at the prime 2" of Tyler Lawson, in which he proves the result stated in the title of this talk.

We just spent a whole week learning techniques for proving that a ring spectrum has more structure. I'm here to rain on the parade. The flavor of the techniques employed in this talk will be a little bit different from those used in the preceding talks, as the methods one uses to prove that things *aren't* something are rather different from the methods one uses to prove that things *are* something.

In the following, all spectra are localized at 2. In particular, our BP will be the 2-local BP; we could also 2-complete it and the main result would still hold.

I wish to show that BP is not E_n for any $n \geq 12$. To do so, I need to find some kind of invariant that will conflict with BP being E_n. If BP were E_n, then its Postnikov tower automatically enriches to a tower of E_n ring spectra. In particular, the map to the bottom layer $BP \to \tau_{\leq 0} BP \cong H\mathbb{Z}_{(2)}$ composed with the map $H\mathbb{Z}_{(2)} \to H\mathbb{F}_2$ induced by the modulo 2 map $\mathbb{Z}_{(2)} \to \mathbb{F}_2$ is automatically a map of E_n ring spectra whenever BP is an E_n ring spectrum.

Recall what this map does on mod 2 homology: the mod 2 homology of $H\mathbb{F}_2$ is just the dual Steenrod algebra, and the homology of BP sits inside as the infinite polynomial algebra generated by the ξ_is: $H_*(BP) \cong \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] \subset \mathbb{F}_2[\xi_1, \xi_2, \dots] \cong H_*(H\mathbb{F}_2)$. Furthermore, the map above induces this inclusion upon taking homology: since it is automatically an E_n map if BP is E_n, this means that for any E_n structure on BP, all of the induced operations on $H_*(BP)$ are determined by the operations on $H_*(H\mathbb{F}_2)$.

But what sort of operations does an E_n structure induce on $H_*(BP)$? For simplicity, I'll restrict to the $n = \infty$ case. In the rest of this talk, let H denote $H\mathbb{F}_2$. Then HBP is an E_∞-H-algebra if BP is an E_∞ algebra, and we obtain power operations on $H_*(BP)$ as follows.

Let A be an E_∞-H-algebra. Then there exist power operations Q^s that act on π_*A. These operations come as elements $Q^s \in \pi_{i+s}(\mathbb{P}_H(S^i))$. I will show how this acts on an element $f \in \pi_i A = [S^n, A]_{\mathrm{Sp}}$. By the free-forgetful adjunctions

$$E_\infty\text{-}H\text{-alg} \underset{U}{\overset{H\wedge -}{\leftrightarrows}} E_\infty\text{-alg} \underset{U}{\overset{\mathbb{P}}{\leftrightarrows}} \mathrm{Sp}$$

we have $[S^i, A]_{\mathrm{Sp}} \cong [\mathbb{P}_H(S^i), A]_{E_\infty\text{-}H\text{-}alg}$ and $[S^{i+s}, \mathbb{P}_H(S^i)] \cong [\mathbb{P}_H(S^{i+s}), \mathbb{P}_H(S^i)]$. So we can define $Q^s f \in \pi_{i+s}A$ as the composite

$$\mathbb{P}_H(S^{i+s}) \xrightarrow{Q^s} \mathbb{P}_H(S^i) \xrightarrow{f} A.$$

These Q^ss are natural, satisfy a version of the Adem relations as well as the Cartan formula, and are natural in maps of E_∞ ring spectra.

(You can also do this for E_n-H-algebras in Ho(Sp); E_n algebras know about Q^{i+m} on π_i for $m \leq n-1$, and these satisfy e.g. the Adem relations when $m \leq n-2$.)

The Q^ss present a potential obstruction to E_∞ structures on BP: if BP admits an E_∞ structure, then the subalgebra $\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] \subset \mathbb{F}_2[\xi_1, \xi_2, \dots]$ must be closed under the action of the Q^i. However, this is true: it follows readily from the Cartan formula.

At this point, we could just give up. Or we could realize that we have only used a small portion of the structure available to us: while we cannot rule out the existence of an E_∞ structure on BP merely using primary power operations, there are also secondary power operations, tertiary power operations, and so on. These come from the coherence we ask for when we demand that BP be E_∞ instead of just H_∞, which is more than enough to obtain the action of the primary power operations Q^i.

Lawson's basic idea is to find a secondary power operation under which $H_*(BP)$ is not closed in $H_*(H)$. Since this power operation is actually defined for E_{12}-algebras and not just E_∞-algebras, this actually shows that BP cannot even be E_{12}. This is not easy, and there are two main problems that need to be overcome:

(1) one must find a secondary power operation which stands a chance of being an obstruction, and

(2) one must actually compute said secondary power operation.

In this talk, I will focus on the second problem, though it is the first problem which has the greater relation to obstruction theory.

To address the first problem, Lawson actually computes with the Goerss-Hopkins obstruction groups for $H_*(BP)$, using the version of the Goerss-Hopkins obstruction theory based on H for the constant simplicial E_∞ operad, and finds nontrivial obstruction classes. In particular, suitably interpreting one of these led to the computation of the secondary operation he used to show that BP is not E_{12}.

19.1. Secondary operations. Let C be a category enriched over Top_*, e.g. Top_* itself. Suppose I have some maps $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ such that $f_2 \circ f_1$ and $f_3 \circ f_2$ are nullhomotopic. Fix particular nullhomotopies $H_1 : f_2 f_1 \implies *$ and $H_2 : f_3 f_2 \implies *$. Then we may define a bracket $\left\langle f_3 \overset{H_2}{\leftarrow} f_2 \overset{H_1}{\leftarrow} f_1 \right\rangle$ (this is like a Toda bracket or Massey product, but we keep track of the extra information of the specific nullhomotopies) to be the element of $\pi_1 \mathrm{Map}_C(X_0, X_3)$, based at the null map, defined by $f_3(H_1)^{-1} \circ (H_2 f_1)$. (This is loop composition, and H_1^{-1} is doing the homotopy backwards.) If we were working in pointed spaces, then we can think of it more geometrically as an element of $\pi_1 \mathrm{Map}_C(X_0, X_3) = \pi_0 \mathrm{Map}(\Sigma X_0, X_3)$; specifically, think of it as a map out of ΣX_0 where you do $f_3 H_1$ on the top part of the cone and $H_2 f_1$ on the bottom half (so cone height is the parameter in the homotopy).

Then let $\langle f_3, f_2, f_1 \rangle$ denote the set of all $\langle f_3 \leftarrow f_2 \leftarrow f_1 \rangle$, where this varies over all choices of nullhomotopies.

Notation 19.1. Given a collection of graded symbols $\{y_i\}$ where $|y_i| = d_i$, define $\mathbb{P}_H(\{y_i\}) := \mathbb{P}_H(\bigvee_i S^{d_i})$. Then we also write y_i for the element in $\pi_{d_i} \mathbb{P}_H(\{y_i\})$ corresponding to the map $S^{d_i} \to \bigvee_i S^{d_i} \to \mathbb{P}_H(\bigvee_i S^{d_i})$.

Let x be an element with $|x| = 2$. Let C be the category of E_∞-H-algebras under $\mathbb{P}_H(x)$. This is really just the same thing as E_∞-H-algebras with a specified point $x \in \pi_2(R)$. (There's a subtlety, however: we need to make this enriched over *pointed* topological spaces, i.e. to specify null maps. This may be accomplished by replacing C with a category of objects of C that are possibly pointed and possibly augmented, and use either the point or the augmentation to define the null maps. I won't go into this.)

We are now in the context to define secondary operations. I will only define the example of a secondary operation that we will need; the general case should be clear from this.

Lawson finds a sequence of maps

$$\mathbb{P}_H(x, z_{30}) \xrightarrow{R} \mathbb{P}_H(x, y_5, y_7, y_9, y_{13}, y_8, y_{10}, y_{12}) \xrightarrow{Q} \mathbb{P}_H(x)$$

(here subscripts indicate degrees of the symbols y_i etc.) such that $Q \circ R$ is null. This is really just a relation between power operations. The map Q is determined by several elements of $\pi_i \mathbb{P}_H(x)$, picked out by the y_i; these are just polynomials in compositions of the Q^s applied to x; for example, y_5 is sent to $Q^3 x$ and y_{12} is sent to $Q^{10} x + (Q^4 x)^2$. The map R is determined by a single element of $\mathbb{P}_H(x, y_5, \dots)$, and this is a single big power operation that takes in x and the y_i as inputs. The fact that $Q \circ R$ is null says that when we plug the operations defining Q into the big operation R, we get zero. So in this case R may be viewed as a relation between the operations defining Q.

I will not write down the entire relation that Lawson finds; it is quite large. This is the relation that he used the Goerss-Hopkins obstruction theory to find. It is the first possible secondary obstruction to BP having an E_∞ structure, and it is unlikely that one would come across it simply by guess and check.

(If you try to find an obstruction by guess and check, you will find that things like the Nishida relations will ruin your day.)

An element $\alpha \in \pi_2 A$ induces a map $\alpha : \mathbb{P}_H(x) \to A$ which we can fit into the diagram:

$$\begin{array}{c} \mathbb{P}_H(x, z_{30}) \\ \downarrow R \\ \mathbb{P}_H(x, y_5, y_7, y_9, y_{13}, y_8, y_{10}, y_{12}) \\ \downarrow Q \\ \mathbb{P}_H(x) \xrightarrow{\quad\alpha\quad} A \end{array}$$

αQ is nullhomotopic if and only if the operations determined by Q are zero on α.

Suppose αQ is null. Then we can define

$$\langle \alpha, Q, R \rangle \subset \pi_1 \operatorname{Map}_C(\mathbb{P}_H(x, z_{30}), A) \cong \pi_0 \operatorname{Map}_C(\mathbb{P}_H(x, z_{31}), A) \cong \pi_{31} A.$$

Viewing this as a function of such α, it is the secondary operation defined by Q and R. (This is not quite true. Usually a secondary operation would be defined by a choice of nullhomotopy of $Q \circ R$, and as such would have smaller indeterminacy. This is immaterial to us, because we will not need to narrow down the indeterminacy further.)

Applying this secondary operation to the element $\xi_1^2 \in \pi_2(H \wedge H)$ will provide the desired contradiction to the possibility of an E_∞ structure on BP.

Theorem 19.2 (Lawson)**.** *$\langle \xi_1^2, Q, R \rangle$ is defined (i.e., $Q(\xi_1^2) = 0$); it has no indeterminacy modulo decomposables, and is $\equiv \xi_5$ modulo decomposables.*

Since ξ_5 is not in the image of $H_*(BP) \subset H_*(H)$ modulo decomposables (indeed, the image is trivial modulo decomposables), this shows that this inclusion cannot respect this secondary power operation, and therefore that BP cannot be E_∞.

This actually shows that BP cannot be E_{12}, since the largest operation you need to define this secondary operations is a Q^{20} acting on an element of degree 10; this appears at E_{11} and begins to satisfy the relation used to define the secondary operation at E_{12}.

This also shows that no truncated Brown-Peterson spectrum $BP\langle n \rangle$ can be given an E_{12} structure for $n \geq 4$, since one has $H_*(BP\langle n \rangle) = \mathbb{F}_2[\xi_1^2, \ldots, \xi_{n+1}^2, \xi_{n+2}, \ldots] \subset H_*(H)$.

For the rest of the talk, I will discuss the proof of this theorem.

19.2. Method of attack.

(1) Use juggling formulae (e.g. the relations used in (19.1) to shift around terms in these brackets) for these operations, plus $\xi_1^2 = p(b_1)$ via $p : MU \to H$ to reduce to some "functional power operation".

(2) Use a Peterson-Stein relation to reduce further to a primary power operation in $H \wedge_{MU} H$.

(3) Use a further reduction to a primary operation in $SL_1(MU)$, and you can tie this up to how the MU-power operations act on $\pi_* MU$ to save the day.

19.3. Details. For (1), the diagram is as follows:

$$\begin{array}{ccccc} \mathbb{P}_H(x, y_i) & \xrightarrow{Q} & \mathbb{P}_H(x) & & \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle b_1} & \searrow^{\xi_1^2} & \\ \mathbb{P}_H(x, y_4) & \xrightarrow{f} & H \wedge MU & \xrightarrow{p} & H \wedge H \end{array}$$

where f is defined by $x \mapsto b_1 \in H_2(MU)$ and $y_4 \mapsto b_2 \in H_4(MU)$. The idea is that, while Q does not vanish on b_1 like it does on $p(b_1) = \xi_1^2$, it at least factors through $\mathbb{P}(x, y_4)$, via a map μ sending $y_{10} \mapsto Q^6 y_4$ and killing everything else.

Since $\xi_1^2 = p(b_1)$, we have

$$\left\langle \xi_1^2, Q, R \right\rangle = \langle pb_1, Q, R\rangle \approx \langle p, b_1 Q, R\rangle = \langle p, f\mu, R\rangle \approx \langle p, f, \mu R\rangle \tag{19.1}$$

$$\text{and it turns out this is } \approx Q^{16}(\langle p, f, \overline{Q}\rangle).$$

Here I use $\approx$ to denote equality modulo decomposables; it is implicit here that all of the steps in the above are defined with zero indeterminacy modulo decomposables. I will define the operation $\overline{Q}$ shortly.

Since $Q^{16}(\xi_4) \equiv \xi_5$ modulo decomposables, I just need to show that

$$\left\langle p, f, \overline{Q} \right\rangle \equiv \xi_4 \quad \text{(mod decomposables).} \tag{19.2}$$

The maps in this bracket are defined in the following diagram:

$$\begin{array}{ccccccc} \mathbb{P}_H(x, z_{14}) & \xrightarrow{\overline{Q}} & \mathbb{P}_H(x, y_4) & \xrightarrow{f} & H \wedge MU & \xrightarrow{p} & H \wedge H \\ x & \longmapsto & x & \longmapsto & b_1 & \longmapsto & \xi_1^2 \\ z_{14} & \longmapsto & Q^{10} y_4 + x^2 Q^6 y_4 & & & & \\ & & y_4 & \longmapsto & b_2 & \longmapsto & 0 \end{array}$$

19.4. Reduction to $H \wedge_{MU} H$. The trick is that I can extend the composition to

$$\mathbb{P}_H(x, z_{14}) \xrightarrow{\overline{Q}} \mathbb{P}_H(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

(Think of this as the suspension of MU as an augmented H-algebra.) Then there is a "Peterson-Stein" relation of the form $i \left\langle p, f, \overline{Q} \right\rangle = \langle i, p, f\rangle\, \overline{Q}$.

To make use of this relation, I need to know three things. First, I need to know what i does on homotopy, and whether I can detect that $\left\langle p, q, \overline{Q} \right\rangle$ takes the right value after postcomposing with i. Second, I need to be able to compute $\langle i, p, f\rangle$. Third, I need to be able to compute $\overline{Q}$ on this.

First I will try to understand what i does. There is a Künneth spectral sequence

$$\mathrm{Tor}_{**}^{\pi_* MU}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_*(H \wedge_{MU} H).$$

We know that $MU_* \cong \mathbb{Z}[x_i]$, and so $\mathrm{Tor}^{\pi_* MU}(\mathbb{F}_2, \mathbb{F}_2) \cong \Lambda(\sigma x_i)$ with $|\sigma x_i| = (1, 2i)$ (this grading is (homological degree, internal degree))[8]. This spectral sequence is forced to degenerate, so

$$\pi_*(H \wedge_{MU} H) \cong \Lambda_{\mathbb{F}_2}(\sigma x_i).$$

$H \wedge_{MU} H \cong H \wedge_{H \wedge MU} (H \wedge H)$ which gives a different sort of Künneth spectral sequence:

$$\mathrm{Tor}^{H_*(MU)}(\mathbb{F}_2, H_*(H)) \implies \pi_*(H \wedge_{MU} H).$$

The E_2-term of this Künneth spectral sequence is $\Lambda[\xi_k] \otimes \Lambda[\sigma b_n \mid n \neq 2^k - 1]$, and by comparison with the other Künneth spectral sequence must collapse.

The map $i : H \wedge H \to H \wedge_{MU} H$ induces a map of Künneth spectral sequences from which is can be seen that, up to decomposables, $i_*(\xi_i) \equiv \sigma x_{2^{i-1}-1}$. In particular, i_* is an injection on indecomposables, so that to show (19.2) it suffices to show that $i \langle p, f, Q \rangle = \sigma x_7 \in \pi_* H \wedge_{MU} H$. Therefore it suffices to show that

$$\langle i, p, f \rangle \overline{Q} = \sigma x_7.$$

To evaluate $\langle i, p, f \rangle$, we will use the following result of Lawson, which holds for elementary reasons:

Fact 19.3. *Let $R \to H$ be an E_∞ map and let $x \in H_n R$ map to zero in $H_n H$. Then $\langle i, p, x \rangle \subset \pi_{n+1} H \wedge_R H$ contains an element which is detected by the element σx in filtration one of the Künneth spectral sequence*

$$\mathrm{Tor}^{H_*(R)}(\mathbb{F}_2, H_* H) \implies \pi_*(H \wedge_R H).$$

In particular, we deduce that

$$\langle i, p, f \rangle \equiv \sigma x_2$$

up to decomposables.

Now it is just left to show that

$$\overline{Q}(\sigma x_2) \equiv \sigma x_7 \quad \text{(mod decomposables)}.$$

Up to decomposables, $\overline{Q}(\sigma x_2) \equiv Q^{10}(\sigma x_2)$, so we just have to compute $Q^{10}(\sigma x_2)$. At this point this is looking similar to what we saw in the last talk, as in that talk it was shown that, after mapping down to BP, v_3 appears in the total MU-power operation applied to x_2; since $x_7 \mapsto v_3$ and this is an isomorphism on indecomposables in this degree, we see that, up to decomposables, the total MU power operation on x_2 picks up x_7. We need to figure out how to turn this into a power operation in $H \wedge_M UH$. We will show that the suspension σ is compatible in some way with power operations.

To do this, I will define a map of E_∞ spaces $SL_1(MU) \to \Omega(\Omega^\infty_\otimes(H \wedge_{MU} H))$ as follows: there is a diagram of E_∞ ring spectra:

[8] here σ means suspension in an algebraic sense (degree-shift)

$$\begin{array}{ccc} MU & \longrightarrow & H \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \wedge_{MU} H \end{array}$$

upon applying SL_1, this gives a homotopy coherent diagram of E_∞ spaces:

$$\begin{array}{ccc} SL_1(MU) & \longrightarrow & SL_1(H) \\ \downarrow & & \downarrow \\ SL_1(H) & \longrightarrow & SL_1(H \wedge_{MU} H) \end{array}$$

since $SL_1(H)$ is contractible, this determines a map of E_∞ spaces $SL_1(MU) \to \Omega(SL_1(H \wedge_{MU} H))$ which may be composed with loops on the inclusion $SL_1(H \wedge_{MU} H) \to \Omega^\infty_\otimes(H \wedge_{MU} H)$ to obtain the desired map.

This induces a map $g : H \wedge \Sigma(\Sigma^\infty_+ SL_1(MU)) \to H \wedge_{MU} H$, and further a map $g_* : H_*(SL_1(MU)) \to \pi_{*+1} H \wedge_{MU} H$. We have $\pi_*(SL_1(MU)) \cong \pi_* MU$ for $* > 0$ and on the Hurewicz image, this g_* is just σ. Furthermore, since it comes from a map of E_∞ spaces, g_* is compatible with power operations, so now we just have to compute this power operation in $H \wedge \Sigma(\Sigma^\infty_+ SL_1(MU))$, modulo the kernel of g_*, which is large. Modulo this kernel, power operations on the Hurewicz image in $H_*(SL_1(MU))$ may be reduced to power operations on MU. This is reasonable because both of these sorts of power operations come from the multiplicative E_∞ structure on $\Omega^\infty MU$, and you just have to figure out how to translate between π_* and H_* operations. More precisely, we have the following theorem of Lawson:

Proposition 19.4. Let $y \in \pi_{2n}(MU)$, $n > 0$. Suppose, in the notation of Gabe's talk, that $P_{C_2,MU}(y) = \sum_i c_i \xi^i$. Then, letting $[y] \in H_{2n}(SL_1(MU))$ denote the Hurewicz image of $y \in \pi_{2n} MU \cong \pi_{2n} SL_1(MU)$, we have

$$Q^{2k}([y]) = [c_{k-n}] \pmod{\ker g_*}.$$

Combining this result with the MU-power operation calculated in Gabe's talk, we obtain the desired result:

$$Q^{10}([x_2]) \equiv [x_7] \pmod{\ker g_*}$$

in $H_*(MU)$.

If we wanted to generalize this result to odd primes, we'd need to do two things: find a potential secondary obstruction, and compute it. The first step would depend on a analogous Goerss-Hopkins obstruction theory calculation to the one that Tyler made use of at the prime 2. As at the prime 2, one can reduce the second step to the calculation of an MU-power operation in $\pi_* MU$. Lawson believes that the method he used in the appendix of his preprint to calculate the necessary power operation at 2, which is simpler than the Johnson-Noel method because it only calculates up to decomposables, can be generalized to the odd primary case, and one of his students is working on this.

Talk 20: Further directions (Maria Basterra)

My thesis was trying to fix what we thought was wrong with Kriz' proof that BP was an E_∞-ring spectrum. The issue was how he defined the E_∞ cohomology theory – that's what led to the definition of TAQ. He had a theory of Postnikov towers; at the same time this was happening, EKMM was being developed; they would change the definition of $\wedge$ every other week, but at least there was a smash product. Other people were thinking about obstruction theory, Γ-homology; Robinson, Pirashvili was talking about stabilization, etc. Obstruction theory developed with Postnikov towers is completely different from the obstruction theory developed by Whitehouse and Robinson about the n-stages.

Here are some things I'd like to know:

- Is BP an E_∞-ring spectrum at an odd prime? (probably not. . .)
- What is the relationship between the Postnikov tower obstruction theory and the "stages" of Robinson? We've talked about all these cohomology theories that house obstructions, and spent some time trying to prove the groups are the same, but they don't appear in the same way; so what's the relationship? Clark Barwick ("Operator categories") has some ideas about this.

  You can draw a diagram of "arity" (i.e. $A_1, A_2, \dots, A_\infty = E_1$) vs. "commutativity" (i.e. $E_1, \dots, E_\infty$).

A_1				
A_2				
A_3		??		
$\vdots$				
A_∞	E_2	E_3	$\dots$	E_∞

  This is a proposal to get an "arity" filtration of the E_n-operad similar to the A_k-filtration of $E_1 = A_\infty$.

  If all the intermediate stages inside are represented by operads (with maps from each operad down and to the right), maybe you'll have a way to relate algebras over these different things. Going down is clear; going right there's some kind of wreath product construction that Clark understands. We're expecting Robinson's stages to lie somewhere in the middle; the n-stage is definitely one row down from the $(n-1)$-stage, but it's unclear how many columns to the right of the $(n-1)$-stage it lies. Clark conjectures that the n-stages lie in the sub-diagonal below the main diagonal.

  There's already a notion of wreath product of operads $E_1 \wr E_1 = E_2$, etc.; there's also a formal procedure of truncation.

- What about an equivariant version of all of this? There are lots of people thinking about this – Basterra, Blumberg, Hill, Lawson, Mandell. The aim is to build a theory of TAQ of equivariant G-(ring spectra) or equivariant (G-ring) spectra. The first problem is what this means, exactly. Everything is harder because you have to make choices: suspension is not so trivial, there are connectivity issues. The idea is to get a theorem that says you can describe ring spectra by means of operads. So you have to define operads in this setting; Blumberg and Hill have ideas of how to do this. There are lots of versions of E_∞-operads, called $\mathbb{N}_\infty$; it depends on how much information you want to encode. You can define indecomposables in this setting, and I think we can prove that taking indecomposables is the same as stabilization. Here's another circumstance where we could decide to define TAQ as stabilization. But for the purposes that we/others are interested in, we want to define it as (derived) indecomposables.

What would play the role of BP in this setting? (That is, something where you want to know what structure it has.) Maybe Goerss-Hopkins stuff to do equivariant tmf.

- Then there's the ∞-category world. This is the opposite situation: Lurie has described topological Quillen cohomology in an ∞-category of algebras of an appropriate nice ∞-operad (Dylan: "anything you want") as stabilization. But people want to consider the same notion without starting at a stable category. Harpaz, Nuiten, and Prasma have tangent categories and algebras over operads; so they are developing this theory, but coming in the opposite way – we defined a theory in terms of indecomposables and then proved that it's stabilization, whereas they defined things in terms of stabilization, and try to show that that's indecomposables.
- Mike Hill also has an idea about equivariant Γ-homology.
- Motivic stuff.

Sarah: We want a good theory of secondary operations of power operations. Right now it's pretty ad hoc. This is one of Tyler's questions in his problem list in his paper.

We want to study Dyer-Lashoff operations acting on the p-primary MU dual Steenrod algebra $\pi_*(H\mathbb{F}_p \wedge_{MU} H\mathbb{F}_p)$. A homotopical framework for secondary operations?

Dylan: There's a free-forgetful adjunction $U : \mathrm{Alg}_{E_\infty/H\mathbb{F}_2} \rightleftarrows \mathrm{Sp} : F$. Then $\mathrm{End}(U)$ is a set which is actually an infinite loop space, so you get a spectrum, and it's really an A_∞-ring spectrum. As spectra, there is an equivalence

$$\mathrm{End}(U) \simeq \mathrm{holim}_n\, \Sigma^{-n}\overline{\mathbb{P}}_{H\mathbb{F}_2}(H \wedge S^n).$$

(This is "easy" Goodwillie calculus – use co-linearization.) The homotopy groups of this is (a completed version of) Dyer-Lashof operations. Massey products are secondary operations. $\overline{\mathbb{P}}(X)$ is something like $\bigvee X^{\wedge k}_{h\Sigma_k}$. If I just take the $k = 2$ piece and call that $\mathbb{P}^2$, $\mathrm{holim}_n\, \Sigma^{-n}\mathbb{P}^2_{H\mathbb{F}_2}(H \wedge S^n)$ is (some suspension of) the Tate spectrum $H^{t\Sigma_2}$. All of this is stolen from lecture notes of Lurie. (I know people arbitrarily attribute things to Lurie even when he wasn't the first. While it is true that connections between Tate constructions, Goodwillie calculus, and power operations abound in the literature, going back to Singer and also Kuhn-McCarty, I have found no other reference that proceeds in the way I described above.) Anyway: I just really want to advertise this A_∞ ring as an object of study. [A warning: this approach for defining operations more generally will only ever see *additive* operations. Since we're looking at $H\mathbb{F}_2$, all the operations are additive anyway, and we can't tell the difference.]

??: People want to take a spectrum and do things like adjoin roots of unity etc., and in general there's no guarantee this preserves E_∞ structure structure. If you have an E_∞ thing and Bousfield localize, then you get an E_∞ thing, but this is not true equivariantly.

Özgür: Pirashvili has an obstruction theory to formality of dga's; the obstruction groups are the same as our obstruction theories. The obstructions are somewhat identifiable; define an ∞-dga using some structure maps, and those maps correspond to cochains. So for dga's people actually knew what those obstructions were.

Paul: $BP\langle 1\rangle$ is E_∞ and $BP\langle 2\rangle$ is E_∞. This has to stop at some point. What is this point? In general $BP\langle n\rangle$ is assembled out of Morava E-theories, which are really nice. There has to be some story about how there's a problem in assembling these things.

Dylan: Math is hard.

Maria: Math is beautiful. Life is hard. □