A course on

Lie Algebras

and their Representations

Taught by C. Brookes

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Disclaimer

These are my notes from Prof. Brookes' Part III course on Lie algebras, given at Cambridge University in Michaelmas term, 2012. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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Lecture 1: October 5

Chapter 1: Introduction

Groups arise from studying symmetries. Lie algebras arise from studying infinitesimal symmetries. Lie groups are analytic manifolds with continuous group operations. Algebraic groups are algebraic varieties with continuous group operations.

Associated with a Lie group G is the tangent space at the identity element T_1G; this is endowed with the structure of a Lie algebra. If $G = GL_n(\mathbb{R})$, then $T_1G \cong M_{n\times n}(\mathbb{R})$. There is a map

$$exp : \text{nbd. of } 0 \text{ in } M_n(\mathbb{R}) \to \text{nbd. of } 1 \text{ in } GL_n(\mathbb{R}).$$

This is a diffeomorphism, and the inverse is log.

For sufficiently small x, y, we have $exp(x)exp(y) = exp(\mu(x, y))$ for some power series

$$\mu(x, y) = x + y + \lambda(x, y) + \text{terms of higher degree},$$

where λ is a bilinear, skew-symmetric map $T_1G \times T_1G \to T_1G$. We write $[x, y] = 2\lambda(x, y)$, so that

$$exp(x)exp(y) = exp(x + y + \frac{1}{2}[x, y] + \cdots).$$

The bracket is giving the first approximation to the non-commutativity of exp.

Definition 1.1. A Lie algebra L over a field k is a k-vector space together with a bilinear map $[-, -] : L \times L \to L$ satisfying

(1) $[x, x] = 0$ for all $x \in L$;
(2) Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Remark 1.2.

(1) This is a non-associative structure.
(2) In this course, k will almost always be $\mathbb{C}$. Later (right at the end), I may discuss characteristic p.
(3) Assuming the characteristic is not 2, then condition (1) in the definition is equivalent to
$(1)'$ $[x, y] = -[y, x]$.
(Consider $[x + y, x + y]$.)
(4) The Jacobi identity can be written in all sorts of ways. Perhaps the best way makes use of the next definition.

Definition 1.3. For $x \in L$, the adjoint map $ad_x : L \to L$ sends $y \mapsto [x, y]$.

Then the Jacobi identity can be written as

Proposition 1.4.

$$ad_{[x,y]} = ad_x \circ ad_y - ad_y \circ ad_x.$$

Example 1.5. The basic example of a Lie algebra arises from using the commutator in an associative algebra, so $[x, y] = xy - yx$.

If $A = M_n(k)$, then the space of $n \times n$ matrices has the structure of a Lie algebra with Lie bracket $[x, y] = xy - yx$.

Definition 1.6. A Lie algebra homomorphism $\varphi : L_1 \to L_2$ is a linear map that preserves the Lie bracket: $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

Note that the Jacobi identity is saying that $ad : L \to End_k(L)$ with $x \mapsto ad_x$ is a Lie algebra homomorphism when $End_k(L)$ is endowed with Lie bracket given by the commutator.

Notation 1.7. We often write $\mathfrak{gl}_n(L)$ instead of $End_k(L)$ to emphasize we're considering it as a Lie algebra. You might write $\mathfrak{gl}(V)$ instead of $End_k(V)$ for a k-vector space V.

Quite often if there is a Lie group around one writes $\mathfrak{g}$ for T_1G.

Definition 1.8. The *derivations* $\mathrm{Der}_k A$ of an associative algebra A (over a base field k) are the linear maps $D : A \to A$ that satisfy the Leibniz identity $D(ab) = aD(b) + bD(a)$.

For example, $D_x : A \to A$ sending $a \mapsto xa - ax$ is a derivation. Such a derivation (one coming from the commutator) is called an *inner derivation*. Clearly if A is commutative then all inner derivations are zero.

Example/ Exercise 1.9. Show that

$$\mathrm{Der}_k k[X] = \{f(x)\tfrac{d}{dx} \ : \ f(x) \in k[X]\}.$$

If you replace the polynomial algebra with Laurent polynomials, you get something related to a Virasoro algebra.

One way of viewing derivations is as the first approximation to automorphisms. Let's try to define an algebra automorphism $\varphi : A[t] \to A[t]$ which is $k[t]$-linear (and has $\varphi(t) = t$), where $A[t] = \sum_{i=0}^{\infty} At^i$. Set

$$\varphi(a) = a + \varphi_1(a)t + \varphi_2(a)t^2 + \cdots$$
$$\varphi(ab) = ab + \varphi_1(ab)t + \varphi_2(ab)t^2 + \cdots .$$

For φ to be a homomorphism we need $\varphi(ab) = \varphi(a)\varphi(b)$. Working modulo t^2 (just look at linear terms), we get that φ_1 is necessarily a derivation. On the other hand, it is not necessarily the case that we can "integrate" our derivation to give such an automorphism.

For us the important thing to notice is that $\mathrm{Der}_k A$ is a Lie algebra using the Lie bracket inherited from commutators of endomorphisms.

Lecture 2: October 8

Core material: Serre's *Complex semisimple Lie algebras*.

RECALL: Lie algebras arise as (1) the tangent space of a Lie group; (2) the derivations of any associative algebra; (3) an associative algebra with the commutator as the Lie bracket.

Definition 2.1. If L is a Lie algebra then a k-vector subspace L_1 is a *Lie subalgebra* of L if it is closed under the Lie bracket.

Furthermore, L_1 is an ideal of L if $[x, y] \in L_1$ for any $x \in L_1, y \in L$. In this case we write $L_1 \triangleleft L$.

Exercise 2.2. Show that if $L_1 \triangleleft L$ then the quotient vector space L/L_1 inherits a Lie algebra structure from L.

Example 2.3. $\mathrm{Der}_k(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. The inner derivations $Innder(A)$ form a subalgebra of $\mathrm{Der}_k(A)$.

Exercise 2.4. Are the inner derivations an ideal of $\mathrm{Der}_k(A)$?

Remark 2.5. The quotient $\mathrm{Der}_k(A)/Innder_k(A)$ arises in cohomology theory. The cohomology theory of associative algebra is called Hochschild cohomology. $H^1(A, A) = \mathrm{Der}_k(A)/Innder_k(A)$ is the first Hochschild cohomology group.

The higher Hochschild cohomology groups arise in deformation theory/ quantum algebra. We deform the usual product on $A[t]$ (or $A[[t]]$) inherited from A with t central to give other 'star products' $a * b = ab + \psi_1(a, b)t + \psi_2(a, b)t^2$. We want our product to be associative; associativity forces Hochschild cohomology conditions on the ψ_i.

Definition 2.6. A Lie algebra L is *abelian* if $[x, y] = 0$ for all $x, y \in L$. (Think abelian $\implies$ trivial commutator.)

Example 2.7. All one-dimensional Lie algebras have trivial Lie brackets.

Example 2.8. Every one-dimensional vector subspace of a Lie algebra is an abelian subalgebra.

Definition 2.9. A Lie algebra is *simple* if

(1) it is not abelian;
(2) the only ideals are 0 and L.

Note the slightly different usage compared with group theory where a cyclic group of prime order is regarded as being simple.

One of the main aims of this course is to discuss the classification of finite-dimensional complex simple Lie algebras. There are four infinite families:

- type A_n for $n \geq 1$: these are $\mathfrak{sl}_{n+1}(\mathbb{C})$, the Lie algebra associated to the special linear group of $(n + 1) \times (n + 1)$ matrices of determinant 1 (this condition transforms into looking at trace zero matrices);

- type B_n for $n \geq 2$: these look like $\mathfrak{so}_{2n+1}(\mathbb{C})$, the Lie algebra associated with $SO_{2n+1}(\mathbb{C})$;
- type C_n for $n \geq 3$: these look like $\mathfrak{sp}_{2n}(\mathbb{C})$ (symplectic $2n \times 2n$ matrices);
- type D_n for $n \geq 4$: these look like $\mathfrak{so}_{2n}(\mathbb{C})$ (like B_n but of even size).

For small n, $A_1 = B_1 = C_1$, $B_2 = C_2$, and $A_3 = D_3$, which is why there are restrictions on n in the above groups. Also, D_1 and D_2 are not simple (e.g. D_1 is 1-dimensional abelian).

In addition to the four infinite families, there are five exceptional simple complex Lie algebras: E_6, E_6, E_8, F_4, G_2, of dimension $78, 133, 248, 52, 14$ respectively. G_2 arises from looking at the derivations of Cayley's octonions (non-associative).

Definition 2.10. Let L_0 be a real Lie algebra (i.e. one defined over the reals). The *complexification* is the complex Lie algebra

$$L = L_0 \otimes_{\mathbb{R}} \mathbb{C} = L_0 + iL_0$$

with the Lie bracket inherited from L_0.

We say that L_0 is a real form of L. For example, if $L_0 = \mathfrak{sl}_n(\mathbb{R})$ then $L = \mathfrak{sl}_n(\mathbb{C})$.

Exercise 2.11. L_0 is simple $\iff$ the complexification L is simple OR L is of the form $L_1 \times \overline{L_1}$, in which case L_1 and $\overline{L_1}$ are each simple.

In fact, each complex Lie algebra may be the complexification of several non-isomorphic real simple Lie algebras.

Before leaving the reals behind us, note the following theorems we will not prove:

Theorem 2.12 (Lie). *Any finite-dimensional real Lie algebra is isomorphic to the Lie algebra of a Lie group.*

Theorem 2.13. *The categories of finite-dimensional real Lie algebras, and of connected simply-connected Lie groups, are equivalent.*

Chapter 2: Elementary properties, nilpotent and soluble Lie algebras

Remark 2.14. Most of the elementary results are as you would expect from ring theory; e.g. the isomorphism theorems.

If θ is a Lie algebra homomorphism $\theta : L_1 \to L_2$, then $\ker \theta$ is an ideal of L_1, and $\operatorname{im} \theta$ is a subalgebra of L_2. We can define the quotient Lie algebra $L/\ker \theta$ and it is isomorphic to $\operatorname{im} \theta$.

Definition 2.15. (2.1) The *centre* $Z(L)$ is a Lie algebra defined as

$$\{x \in L \,:\, [x, y] = 0 \;\forall y \in L\}.$$

Exercise 2.16. Classify all 2-dimensional Lie algebras.

Lecture 3: October 10

Definition 3.1. (2.2) The *derived series* of a Lie algebra is defined inductively (as for groups):

$$L^{(0)} = L,$$
$$L^{(1)} = [L, L],$$
$$L^{(i)} = [L^{(i-1)}, L^{(i-1)}],$$

and $[A, B]$ is the set of finite sums of $[a, b]$ for $a \in A, b \in B$ (that is, a typical element is $\sum_i [a_i, b_i]$).

Note that $L^{(i)} \triangleleft L$ (i.e. these are ideals).

Thus, L is abelian $\iff L^{(1)} = 0$.

Definition 3.2. (2.3) A Lie algebra is *soluble* if $L^{(r)} = 0$ for some r. The smallest such r is called the *derived length* of L.

Lemma 3.3. *(2.4)*

1. *Subalgebras and quotients of soluble Lie algebras are soluble.*
2. *If $J \triangleleft L$ and if J and L/J are soluble, then L is soluble.*
3. *Soluble Lie algebras cannot be simple.*

Proof. (1) If L_1 is a subalgebra of L then $L_1^{(r)} \leq L^{(r)}$ (by induction). If $J \triangleleft L$ then $(L/J)^{(r)} = (L^{(r)} + J)/J$.

(2) If L/J is soluble then $L^{(r)} \leq J$ for some r. If J is soluble then $J^{(s)} = 0$ for some s. But $L^{(r+s)} = (L^{(r)})^{(s)} \leq J^{(s)} = 0$.

(3) By definition if L is simple it is nonabelian and so $L^{(1)} \neq 0$. But if L is soluble and nonzero then $L \supsetneq L^{(1)}$. Thus $L^{(1)}$ would be a nonzero ideal... which is a contradiction. $\square$

Example 3.4. (2.5a) One-dimensional subspaces of Lie algebras are abelian, and abelian Lie algebras are soluble.

Example 3.5. (2.5b) All 2-dimensional Lie algebras are soluble. Take a basis $\{x, y\}$. Then $L^{(1)}$ is spanned by $[x, x], [x, y], [y, x], [y, y]$, and hence by $[x, y]$. Thus $L^{(1)}$ is either 0- or 1-dimensional according to whether $[x, y] = 0$ or not. So $L^{(1)}$ is abelian, and $L^{(2)} = 0$. We've essentially classified 2-dimensional Lie algebras (2 types, depending on whether $[x, y] = 0$ or not).

Exercise 3.6. Classify 3-dimensional Lie algebras. If you get stuck, look in Jacobson.

Example 3.7. (2.5c) Let L have basis $\{x, y, z\}$ with $[x, y] = z$, $[y, z] = x$, $[z, x] = y$. If $k = \mathbb{R}$, this is $\mathbb{R}^3$ with $[x, y] = x \wedge y$ (formal vector product). Then

$$L = L^{(1)} = L^{(2)} = \cdots$$

and L is not soluble. The centre (stuff that commutes with everything) $Z(L) = 0$. Under the adjoint map $ad : L \to \mathfrak{gl}_3$,

$$x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$y \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$z \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

You can see that the image of ad is the space of 3×3 skew-symmetric matrices. This is the tangent space at 1 for O_3 if $k = \mathbb{R}$.

Observe that $Z(L)$ is the kernel of ad. When the centre is zero, ad is injective. (This is why $L \cong$ the space of skew-symmetric 3×3 matrices in the previous example.)

Example 3.8. (2.5d) Let

$$\mathfrak{n}_n = \{ \text{ strictly upper-triangular } n \times n \text{ matrices over } k \ \}.$$

This is a Lie subalgebra of $\mathfrak{gl}_n(k)$. It is soluble but nonabelian if $n \geq 3$. (It is a prototype for nilpotent Lie algebras.)

$$\mathfrak{b}_n = \{ \text{ upper triangular } n \times n \text{ matrices } \}$$

Clearly $\mathfrak{n}_n \subset \mathfrak{b}_n$. Check that it's actually an ideal. $\mathfrak{b}/\mathfrak{n}$ is abelian, and $\mathfrak{b}$ is soluble.

Definition 3.9. (2.6) A finite-dimensional Lie algebra L is *semisimple* if L has no nonzero soluble ideals.

Thus simple $\implies$ semisimple (since soluble Lie algebras can't be simple).

Definition 3.10. (2.7) The *lower central series* of a Lie algebra L is defined inductively:

$$L_{(1)} = L,$$
$$L_{(i)} = [L, L_{(i-1)}] \text{ for } i > 1.$$

L is *nilpotent* if $L_{(r)} = 0$, and the *nilpotency class* is the least such r.

Note that each $L_{(i)} \triangleleft L$, and that the numbering starts at 1 instead of 0 as with the derived series. Note that $\mathfrak{n}_n$ is nilpotent in Example **2.5d**, and $\mathfrak{b}_n$ is non-nilpotent if $n \geq 2$.

Lemma 3.11. *(2.8)*

(1) *Subalgebras and quotients of nilpotent Lie algebras are nilpotent.*
(2) *L is nilpotent of class $\leq r \iff [x_1, [x_2, \cdots, x_r]] = 0$ for all $x_i \in L$.*
(3) *There are proper inclusions of classes*

$$\{ \text{ abelian Lie algebras } \} \subsetneq \{ \text{ nilpotent Lie algebras } \} \subsetneq \{ \text{ soluble Lie algebras } \}$$

Proof. *(1) Show that homomorphic images of nilpotent Lie algebras are nilpotent.*

(2) Definition.

(3) If $L_{(r)} = 0$ then $L_{(2^r)} = 0$ and so $L^{(r)} = 0$ by **2.9***(3).*

□

Note that $\mathfrak{b}$ is non-nilpotent, but $\mathfrak{n}$ and $\mathfrak{b}/\mathfrak{n}$ are nilpotent.

PROPOSITION 3.12. *(2.9)*

(1) $[L_{(i)}, L_{(j)}] \subset L_{(i+j)}$.
(2) *Lie brackets of r elements of L in any order lies in $L_{(r)}$. e.g.* $[[x_1, x_2], [x_5, x_4]]$.
(3) $L^{(k)} \subset L_{(2^k)}$ *for all k.*

Proof. *(1) Induction on j. $(L_{(i)}, L_{(1)}) = (L_{(i)}, L) \subset L_{(i+1)}$ by definition.*

Assume the result is true for all i, and for a specific j. Want $[L_{(i)}, L_{(j+1)}] \subset L_{(i+j+1)}$. But

$$\begin{aligned}
[L_{(i)}, L_{(j+1)}] &= [L_{(i)}, [L_{(j)}, L]] \\
&= -[L_{(j)}, \underbrace{[L, L_{(i)}]}_{\subset L_{(i+1)}}] - [L, \underbrace{[L_{(i)}, L_{(j)}]}_{\subset L_{(i+j)}}] \\
&\subset [L_{(i+1)}, L_{(j)}] + [L, L_{(i+j)}] \\
&\subset L_{(i+1+j)} + L_{(1+i+j)}
\end{aligned}$$

where the first term in the last line is by induction (and the second term in the last line is by definition).

(2) By induction on r. If $r = 1$, $L_{(1)} = L$ and there is nothing to do. Write any bracket of r terms as $[y, z]$ where y contains brackets of i terms and z contains brackets fo j terms, where $i + j = r$ and $i, j > 0$. By induction, $[y, z] \in [L_{(i)}, L_{(j)}]$ which is $\subset L_{(i+j)} = L_{(r)}$ by part (1).

(3) By induction on k. For $k = 0$, we have $L = L^{(0)}$ and $L_{(2^0)} = L_{(1)} = L$.

Now assume inductively that $L^{(k)} \subset L_{(2^k)}$ for some k. Then

$$L^{(k+1)} = [L^{(k)}, L^{(k)}] \subset L_{(2^{k+1})}$$

by part (1). □

DEFINITION 3.13. (2.10) The *upper central series* is defined inductively:

$$Z_0(L) = 0;$$
$$Z_1(L) = Z(L) \text{ is the centre;}$$

$$Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L)).$$

Alternative definition:

$$Z_i(L) = \{x \in L : [x, y] \in Z_{i-1} \ \forall y \in L\}$$

Exercise: does this work and why?

Definition 3.14. (2.11) A *central series* in L consists of ideals J_i,

$$0 = J_1 < \cdots < J_r = L$$

for some r, with $[L, J_i] \leq J_{i-1}$ for each i.

Exercise 3.15. If we have such a central series then there are Z_i with $L_{r+i-1} \leq J_i \leq Z_i(L)$. So the lower central series is the one that goes down as fast as it can, and the upper central series is the one that goes up as fast as it can.

Exercise 3.16. If L is a Lie algebra then $\dim(L/Z(L)) \neq 1$. (similar theorem in group theory)

Lecture 4: October 12

Last time we met $\mathfrak{n}_n$ (strictly upper triangular $n \times n$ matrices), and $\mathfrak{b}_n$, the upper triangular $n \times n$ matrices. These were the "prototype" examples of nilpotent and soluble Lie algebras. Our next target is to show that, if we have a Lie subalgebra L of $\mathfrak{gl}(V)$, where V is an n-dimensional vector space:

(1) if each element of L is a nilpotent endomorphism then L is a nilpotent Lie algebra (Engel);
(2) we can pick a basis of V so that so that if L is soluble then $L \leq \mathfrak{b}_n$, and if L is nilpotent then $L \leq \mathfrak{n}_n$. (Lie/ Engel).

Another way of putting this is in terms of flags.

Definition 4.1. A chain

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

with $\dim V_j = j$ is called a *flag* of V.

Restating (2), we have that, if L is a soluble Lie subalgebra of $\mathfrak{gl}(V)$ then there is a flag with $L(V_i) \leq V_i$ for each i, and if L is nilpotent then there is a flag with $L(V_i) \leq V_{i-1}$ for each i.

A theorem of Engel. Before proving Engel's family of results we need some preparation.

Definition 4.2. The *idealiser* of a subset S of a Lie algebra L is

$$Id_L(S) = \{y \in L : [y, S] \subset S\}$$

This is also called the *normaliser*.

If S is a subalgebra then $S \leq Id_L(S)$.

We say that L *has the idealiser condition* if every proper subalgebra has a strictly larger idealiser.

LEMMA 4.3. *(2.13) If L is a nilpotent Lie algebra, then it has the idealiser condition.*

Proof. Let K be a proper Lie subalgebra of L, and let r be minimal such that $Z_r(L) \not\leq K$. (Note: for nilpotent L, $L = Z_s(L)$ for some s. Exercise: if s is the smallest such index, then the nilpotency class is $s+1$.)

We have $r \geq 1$, and $[K, Z_r] \leq [L, Z_r] \leq Z_{r-1} \leq K$ (last $\leq$ by minimality of r). So $K < K + Z_r \leq Id_L(K)$. $\square$

DEFINITION 4.4. A *representation* ρ of a Lie algebra L is a Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$ for some vector space V. If $\dim V$ is finite then $\dim V$ is the degree of ρ.

For example, the adjoint map $ad : L \to \mathfrak{gl}(L)$ is a representation; it is sometimes called the *regular representation*. We want to consider nilpotent endomorphisms.

LEMMA 4.5. *(2.15) If $x \in L \leq \mathfrak{gl}(V)$ and $x^r = 0$ (composition of endomorphisms), then $(ad_x)^m = 0$ for some m, in $End_*(L)$.*

Proof. Let θ be premultiplication by x in $End_k(L)$, and φ postmultiplication. Then $ad_x = \theta - \varphi$. By assumption, $\theta^r = 0 = \varphi^r$. Also note that $\theta\varphi = \varphi\theta$. So $(ad_X)^{2r} = 0$. $\square$

The key stage in proving Engel's results is:

PROPOSITION 4.6. *(2.16) Let $L \leq \mathfrak{gl}(V)$ where V is an n-dimensional vector space. Suppose each $x \in L$ is a nilpotent endomorphism of V. Then there exists nonzero $v \in V$* such that $L(v) = 0$. (i.e. there is a common eigenvector)

Proof. Induction on $\dim L$. If $\dim L = 1$, say L is spanned by x, then we can take v to be an eigenvector of x, noting that 0 is the only eigenvalue of a nilpotent endomorphism. Assume $\dim L > 1$.

Claim: L satisfies the idealiser condition. Let $0 \neq A \subsetneq L$ be a subalgebra; we want to show $[A + x, A] \subset A$ for some $x \notin A$. Consider $\rho : A \to \mathfrak{gl}(L)$ taking $a \mapsto (ad_a : x \to [a, x])$. Since A is a subalgebra, there is an induced representation

$$\overline{\rho} : A \to \mathfrak{gl}(L/A) \text{ where } a \mapsto (\overline{ad_a} : x + A \to [a, x] + A).$$

By **2.15**, each ad_a is nilpotent and so $\overline{ad_a}$ is nilpotent. Note that $\dim \overline{\rho}(A) \leq \dim A < \dim L$. Because $\dim L/A < \dim L$, we may inductively assume the Proposition is true for $\overline{\rho}(A) \subset \mathfrak{gl}(L/A)$; therefore, there exists $x \in L/A$ with $\overline{\rho}(a)(x + A) = A$ for all $a \in A$. That is, $[a, x] \in A$ for all $a \in A$ and $x \in Id_L(A)\backslash A$. Thus the idealiser is strictly larger.

Now let M be a maximal (proper) subalgebra of L. By the claim, $Id_L(M) = L$, and thus M is an ideal of L. So $\dim(L/M) = 1$ – otherwise, L/M would have a proper non-zero subalgebra; pulling back to L would be contradicting maximality of M. Thus L is the span of M and x, for some $x \in L$.

Consider $U = \{u \in V : M(u) = 0\}$. By induction, since $\dim M < \dim L$, we know that $U \neq \{0\}$. (Elements of M have a common eigenvector.) For $u \in U, m \in M$,

$$m(x(u)) = ([m, x] + xm)(u) = 0$$

since $[m, x] \in M$ (because $M \triangleleft L$). So $x(u) \in U$, for all $u \in U$. Take $v \neq 0$ in U such that $x(v) = 0$. But L is the span of M and x, so $L(v) = 0$ as required. □

Corollary 4.7. *(2.17) For $L \leq \mathfrak{gl}(V)$ as in* **2.16**. *There exists a flag $0 = V_0 \subsetneq \cdots \subsetneq V_n = V$ such that $L(V_j) \leq V_{j-1}$ for each j. In particular, L is nilpotent.*

Proof. By induction. Use **2.16** to produce v. Take $V_1 = \langle v \rangle$. Then consider the image of L in $\mathfrak{gl}(V/V_1)$ and apply the inductive hypothesis. □

Theorem 4.8 (Engel). *(2.18) For a finite-dimensional Lie algebra L :*

$$L \text{ is nilpotent} \iff ad_x \text{ is nilpotent } \forall x \in L.$$

Proof. ($\implies$) follows from **2.9**.

($\impliedby$) $ad : L \to \mathfrak{gl}(L)$. If ad_x is nilpotent for all x then the image $ad(L)$ satisfies the conditions of **2.16** and **2.17** and so $ad(L)$ is nilpotent. But $ad(L) \cong L/\ker(ad) = L/Z(L)$. So L is nilpotent. □

Lecture 5: October 15

Exercise 5.1. For finite-dimensional Lie algebras the following are equivalent.

(1) L is nilpotent
(2) L satisfies the idealiser condition
(3) maximal (proper) subalgebras of L are ideals of L.

There is a similar statement for finite groups, replacing idealiser with normaliser.

Theorem 5.2 (Engel). *(2.19) Suppose L is a Lie subalgebra of $\mathfrak{gl}(V)$, where V is a finite-dimensional vector space, and each $x \in L$ is a nilpotent endomorphism.*

Then L is a nilpotent Lie algebra.

Before going on to consider Lie's theorem about common eigenvectors for soluble Lie algebras, let's digress and think about derivations of Lie algebras.

Definition 5.3. $D \in End_k(L)$ is a derivation of a Lie algebra L if:

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

The space $Der(L)$ of derivations forms a Lie subalgebra of $\mathfrak{gl}(L)$.

The inner derivations are those of the form ad_z for $z \in L$.

Thus the image of $ad : L \to \mathfrak{gl}(L)$ is precisely the space of inner derivations; one could say $Innder(L) = ad(L)$.

PROPOSITION 5.4. *(2.21) Let L be a finite-dimensional nilpotent Lie algebra. Then $Der(L) \neq ad(L)$ (i.e. there is an "outer" derivation).*

EXERCISE 5.5. Show that $Der(L) = ad(L)$ if L is soluble and nonabelian of dimension 2.

REMARK 5.6. We'll see later that, for semisimple finite-dimensional Lie algebras of characteristic zero, then $Der(L) = ad(L)$.

Thus for semisimple Lie algebras, $ad : L \to Der(L)$ is an isomorphism. (Recall that $\ker(ad) = Z(L)$, and this is a soluble ideal. In the semisimple case, this has to be zero, which gives injectivity.)

REMARK 5.7. $Der(L)/ad(L)$ is the first Lie algebra cohomology group.

Proof of Proposition **2.21**. Assume L is nilpotent. So it has the idealiser condition (by **2.13**). So if M is a maximal subalgebra, it is an ideal of L (saw this in the proof of Engel's theorem), of codimension 1 ($\dim L/M = 1$) and $M \geq L^{(1)}$. Choose $x \in L \backslash M$. So L is spanned by M and x. Let

$$C = \{y \in L \,:\, [y, M] = 0\}.$$

Then $0 \subsetneq Z(M) \subset C \triangleleft L$ (first proper inclusion by the nilpotency of M). *Nilpotency + Proposition* **2.9***(2) says that $[m_1, [m_2, [m_3, \cdots]]] = 0$ for any chain of r things. So just pick a chain of $r-1$ things.* (C is an ideal by Jacobi identity and the fact that $M \triangleleft L$.) Set r to be the biggest index such that $L_{(r)} \geq C$. Pick $c \in C \backslash L_{(r+1)}$. The map

$$D : m + \lambda x \mapsto \lambda c$$

is linear, well-defined, and $D(M) = 0$. You need to check that D is a derivation.

We need to show this is not inner. Suppose $D = ad_t$ for some $t \in L$. Then $D(M) = 0$ means that $[t, M] = 0$ and so $t \in C \leq L_{(r)}$. Thus $c \in D(x) = [t, x] \in L_{(r+1)}$, which is a contradiction. $\square$

We'll come back to nilpotent Lie algebras in due course, when we'll define Cartan subalgebras of finite-dimensional Lie algebras. By definition, these are nilpotent subalgebras that are equal to their own idealiser (in L). We find that those are precisely the centralisers of "regular" elements of L, and this is where some geometry appears – the set of regular elements form a connected, dense open subset of L. When L is semisimple these Cartan subalgebras are abelian.

The basic result for Lie's theorem is

Proposition 5.8. *(2.22) Let $k = \mathbb{C}$ (we need algebraic closure). Let L be a soluble Lie subalgebra of $\mathfrak{gl}(V)$. If $V \neq 0$ then it contains a common eigenvector for all the elements of L.*

Proof. Induction on $\dim L$. If $\dim L \leq 1$, we're done by the fact that eigenvectors exist over an algebraically closed field.

Suppose $\dim L > 1$ and the result is true for smaller dimension L. Since L is soluble and non-zero, then $L^{(1)} \subsetneq L$. Let M be a maximal subalgebra in L containing $L^{(1)}$. Then $M \lhd L$, and $\dim L/M = 1$. As before, pick $x \in L \backslash M$, and so L is spanned by M and x. M is strictly smaller than L, and it inherits solubility, so apply the inductive hypothesis. Find a nonzero eigenvector $u \in V$ with $m(u) = \lambda(m)u$ for all $m \in M$ Let $u_1 = u$ and $u_{i+1} = x(u_i)$. Set U_i to be the span of the u_i. Set n to be maximal such that $u_1, \cdots, u_n$ are linearly independent.

Claim 5.9. *Each $M(U_i) \leq U_i$ for each i, and $m(u_i) \equiv \lambda(m)u_i \pmod{U_{i-1}}$ for $i \leq n$.*

(Thus if we use the basis $u_1, \cdots, u_n$ of U_n then $m \in M$ is represented by a triangular matrix, with $\lambda(m)$ on all the diagonal entries.)

Proof. By induction. If $i = 1$ use the single eigenvector from before.

If $m(u_i) = \lambda(m)u_i \pmod{U_{i-1}}$ then $x(m(u_i)) \equiv \lambda(m)x(u_i) \pmod{U_i}$, and $m(u_{i+1}) = m(x(u_i)) = (\underbrace{[m,x]}_{=mx-xm} + xm)(u_i) \equiv \lambda(m)u_{i+1} \pmod{U_i}$. □

So, with this good choice of basis we can make every $m \in M$ into a triangular matrix. For $m \in L^{(1)}$ these matrices have trace zero (commutators have trace zero), but *trace* $= n \cdot \lambda(m)$. Since $\operatorname{char} k = 0$, we have $\lambda(m) = 0$ for $m \in L^{(1)}$.

Claim 5.10. *All the u_i's are eigenvectors: $m(u_i) = \lambda(m)u_i$.*

Proof of claim. Again by induction. So assume $m(u_i) = \lambda(m)u_i$. Then we have

$$m(u_{i+1}) = ([m,x] + xm)(u_i) = xmu_i = \lambda(m)u_{i+1}.$$

□

Thus U_n is an eigenspace for M, invariant under x: $x(U_n) \leq U_n$. So pick an eigenvector for x in U_n; this is necessarily a common eigenvector for L. □

Lecture 6: October 17

Last notes about solubility and Lie's theorem.

Corollary 6.1 (Lie's theorem). *(2.23) For a soluble complex Lie subalgebra L of $\mathfrak{gl}(V)$ there is a flag $\{V_i\}$ such that $L(V_j) \leq V_j$.*

Proposition 6.2. *(2.24) Let L be a finite-dimensional soluble complex Lie algebra. Then there is a chain of ideals*

$$0 = J_0 < J_1 < \ldots < J_n = L$$

with $\dim J_i = i$.

Proof. Apply Corollary **2.23** to $ad(L)$; the subspaces in the flag are ideals of L. □

Proposition 6.3. *(2.25) If L is a soluble complex Lie algebra, then $L^{(1)}$ is nilpotent. Indeed, if $x \in L^{(1)}$ then $ad_x : L \to L$ is nilpotent.*

Proof. *Take a chain of ideals as in Proposition* **2.24***, and choose a basis of L such that $x_i \in J_i$ for every i. Then ad_x is upper triangular with respect to this basis, and so $[ad_x, ad_x]$ is strictly upper triangular, hence nilpotent.* □

Definition 6.4. $\rho : L \to \mathfrak{gl}(V)$ for a vector space is an *irreducible representation* if no subspace $0 \leq W < V$ is invariant under $\rho(L)$.

Corollary 6.5 (Corollary to Lie's theorem). *(2.27) Irreducible representations of a finite-dimensional soluble complex Lie algebra are all 1-dimensional.*

Proof. Suppose ρ is irreducible. Then the common eigenvector of $\rho(L)$ from Lie's theorem spans a 1-dimensional subspace $V_1 \leq V$ with $\rho(L)V_1 \leq V_1$. Irreducibility implies that $V_1 = V$. □

Remark 6.6. Recall that not all irreducible representations of finite soluble groups are 1-dimensional.

If $G \subset GL(V)$ is a soluble subgroup, and k is algebraically closed (say, $\mathbb{C}$), there exists a subgroup of finite index which, w.r.t. some basis, is represented by triangular matrices.

Remark 6.7 (Cartan's Criterion). Let V be a finite-dimensional vector space, $L \leq \mathfrak{gl}(V)$. Then L is soluble iff $tr(\underbrace{x \circ y}_{\substack{\text{composition}\\ \text{of end.}}}) = 0$ for all $x \in L, y \in L^{(1)}$. The $\Longrightarrow$ direction you can do as an exercise. Note that $L \times L \to k$ sending $(x, y) \mapsto tr(xy)$ is a symmetric bilinear form (called a "trace form").

Remark 6.8. We'll meet the Killing form

$$B : L \times L \to k \text{ where } (x, y) \mapsto tr(ad_x ad_y)$$

which plays a fundamental role in the theory of semisimple Lie algebras, because it is nondegenerate when L is semisimple. It also has the property that

$$B([x, y], z) = B(x, [y, z])$$

(i.e. it is "invariant").

Chapter 3: Cartan subalgebras and the Killing form

Definition 6.9. Define

$$L_{\lambda,y} = \{x \in L \,:\, (ad_y - \lambda \cdot Id)^r x = 0 \text{ for some } r\}$$

is the generalised λ-eigenspace. Note that $y \in L_{0,y}$ since $[y,y] = 0$.

Lemma 6.10. *(3.2)*

(1) *$L_{0,y}$ is a Lie subalgebra of L.*
(2) *If L is finite-dimensional and $L_{0,y} \leq$ a subalgebra A of L, then $I_L(A) = A$. In particular, using (1), we know that $L_{0,y} = I_L(L_{0,y})$.*

Proof. (1) $L_{0,y}$ is a subspace. If $a, b \in L_{0,y}$ with $(ad_y)^r(a) = 0 = (ad_y)^s(b)$ then Leibniz applies – recall ad_y is a derivation – to give

$$(ad_y)^{r+s}([a,b]) = \sum_{i=0}^{r+s} \binom{r+s}{i} [(ad_y)^i(a), (ad_y)^{r+s-i}(b)] = 0$$

Exercise: show that

$$[L_{\lambda,y}, L_{\mu,y}] \subset L_{\lambda+\mu,y}$$

if $\lambda, \mu \in \mathbb{C}$ and Lemma **3.2**(1) is a special case of this.

(2) Let the characteristic polynomial of ad_y (in variable t) be

$$\det(t \cdot Id - ad_y) = t^m f(t)$$

with $t \nmid f(t)$. Note that $m = \dim L_{0,y}$. $ad_y(y) = 0$ and so 0 is an eigenvalue of ad_y; so $m \geq 1$.

t^m and $f(t)$ are coprime, and so we can write

$$1 = q(t) \cdot t^m + r(t) f(t)$$

for some polynomials $g(t)$ and $r(t)$.

Let $b \in I_L(A)$. So

$$b = q(ad_y)(ad_y)^m(b) + r(ad_y)f(ad_y)(b). \tag{6.1}$$

But $m \geq 1$ and $y \in L_{0,y} \leq A$. So the first term in (6.1) is in A since $b \in Id_L(A)$. By remembering what the characteristic polynomial is, Cayley-Hamilton says that

$$(ad_y)^m f(ad_y)(b) = 0.$$

So $f(ad_y)(b)$ is killed when you apply sufficiently many ad_y's to it; that means that $f(ad_y)(b) \in L_{0,y} \subset A$. And so the second term in (6.1) is in A. Therefore, both terms of b are in A, so $b \in A$. Hence $Id_L(A) = A$.

$\square$

Definition 6.11. The *rank* of a Lie algebra L is the least m such that

$$\det(t \cdot Id - ad_y) = t^m f(t) \text{ where } t \nmid f(t)$$

for some $y \in L$.

Thus the rank of L is the minimal dimension of some $L_{0,y}$.

DEFINITION 6.12. An element is *regular* if $\dim L_{0,y} = rank(L)$.

EXERCISE 6.13. What is the rank of $\mathfrak{sl}_2(\mathbb{C})$ (trace zero matrices)? What about $\mathfrak{sl}_3(\mathbb{C})$?

LEMMA 6.14. *(3.4)*

$$L = \bigoplus_{\lambda} L_{\lambda,y}$$

where the sum is over the eigenvalues of ad_y.

Proof. Standard linear algebra. □

EXERCISE 6.15. (3.5) Let θ and φ be in $End(V)$; let $c \in k$ and let $\theta + c\varphi$ have characteristic polynomial

$$f(t,c) = t^n + f_1(c)t^{n-1} + \ldots + f_n(c)$$

Show that each f_i is a polynomial of degree $\leq i$ (in c).

LEMMA 6.16. *Let K be a subalgebra of L, and $z \in K$ with $L_{0,z}$ minimal in the set $\{L_{0,y} : y \in K\}$ (you can't find anything in $L_{0,z}$ that's also inside the set). Suppose $K \leq L_{0,z}$. Then $L_{0,z} \leq L_{0,y}$ for all $y \in K$.*

DEFINITION 6.17. A Cartan subalgebra (CSA) of L is a nilpotent subalgebra which is equal to its own idealiser.

THEOREM 6.18. *Let H be a subalgebra of a complex Lie algebra L. Then H is a CSA iff H is a minimal subalgebra of type $L_{0,y}$ (there isn't another one of that type inside it).*

LECTURE 7: OCTOBER 19

LEMMA 7.1. *(3.6) Let K be a subalgebra of a complex Lie algebra L, $z \in K$ with $L_{0,z}$ minimal in the set $\{L_{0,y} : y \in K\}$. Suppose $K \leq L_{0,z}$. Then $L_{0,z} \leq L_{0,y}$ for all $y \in K$.*

THEOREM 7.2. *(3.8) A complex Lie algebra H is a CSA $\iff$ H is a minimal subalgebra of the form $L_{0,y}$. In particular CSA's exist.*

Proof of Theorem **3.8** *from Lemma* **3.6**. Suppose H is minimal of type $L_{0,z}$. Then $I_L(H) = H$ by Lemma **3.2**. Take $H = K$ in Lemma **3.6** and we deduce $H = L_{0,z} \leq L_{0,y}$ for all $y \in H$. Thus each $ad_y|_H : H \to H$ is nilpotent. Hence H is nilpotent by Engel's theorem.

Conversely, say H is a CSA. Then $H \leq L_{0,y}$ for all $y \in H$ (using Engel's Theorem **2.18**, since H is nilpotent). Suppose we have strict inequality for all $y \in H$. Choose $L_{0,z}$ as small as possible (this is not an infinite descending chain because we're assuming everything is finite-dimensional). By Lemma **3.6** with $K = H$, $L_{0,z} \leq L_{0,y}$ for all $y \in H$, and so ad_y acts nilpotently on $L_{0,z}$. Thus $\{ad_y : y \in H\}$ induces a Lie subalgebra of $\mathfrak{gl}(L_{0,z}/H)$ with every element nilpotent.

So by Proposition **2.16** (existence of common eigenvector), there exists $x \in L_{0,z} \backslash H$ with $[H, H + x] \leq H$. So $[x, H] \leq H$ and hence $x \in I_L(H) \backslash H$, a contradiction to the CSA condition that $H = Id_L(H)$.

So we can't have strict inequality, and H must be of the form $L_{0,y}$ for some y. It must be minimal, as any proper subalgebra of H satisfies the idealiser condition. So by Lemma **3.2**, the subalgebra couldn't be a $L_{0,z}$. $\square$

Proof of Lemma **3.6**. Fix $y \in K$. Consider the set $S = \{ad_{z+cy} : c \in k\}$. Write $H = L_{0,z}$. Each $z + cy \in K \leq H$ (by supposition). So $S(H) \subset H$ and so elements of S induce endomorphisms of H and L/H. Say $f(t, c)$ and $g(t, c)$ are characteristic polynomials of ad_{z+cy} on H and L/H, respectively. If $\dim H = m$, $\dim L = n$, we have

$$f(t, c) = t^m + f_1(c)t^{m-1} + \cdots + f_m(c)$$
$$g(t, c) = t^{n-m} + g_1(c)t^{n-m-1} + \cdots + g_{n-m}(c)$$

where f_i, g_i are polynomials of degree $\leq i$ (see Exercise **3.5**). Now ad_z has no zero eigenvalues on L/H (because H is the generalised zero eigenspace of things killed by ad, and you're getting rid of that) and so $g_{n-m}(0) \neq 0$. Hence we can find $c_1, \cdots, c_{m+1} \in k$ with $g_{n-m}(c_j) \neq 0$ for each j. Then $g_{n-m}(c_j) \neq 0$ implies that ad_{z+c_jy} has no zero eigenvalues on the quotient L/H. Hence, $L_{0,z+c_jy} \leq H$ *(if there was some generalized eigenvector not in H, then there would be some nonzero generalized eigenvector in $L_{0,z+c_jy}/H$ and hence some eigenvector in $L_{0,z+c_jy}/H$, a contradiction)*.

The choice of z implies $L_{0,z+c_jy} = H$ for each c_j (H was minimal). So 0 is the only eigenvalue of $ad_{z+c_jy}|_H : H \to H$. That is, $f(t, c_j) = t^m$ for $j = 1 \cdots m + 1$. So $f_i(c_j) = 0$ for $j = 1, \cdots, m + 1$. But $\deg f_i < m + 1$ and so f_i is identically 0. So $f(t, c) = t^m$ for all $c \in k$ and so $L_{0,z+cy} \geq H$ for all $c \in k$. But y was arbitrary in K. Replace y by $y - z$ and take $c = 1$. This gives $L_{0,y} \geq H$ for all $y \in K$. $\square$

Theorem 7.3. *(3.9) $L_{0,y}$ is a CSA if y is regular. (Recall that y is regular if $L_{0,y}$ is of minimal dimension.)*

Proof. Immediate. $\square$

Remark 7.4. One could write many of those arguments in terms of the geometry of the space of regular elements.

Proposition 7.5. *(3.10) Let L be a complex Lie algebra. The set L_{reg} of regular elements of L is a connected, dense open subset of L in the Zariski topology.*

Proof. Let m be the rank of L. Then

$$\det(t \cdot Id - ad_y) = t^m(t^{n-m} + \cdots + \underbrace{h(y)}_{\substack{constant\\term}}).$$

Set $V = \{y \in L : h(y) = 0\}$. This is the solution set of a polynomial equation. $h(y)$ is homogeneous of degree $n - m$ given a choice of coordinates. V is Zariski-closed, and it has empty interior since $h(y)$ is not identically zero. Thus $L_{reg} = L\backslash V$ is open and dense. It is also connected *in the Zariski topology* – the complex line including x and $y \in L$ has finite intersection with V, and so x and y must be in the same component (we're using the fact that we're over $\mathbb{C}$).

In the real case, the set of regular elements need not be connected. $\square$

Lemma 7.6. *(3.11) Let $\varphi : L \twoheadrightarrow L_1$ be a surjective Lie algebra homomorphism.*

1. *If H is a CSA of L then $\varphi(H)$ is a CSA of L_1.*
2. *If K is a CSA of L_1, and we define $L_2 = \varphi^{-1}(K)$ then any Cartan subalgebra H of L_2 is a CSA of L.*

Proof. (1) H is nilpotent, so $\varphi(H)$ is nilpotent. We have to show that $\varphi(H)$ is self-idealising. Take $\varphi(x) \in I_{L_1}(\varphi(H))$ (using surjectivity). Then $\varphi([x, H]) = [\varphi(x), \varphi(H)] \subset \varphi(H)$. So $[x, H + \ker\varphi] \leq H + \ker\varphi$, and $x \in I_L(H + \ker\varphi) = H + \ker\varphi$ by **3.2**. So $\varphi(x) \in \varphi(H)$.

(2) From (1), $\varphi(H)$ is a CSA of $\varphi(L_2) = K$. So $\varphi(H) = K$, since K is nilpotent and $\varphi(H)$ is equal to its idealiser *(alternatively, no nilpotent algebra is properly contained in another one, because they're both minimal of the form $L_{0,y}$)*. If $x \in L$ and $[x, H] \leq H$ then $[\varphi(x), \varphi(H)] \leq \varphi(H) = K$. So $\varphi(x) \in I_{L_1}(K)$. This is equal to K since K is a CSA. Hence $x \in L_2 = \varphi^{-1}(K)$. So $x \in I_{L_2}(H)$ and that's equal to H since H is a CSA of L_2. Thus $I_L(H) = H$ and H is a CSA of L. □

Lecture 8: October 22

Examples classes: Thursday 25 Oct. MR9 2-3; MR4 3:30-4:30.

Definition 8.1. The inner automorphism group of a complex Lie algebra L is the subgroup of the automorphism group of L generated by the automorphisms of the form

$$e^{ad_x} = 1 + ad_x + \frac{ad_x^2}{2!} + \cdots$$

for $x \in L$. Note that this exists (if L is finite-dimensional) by usual convergence arguments using norms on operators on finite-dimensional vector spaces.

You need to check that e^{ad_x} *is* an automorphism of L. Within this inner automorphism group we have the subgroup generated by the e^{ad_x} where ad_x is nilpotent. Note that, in this case, e^{ad_x} is a finite sum.

Theorem 8.2. *(3.13) Any two CSA's of a finite-dimensional complex Lie algebra are conjugate under the group*

$$\left\langle e^{ad_x} : ad_x \text{ is nilpotent} \right\rangle$$

(i.e. "conjugate" means here that they are in the same orbit of the above group).

(Not proven here; see Sternberg's book.)

Hard exercise 8.3. Prove this is true for soluble L.

Exercise 8.4. Prove that this is true for any 2 CSA's for any finite-dimensional complex Lie algebra under the full inner automorphism group.

Sketch of proof of Theorem **3.13**. Let H be a CSA.

Claim 8.5. *The set*

$$V = \{x \in H \,:\, \overline{ad_x} : L/H \to L/H \text{ is invertible}\}$$

is nonempty.

Claim 8.6. *Let G be the inner automorphism group of L. Set $W = G \cdot V$ (union of all $g(V)$). Then W is open in L (and non-empty).*

Claim 8.7. *There is a regular element $x \in L_{reg}$ with $H = L_{0,x}$.*

Define an equivalence relation on L_{reg} where $x \sim y \iff L_{0,x}$ and $L_{0,y}$ are conjugate under G.

Claim 8.8. *The equivalence classes are open in L_{reg}. But last time, by* **3.9** *we saw that L_{reg} is connected (where $k = \mathbb{C}$).*

So there is only one equivalence class, and we're done. □

Note, if $k = \mathbb{R}$ then we don't have just one orbit under the action of the inner automorphism group.

We've "proved":

Proposition 8.9. *(3.14) Any CSA is of the form $L_{0,x}$ for some regular x.*

Proposition 8.10. *(3.15) The dimension of any CSA is the rank of L.*

Definition 8.11. Let $\rho : L \to \mathfrak{gl}(V)$ be a representation. The trace form of ρ is

$$\beta(x, y) = Tr(\rho(x)\rho(y)).$$

Lemma 8.12. *(3.17)*

(1) *$\beta : L \times L \to k$ is a symmetric bilinear form on L;*
(2) *$\beta([x, y], z) = \beta(x, [y, z])$ for all $x, y, z \in L$;*
(3) *the radical*

$$R = \{x \in L \,:\, \beta(x, y) = 0 \ \forall y \in L\}$$

is a Lie ideal of L.

Proof. (2) uses the fact that $Tr([u, v]w) = Tr(u[v, w])$ for all $u, v, w \in \mathrm{End}(V)$

(3) R is clearly a subspace, and it is an ideal by (2) □

Definition 8.13. The *Killing form* of L is the trace form B_L of the adjoint representation $ad : L \to \mathfrak{gl}(L)$.

Exercise 8.14. (3.19) For any trace form on L, and any Lie ideal J of L, the orthogonal space of J with respect to the form is an ideal of L. (In particular this applies to the Killing form.)

Exercise 8.15. For $J \triangleleft L$, the restriction of B_L to J is equal to B_J.

Theorem 8.16 (Cartan-Killing criterion). *(3.20) A finite dimensional Lie algebra (with char$(k) = 0$) is semisimple $\iff$ its Killing form is nondegenerate.*

Compare this with Cartan's solubility criterion, $B_L(x, y) = 0\ \forall x \in L, y \in L^{(1)} \implies$ solubility.

Before proving Theorem **3.20** we need to meet semisimple/ nilpotent elements.

Definition 8.17. $x \in \mathrm{End}(V)$ is *semisimple* iff it is diagonalisable (over $\mathbb{C}$). Equivalently, the minimal polynomial is a product of distinct linear factors.

In general, $x \in L$ is semisimple if ad_x is semisimple.

Likewise, $x \in L$ is nilpotent if ad_x is nilpotent.

Remark 8.18. If $x \in \mathrm{End}(V)$ is semisimple, and $x(W) \le W$ for a subspace W, then $x|_W : W \to W$ is semisimple.

If $x, y \in \mathrm{End}(V)$ are semisimple, and $xy = yx$ then x, y are simultaneously diagonalisable, and so $x \pm y$ is semisimple.

Lecture 9: October 24

Lemma 9.1. *(3.22) Let $x \in \mathrm{End}(V)$. Then:*

1. *there are unique $x_s, x_n \in \mathrm{End}(V)$ with x_s semisimple, x_n nilpotent and $x = x_s + x_n$ with $x_s x_n = x_n x_s$;*
2. *there exist polynomials $p(t), q(t)$ with zero constant term such that $x_s = p(x)$, $x_n = q(x)$, x_n and x_s commute with all endomorphisms commuting with x;*
3. *if $A \le B \le V$ such that $x(B) \le A$, then $x_s(B) \le A$ and $x_n(B) \le A$.*

Definition 9.2. The x_s and x_n are the *semisimple and nilpotent parts* of x.

Proof. (3) follows from (1) and (2).

Let the characteristic polynomial of x be $\prod (t - \lambda_i)^{m_i}$. Then the generalised eigenspaces look like:
$$V_i = \ker(x - \lambda_i \cdot Id)^{m_i}$$
for each i. Then $V = \bigoplus V_i$, where $x(V_i) \le V_i$. The characteristic polynomial of $x|_{V_i}$ is $(t - \lambda_i)^{m_i}$. Find a polynomial $p(t)$ with $p(t) \equiv 0 \pmod t$ and $p(t) \equiv \lambda_i \pmod{(t - \lambda_i)^{m_i}}$. This exists by the Chinese remainder theorem. Define $q(t) = t - p(t)$ and set $x_s = p(x), x_n = q(x)$. p and q have zero constant term. On V_i, $x_s - \lambda_i \cdot Id$ acts like a multiple of $(x - \lambda_i \cdot Id)^{m_i}$ and so trivially. So x_s is diagonalisable.

Also $x_n = x - x_s$ acts like $x - \lambda_i \cdot Id$ on V_i, and hence nilpotently on V_i. So x_n is nilpotent.

It remains to prove uniqueness. Suppose $x = s + n$ where s is semisimple and n is nilpotent, and $sn = ns$. *Rewrite this condition as $s(x - s) = (x - s)s$ which shows that x commutes with s.* Then s, n commute with x_s and x_n (since x_s is a polynomial of x). If two semisimple elements commute with each other, then their sums are semisimple (this is because two diagonalisable elements are simultaneously diagonalisable). Sums of nilpotent elements are always nilpotent. So $x_s - s = n - x_n$ are both semisimple and nilpotent, hence zero. □

Exercise 9.3. (3.24) If $x \in \mathfrak{gl}(X)$ as in Lemma **3.22**, $x = x_s + x_n$, then ad_{x_s} and ad_{x_n} are the semisimple and nilpotent parts of ad_x.

If $Z(L) = 0$ (e.g. if L is semisimple), then $L \cong ad(L)$ and there is no ambiguity about the definition of semisimple elements.

Lemma 9.4. *(3.25) Let A and B be subspaces of $L = \mathfrak{gl}(V)$, with $A \leq B$. Let $M = \{x \in L : [x, B] \leq A\}$. If $x \in M$ satisfies $tr(xy) = 0$ for all $y \in M$ then x is nilpotent.*

Proof. Split x into $x = x_s + x_n$. Pick $v_1, \cdots, v_n$ to be a basis of V of eigenvectors of x_s, with eigenvalues $\lambda_1, \cdots, \lambda_n$. Define $e_{ij} \in \mathfrak{gl}(V)$ with $e_{ij}(v_k) = \delta_{ik} v_j$ (this is a matrix with a bunch of zeroes and a single 1). Note that

$$ad_{x_s} e_{ij} = x_s e_{ij} - e_{ij} x_s = (\lambda_j - \lambda_i) e_{ij}.$$

Let E be the $\mathbb{Q}$-vector space spanned by the eigenvalues $\{\lambda_i\}$. It suffices to show that $x_s = 0$ or that $E = 0$, or that the dual of E is zero. Take a linear map $f : E \to \mathbb{Q}$. The goal is to show that $f = 0$. Let

$$y = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} \in \mathrm{End}(V)$$

w.r.t. the basis $\{v_i\}$. Then

$$ad_y(e_{ij}) = y e_{ij} - e_{ij} y = (f(\lambda_j) - f(\lambda_i)) e_{ij} = f(\lambda_j - \lambda_i) e_{ij}$$

as f is linear. Let $r(t)$ be a polynomial with zero constant term and $r(\lambda_j - \lambda_i) = f(\lambda_j - \lambda_i)$ for all i, j. Then we have

$$ad_y = r(ad_{x_s}),$$

but by **3.24**, ad_{x_s} is the semisimple part of ad_x, and so is a polynomial in ad_x with zero constant term. So ad_y is also such a polynomial; ad_y is a polynomial in ad_{x_s}, and that is a polynomial in ad_x. But $ad_x(B) \leq A$ and so $ad_y(B) \leq A$. Thus $y \in M$. The hypothesis says that $0 = tr(xy) = \sum f(\lambda_i)\lambda_i$. Apply f, remembering that it is linear.

$$0 = \sum f(\lambda_i)^2$$

So each $f(\lambda_i) = 0$. Hence f is zero as desired. □

Theorem 9.5 (Cartan's solubility criterion). *(3.26) Suppose $L \in \mathfrak{gl}(V)$. If $tr(xy) = 0$ for all $x \in L^{(1)}, y \in L$, then L is soluble.*

Proof. It is enough to show that $L^{(1)}$ is nilpotent. For that we just show that all elements of $L^{(1)}$ are nilpotent endomorphisms (remember Engel).

Take $A = L^{(1)}$, $B = L$ in Lemma **3.25**. So $M = \{x \in L : [x, L] \leq L^{(1)}\} \geq L^{(1)}$. *(Note that $M \neq L$ because $\leq$ means* ideal in*, not just* contained in*.)* By definition $L^{(1)}$ is generated by things of the form $[x, y]$. Let $z \in M$. But $tr([x, y]z) = tr(x[y, z])$ (cf. **3.17**). So $tr(wz) = 0$ for all $w \in L^{(1)}$, $z \in M$. By Lemma **3.25**, each $w \in L^{(1)}$ is nilpotent. □

Exercise 9.6. Prove the converse.

Proof of **3.20***: L f.d., semisimple $\iff$ B_L nondegenerate.* Let R be the radical of L (the largest soluble ideal). Exercise: convince yourself that there is such a thing.

($\implies$) Let S be the radical of the Killing form, the set $\{x : tr(ad_x ad_y) = 0 \ \forall y \in L\}$. If $x \in S$ then $0 = B_L(x, x) = B_S(x, x)$. By **3.17**, S is an ideal. (If you restrict the Killing form of L to an ideal, then you get the Killing form of the ideal.) By Theorem **3.26**, $ad(S)$ is soluble. So S is a soluble ideal and so $S \leq R$. Thus if $R = 0$ (equivalently, L is semisimple), then $S = 0$. We get nondegeneracy of the Killing form.

($\impliedby$) Conversely, let J be an abelian ideal. Take $y \in J$, $x \in L$. Then $ad_x ad_y(L) \leq J$. Since J is abelian, $(ad_x ad_y)^2(L) = 0$. So $ad_x ad_y$ is nilpotent. Hence, the trace $tr(ad_x ad_y) = 0$. But this is just the Killing form $B_L(x, y)$. So we showed $B_L(x, y) = 0$ for all $x \in L, y \in J$. We showed that J is orthogonal to everything, and so $J \leq S$. If the Killing form is nondegenerate, J must be zero. That is, any abelian ideal is zero. This forces even soluble ideals to be 0, and hence $R = 0$.

□

Lecture 10: October 29

Lemma 10.1. *(3.27) L is a finite-dimensional Lie algebra, as always.*

1. *If L is semisimple then L is a direct sum $\bigoplus L_i$, where the L_i's are simple ideals of L.*
2. *If $0 \neq J \lhd L = \bigoplus L_i$ with $L_i \lhd L$, then J is a direct sum of the L_i's.*
3. *If L is a direct sum of simple ideals, then L is semisimple.*

Proof. (1) By induction on $\dim L$. Let L be semisimple, and $J \lhd L$. Then the orthogonal space $J^\perp$ (wrt the Killing form) is an ideal (using **3.17**). The non-degeneracy of B_L implies $\dim J + \dim J^\perp = \dim L$. But $J \cap J^\perp$ is soluble (using the Cartan solubility criterion on $ad(J \cap J^\perp)$) and so is zero since L is semisimple. So $L = J \oplus J^\perp$, and any ideal of J or $J^\perp$ is an ideal of L *because $[J, J^\perp] \subset J \cap J^\perp$ since they're both ideals*. So J and $J^\perp$ are semisimple, and we can apply induction to them: J and $J^\perp$ are direct sums of simple ideals.

(2) Suppose $J \cap L_i = 0$. Then $[L_i, J] = 0$, and hence $J \leq \bigoplus_{j \neq i} L_j$. Now prove this part by induction on $\dim L$.

(3) If L is a direct sum of ideals, then use (2) to prove that the radical of L is a direct sum of simples, and is therefore zero (recall simple Lie algebras are nonabelian). $\square$

Corollary 10.2. *(3.28)*

(1) *L is semisimple iff L is uniquely a direct sum of simple ideals.*
(2) *Ideals and quotients are semisimple.*
(3) *If L is semisimple, then $L = L^{(1)}$.*

Proposition 10.3. *(3.29) If L is semisimple, then $Der(L) = Inn(L)$ (all derivations are inner).*

Proof. Let D be a derivation of L, and $x \in L$. Then

$$[D, ad_x] = ad_{D(x)}. \tag{10.1}$$

We're dealing with semisimple Lie algebras so the centre $Z(L)$ is zero. $L \cong ad(L) \triangleleft Der(L)$. Thus L may be regarded as an ideal of $Der(L)$. The Killing form B_L is the restriction of $B_{Der(L)}$. Let J be the orthogonal space $ad(L)^{\perp}$ (with respect to $B_{Der(L)}$), an ideal of $Der(L)$. The point is to show that $J = 0$. By the non-degeneracy of $B_{ad(L)} = B_L$, $J \cap ad(L) = 0$. So $[ad(L), J] \leq ad(L) \cap J = 0$. Thus if $D \in J$, then using (10.1), $ad_{D(x)} = 0$ for all $x \in L$. That is, $D(x) \in Z(L) = 0$. Thus, $D(x) = 0$ for all $x \in L$, and $D = 0$. Hence $J = 0$ and so $Der(L) = Ad(L)$. $\square$

Theorem 10.4. *(3.30) Let H be a CSA of a semisimple Lie algebra L. Then,*

(1) *H is abelian;*
(2) *the centraliser $Z_L(H) = \{x : [x, h] = 0 \ \forall h \in H\}$ is H;*
(3) *every element of H is semisimple;*
(4) *the restriction of the Killing form B_L to H is non-degenerate.*

Proof. (4) Let H be a CSA. Then $H = L_{0,y}$ for some regular element $y \in L$ (this was only discussed sketchily... but take it on faith for now). Consider the decomposition of L into generalised eigenspaces of ad_y: $L = L_{0,y} \oplus \sum_{\lambda \neq 0} L_{\lambda,y}$. Let B_L be the Killing form on L. We want to show that $L_{\lambda,y}$ and $L_{\mu,y}$ are orthogonal, if $\lambda + \mu \neq 0$; that is, for $u \in L_{\lambda,y}, v \in L_{\mu,y}$ we need to show that $B_L(u, v) = tr(ad_u ad_v) = 0$. Recall that

$$[L_{\lambda,y}, L_{\mu,y}] \subset L_{\lambda+\mu,y}.$$

The operator $ad_u ad_v$ maps each generalised eigenspace into a different one. So $tr(ad_u ad_v) = 0$. *There can be nothing on the diagonal of the matrix for $ad_u ad_v$, because then that would mean part of some $L_{\lambda,y}$ got mapped into $L_{\lambda,y}$.*

Thus the generalised eigenspaces $L_{\lambda,y}$ and $L_{\mu,y}$ are orthogonal if $\lambda + \mu \neq 0$. So

$$L = L_{0,y} \oplus \bigoplus (L_{\lambda,y} + L_{-\lambda,y}).$$

But B_L is non-degenerate and so its restriction to each summand is non-degenerate. In particular, B_L restricted to $H = L_{0,y}$ is non-degenerate.

(1) Consider the restriction $ad|_H : H \to \mathfrak{gl}(L)$ via $x \mapsto ad_x$. The converse of Cartan's solubility criterion (exercise – think about triangular matrices) implies, since H is soluble *(because nilpotent $\implies$ soluble)* that $tr(ad_{x_1}, ad_{x_2}) = 0$ for all $x_1 \in H$ and $x_2 \in H^{(1)}$.

Thus, $H^{(1)}$ is orthogonal to H w.r.t. the Killing form B_L. But (4) says that $B_L|_H$ is non-degenerate. So $H^{(1)} = 0$, and H is abelian.

(2) The centraliser of H contains H (since H is abelian), and sits inside the idealiser of H. But H is a CSA, and is therefore equal to its idealiser. That forces $H = Z_L(H)$.

(3) Let $x \in H$. We can always decompose $x = x_s + x_n$, for x_s semisimple and x_n nilpotent (using ad_x and $L \cong ad(L)$ for L semisimple). Recall that as part of the definition, x_s and x_n commute with x; also ad_{x_s} and ad_{x_n} are polynomials in ad_x. If $h \in H$ it commutes with x by (1) and so it commutes with x_s and x_n. Thus x_s and $x_n \in Z_L(H) = H$ by part (2). But x_s and x_n commute and ad_{x_n} is nilpotent and so $ad_h ad_{x_n}$ is also nilpotent. So $tr(ad_h ad_{x_n}) = 0$. Thus $B_L(h, x_n) = 0$ for all $h \in H$. But $x_n \in H$ and so non-degeneracy of $B_L|_H$ (by (4)) implies that $x_n = 0$. Thus $x = x_s$ is semisimple. $\square$

Since H is abelian when L is semisimple, ad_{y_1} and ad_{y_2} commute for any $y_1, y_2 \in H$. They are diagonalisable because they are semisimple. So there are common eigenspaces, called *weight spaces*.

Given H, pick a basis $e_1, \cdots, e_n$ of L with respect to which the ad_y's are all diagonal matrices (for $y \in H$). So then, for each y, there is a set of $\lambda_i(y)$ such that $ad_y(e_i) = \lambda_i(y) e_i$. Since $ad_{x+y} = ad_x + ad_y$, we have

$$\lambda_i(x+y)e_i = ad_{x+y}e_i = (ad_x + ad_y)e_i = (\lambda_i(x) + \lambda_i(y))e_i$$

so $\lambda_i(x) + \lambda_i(y) = \lambda_i(x+y)$. The same thing works for scalar multiplication, so each λ_i can be thought of as a linear *function of $y \in H$, such that $L_{\lambda_i}(y) = \{x \in L : ad_y(x) = \lambda_i(y)x\}$ is an eigenspace, for each y. Furthermore, since this space is secretly $span(e_i)$ (which is independent of y), $L_{\lambda_i}(y)$ doesn't depend on y, so we can write it as just L_{λ_i}.*

We knew all along that $L = \bigoplus_i span(e_i)$, but now we can rewrite that as $L = \bigoplus_i L_{\lambda_i}$. This is all the stuff below is saying, except that we *know that there are only finitely many "valid" eigenvalues λ_i, so all but finitely many of the spaces below (i.e. the L_α, where α isn't an eigenvalue) are trivial.*

So... does this imply that all the nontrivial L_α are 1-dimensional? No: if, for some i, j, we have $\lambda_i(y) = \lambda_j(y)$ for all y, then $L_{\lambda_i} = L_{\lambda_j}$ has dimension (at least) 2.

Definition 10.5.

$$L_\alpha = \{x \in L : ad_y(x) = \alpha(y)x \text{ for } y \in H\}$$

where $\alpha : H \to k$ is linear (an element of the dual of H).

Note $H = L_0$ and there is a decomposition

$$L = L_0 \oplus \left(\bigoplus_{\alpha \neq 0} L_\alpha \right).$$

How does this compare to the decomposition $L = L_{0,y} \oplus (L_{\lambda,y} + L_{-\lambda,y})$ for a particular y? Suppose $\lambda_i : H \to \mathbb{C}$ is a linear functional as above. Then, using definitions, $L_{\lambda_i} \subset L_{\lambda_i(y),y}$ for every y. But, for any given y, there is not necessarily a 1-1 correspondence between spaces L_{λ_i} and $L_{\lambda_i(y),y}$. For example, suppose L is 3-dimensional, with different eigenvalue functionals α, β, and γ corresponding to the simultaneously diagonalizing basis, and suppose $\alpha(y) = \beta(y)$ at a particular y. Then the Cartan decomposition will be $L = L_\alpha \oplus L_\beta \oplus L_\gamma$ (three 1-dimensional spaces), but the $L_{\lambda,y}$-decomposition as in **3.30** *will be $L_{\alpha(y),y} \oplus L_{\beta(y),y}$ – the sum of a 1-dimensional space and a 2-dimensional space which is also equal to $L_{\gamma(y),y}$.*

Lecture 11: October 31

Chapter 4: Root systems

Introduction to root systems. Let L be a semisimple complex Lie algebra. The *Cartan decomposition* of L is

$$L = L_0 \oplus \left(\bigoplus_{\alpha \neq 0} L_\alpha \right)$$

with respect to our choice H of CSA. We say the elements of L_α have *weight* α. The $\alpha \neq 0$ for which $L_\alpha \neq 0$ are the *roots* of L (relative to H). Define Φ to be the set of roots of L. We have $L_0 = H$ and $m_\alpha = \dim L_\alpha$ for each weight α, with $(-,-)$ for the Killing form, which we know to be nondegenerate. Write $W^\perp$ for the orthogonal space of W.

Remark 11.1. We'll be discussing the case $k = \mathbb{C}$, but that isn't really necessary. The important thing is that we have the Cartan decomposition. This doesn't necessarily occur if $k = \mathbb{R}$ – not all semisimple real Lie algebras split in this way. Those that do are called *split semisimple*.

Exercise 11.2. Think of a semisimple real Lie algebra which isn't split semisimple.

Lemma 11.3. *(4.2)*

1. *If $x, y \in H$ then $(x, y) = \sum_{\alpha \in \Phi} m_\alpha \alpha(x)\alpha(y)$, where $m_\alpha = \dim L_\alpha$.*
2. *If α, β are weights with $\alpha + \beta \neq 0$ then $(L_\alpha, L_\beta) = 0$.*
3. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*
4. *$(-,-)$ restricted to H is nondegenerate.*
5. *If α is a weight, then $L_\alpha \cap L_{-\alpha}^\perp = 0$. This is a converse to (2): if $x \in L_\alpha$ is nonzero, then it does not kill $L_{-\alpha}$.*
6. If $h \neq 0$ is in H, then $\alpha(h) \neq 0$ for some $\alpha \in \Phi$, and so Φ spans the dual space of H.

Proof. (1) *Remember that $(x, y) := tr(ad_x ad_y)$!* Choose a basis for each weight space – taking their union gives a basis of L. ad_x and ad_y are both represented by diagonal matrices.

(2) We did this when proving Theorem **3.30**(4).

(3) Suppose $\alpha \in \Phi$ but $-\alpha \notin \Phi$. Then $(L_\alpha, L_\beta) = 0$ for all weights β by (2), and so $(L_\alpha, L) = 0$ (by the direct sum representation of L). But the nondegeneracy of the Killing form implies $L_\alpha = 0$, a contradiction.

(4) This is Theorem **3.30**(4).

(5) Take $x \in L_\alpha \cap L_{-\alpha}^\perp$. Then $(x, L_\beta) = 0$ for all weights β. By the same argument we just did, since L is the direct sum of all the L_β's, we have $(x, L) = 0$ and hence $x = 0$ by nondegeneracy.

(6) If $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $(h, x) = \sum m_\alpha \alpha(h)\alpha(x) = 0$ by (1). Nondegeneracy of $(-,-)$ on H forces $h = 0$. Thus, if $h \neq 0$ then there is some $\alpha \in \Phi$ with $\alpha(h) \neq 0$. $\square$

We shall show that we have a finite root system within the dual space H^*. Root systems are subsets of real Euclidean vector spaces, rather than complex ones, and so we are going to be using the real span of Φ in H^*.

Definition 11.4. A subset Φ of a real Euclidean vector space E is a *root system* if

(1) Φ is finite, spans E, and does not contain 0.
(2) For each $\alpha \in \Phi$ there is a reflection s_α with respect to α leaving Φ invariant that preserves the inner product, with $s_\alpha(\alpha) = -\alpha$, and the set of points fixed by s_α is a hyperplane (codimension 1) in E.
(3) For each $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.
(4) For all $\alpha, \beta \in \Phi$, $\frac{2(\beta.\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ where $(-,-)$ is not the Killing form, but rather the inner product on a real vector space H^*.

The dimension of E is the *rank* of the root system.

Note that the reflection s_α has reflecting hyperplane P_α of elements orthogonal to α. Also,

$$s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha.$$

Let E^* be the dual space of E. Let $\alpha^\vee$ be the unique element of E^* which vanishes on P_α and takes value 2 on α. Then

$$s_\alpha(\beta) = \beta - \left\langle \alpha^\vee, \beta \right\rangle \alpha$$

where $\langle -,- \rangle$ is the canonical pairing of E^* and E. Here $\alpha^\vee$ is called the *inverse root.* (Try writing (3) in terms of inverse roots.)

Definition 11.5. A root system Φ is *reduced* if, for each $\alpha \in \Phi$, α and $-\alpha$ are the only two roots proportional to α.

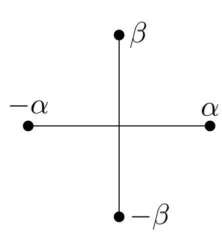
If a root system is not reduced, it contains two proportional roots α and $t\alpha$, with $0 < t < 1$. Applying (3) to $\beta = t\alpha$, we see that $2t \in \mathbb{Z}$, and so $t = \frac{1}{2}$. So the roots proportional to α are $-\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}, \alpha$. The reduced root systems are the only ones we need when looking at semisimple complex Lie algebras. Nonreduced systems arise when you look at the real case.

Definition 11.6. The Weyl group $W(\Phi)$ of a root system Φ is the subgroup of the orthogonal group generated by the s_α, $\alpha \in \Phi$. Note that since Φ is finite, spans E, and each generator leaves Φ invariant, $W(\Phi)$ is a finite reflection group. More generally, we may obtain infinite Coxeter groups generated by 'reflections' w.r.t. nondegenerate symmetric bilinear forms on real vector spaces.

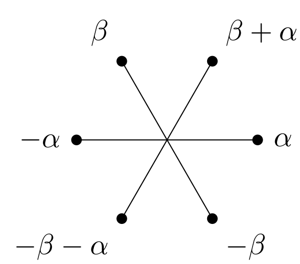
Lecture 12: November 2

Example 12.1 (Root systems of rank 1). The only reduced root system is $\{\alpha, -\alpha\}$. This is type A_1.

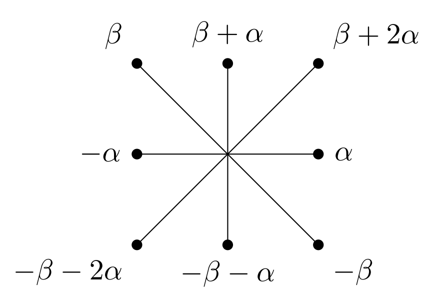
Example 12.2 (Root systems of rank 2). The only reduced root systems of rank 2 are:



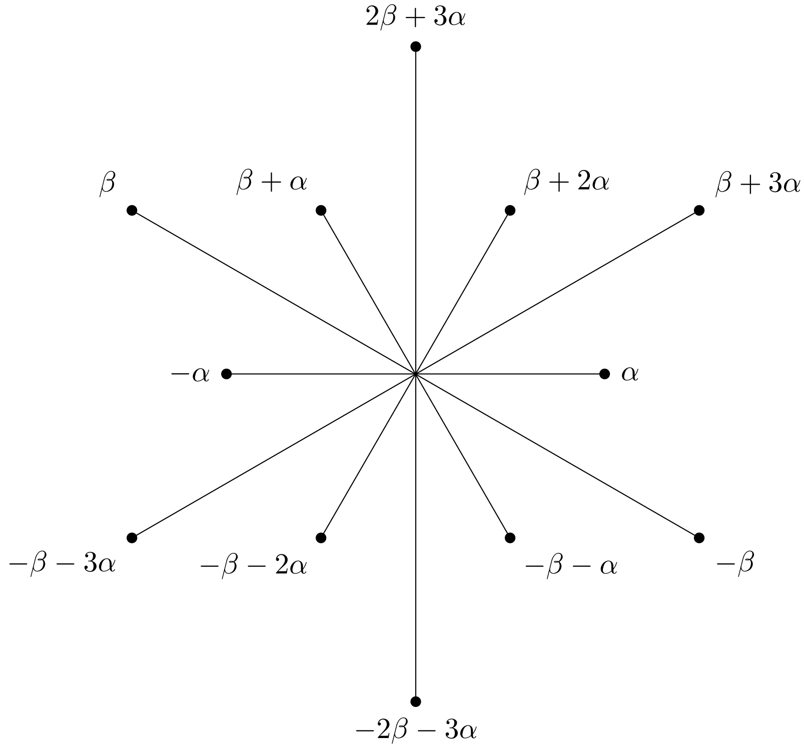
$A_1 \times A_1$



A_2



B_2



$\boxed{G_2}$

The rank 1 root system arises from the Lie algebra $\mathfrak{sl}_2$, which has a Cartan decomposition

$$\mathfrak{sl}_2 = L_0 \oplus L_\alpha \oplus L_{-\alpha}$$

where

$$L_\alpha = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\}$$

$$L_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\}$$

$$L_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right\} = H \text{ (Cartan subalgebra).}$$

We have elements

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in L_0, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in L_\alpha, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in L_{-\alpha}$$

ad_h has 3 eigenvalues: $2, 0, -2$. $\alpha \in H^*$ is the linear form

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \mapsto 2\lambda$$

so $\alpha(h) = 2$. h is the coroot (or inverse root) of α.

In the general case, you look for elements h, e, f and show they form a root system. So in the terminology of the last lecture, instead of showing that Φ is a root system, we will show that its dual is.

DEFINITION 12.3. The α-string of roots through β is the largest arithmetic progression

$$\beta - q\alpha, \cdots, \beta + p\alpha$$

all of whose elements are weights.

LEMMA 12.4. *(4.7) Let $\alpha, \beta \in \Phi$.*

(1)
$$\beta(x) = -\left(\sum_{r=-q}^{p} r m_{\beta+r\alpha} \Big/ \sum_{-q}^{p} m_{\beta+r\alpha}\right) \alpha(x)$$
for all $x \in [L_{-\alpha}, L_\alpha] \subset H = L_0$, where $\dim L_\alpha = m_\alpha$.

(2) *If $0 \neq x \in [L_\alpha, L_{-\alpha}]$ then $\alpha(x) \neq 0$.*

(3) $[L_\alpha, L_{-\alpha}] \neq 0$.

Proof. (1) Let $M = \sum_{r=-q}^{p} L_{\beta+r\alpha}$. So $[L_{\pm\alpha}, M] \leq M$. An element $x \in [L_\alpha, L_{-\alpha}]$ belongs to the derived subalgebra of the Lie algebra $M^{(1)}$. So $ad_x|_M : M \to M$ has zero trace. But $tr(ad_x|_M) = \sum_{r=-q}^{p} m_{\beta+r\alpha}(\beta + r\alpha)(x) = 0$. Rearranging gives (1): the denominator $\sum_{r=-q}^{p} m_{\beta+r\alpha}$ is nonzero since β is a root.

(2) If $\alpha(x) = 0$ then we deduce from (1) that $\beta(x) = 0$, for all roots β. This contradicts **4.2**(6). So if $x \neq 0 \in [L_\alpha, L_{-\alpha}]$, then $\alpha(x) \neq 0$.

(3) Let $v \in L_{-\alpha}$. Then $[h, v] = -\alpha(h)v$ for all $h \in H$ (by definition of the weight). Choose $u \in L_\alpha$, with $(u, v) \neq 0$, by **4.2**(5). Choose $h \in H$ with $\alpha(h) \neq 0$. Set $x = [u, v] \in [L_\alpha, L_{-\alpha}]$. Then $(x, h) = (u, [h, v])$, using the property of the Killing form. This is $-\alpha(h)(u, v) \neq 0$. So $x \neq 0$. $\square$

LEMMA 12.5. *(4.8)*

(1) *$m_\alpha = 1$ for all $\alpha \in \Phi$, and if $n\alpha \in \Phi$ for $n \in \mathbb{Z}$, then $n = \pm 1$.*

(2) *$\beta(x) = \frac{q-p}{2}\alpha(x)$ for all $x \in [L_\alpha, L_{-\alpha}]$.*

Proof. (1) Let u, v, x be as in Lemma **4.7**(3). Take A to be the Lie algebra generated by u and v, and N to be the vector space span of v, H and $\sum_{r>0} L_{r\alpha}$. Then $[u, N] \leq H \oplus \sum L_{r\alpha} \leq N$. Similarly, $[v, N] \leq [v, H] \oplus \sum_{r>0}[v, L_{r\alpha}] \leq N$. Thus $[A, N] \leq N$. So $[x, N] \leq N$ and we can consider $ad_x|_N : N \to N$.

As previously, $x = [u, v]$ is in $A^{(1)}$, and we have

$$\begin{aligned} 0 = tr(ad_x|_N) &= -\alpha(x) + \sum_{r>0} m_{r\alpha} r\alpha(x) \\ &= (-1 + \sum r m_{r\alpha})\alpha(x) \end{aligned}$$

But $\alpha(x) \neq 0$ by **4.7**(2). So $\sum r m_{r\alpha} = 1$. Thus $m_\alpha = 1$ for all α. If $n\alpha$ is a root for $n \in \mathbb{N}$, then $n = \pm 1$.

(2) This follows from (1) and **4.7**(1). ☐

Exercise 12.6. If $\beta + n\alpha \in \Phi$ then $-q \le n \le p$ (i.e. the only roots of the form $\beta + n\alpha$ are the things in the α-string; there are no gaps). *Hint: Apply* **4.8***(2) to two α-strings.*

Lemma 12.7. *(4.9) If $\alpha \in \Phi$ and $c\alpha \in \Phi$ where $c \in \mathbb{C}$, then $c = \pm 1$.*

Proof. Set $\beta = c\alpha$. The α-string through β is $\beta - q\alpha, \cdots, \beta + p\alpha$. Choose $x \in [L_\alpha, L_{-\alpha}]$ so that $\alpha(x) \neq 0$ (from two lemmas ago). Then $\beta(x) = \frac{q-p}{2}\alpha(x)$ by **4.8**(2). So $c = \frac{q-p}{2}$. But $q - p$ is odd; otherwise we're done by **4.8**(1). So $r = \frac{1}{2}(p - q + 1) \in \mathbb{Z}$. Note that $-q \le r \le p$. Hence Φ contains $\beta + r\alpha$ (since p and q were the endpoints of the α-string). That is,

$$\beta + r\alpha = \frac{1}{2}(q - p + p - q + 1)\alpha = \frac{1}{2}\alpha \in \Phi$$

So Φ contains $\frac{1}{2}\alpha$ and $2(\frac{1}{2}\alpha)$, which contradicts **4.8**(1). ☐

Lemma 12.8. *(4.10)*

(1) *For $\alpha \in \Phi$ we can choose $h_\alpha \in H$, $e_\alpha \in L_\alpha$, $f_\alpha = e_{-\alpha} \in L_{-\alpha}$ such that:*
   (a) *$(h_\alpha, x) = \alpha(x)$ for all $x \in H$*
   (b) *$h_{\alpha\pm\beta} = h_\alpha \pm h_\beta$, $h_{-\alpha} = -h_\alpha$ and the h_α ($\alpha \in \Phi$) span H;*
   (c) *$h_\alpha = [e_\alpha, e_{-\alpha}]$, and $(e_\alpha, e_{-\alpha}) = 1$*

(2) *If $\dim L = n$ and $\dim H = \ell$ then the number of roots is $2s := n - \ell$, and $\ell \le s$.*

Proof. (1a) Define h^* by $h^*(x) = (h, x)$ for all $x \in H, h \in H$. Thus each $h^* \in H^*$, the dual of H. $h \mapsto h^*$ is linear, and the map is injective by the nondegeneracy of the restriction of the Killing form B_L to H (see **3.30**). Hence it is surjective, and we can pick h_α to be the preimage of α.

Idea: Define $H \to H^*$ via $h \mapsto (h, -)$. This is injective by nondegeneracy of B_L, hence an isomorphism. So α has a preimage; call it h_α.

(1b) follows as $h \mapsto h^*$ is linear, and because the $\alpha \in \Phi$ actually span H^* by **4.2**(6).

(1c) By **4.2**, there exists $e_{\pm\alpha} \in L_{\pm\alpha}$ with $(e_\alpha, e_{-\alpha}) \neq 0$. Adjust by scalar multiplication to make $(e_\alpha, e_{-\alpha}) = 1$. Let $x \in H$. Then consider the Killing form $([e_\alpha, e_{-\alpha}], x) = (e_\alpha, [e_{-\alpha}, x])$ (by properties of a trace form). This is $\alpha(x)(e_\alpha, e_{-\alpha}) = (h_\alpha, x)$. So $[e_\alpha, e_{-\alpha}] = h_\alpha$.

Idea: pick $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ basically at random, except $(e_\alpha, e_{-\alpha}) \neq 0$. You have to show that $[e_\alpha, e_{-\alpha}]$ satisfies the "property of being h_α" (i.e. $(h_\alpha, h) = \alpha(x)$).

(2) Each weight space $\neq H$ has dimension 1. So the number of roots is an even number $2s = n - \ell$ (counting dimensions), and the elements of the form h_α span H, and so $\ell \le s$. ☐

Lecture 13: November 5

We're going to show that that h_α's (from **4.10**) may be embedded in a real Euclidean space to form a root system.

Exercise 13.1. If $\alpha, \beta \in \Phi$ with $\alpha + \beta$ and $\alpha - \beta$ not in Φ then prove that $(h_\alpha, h_\beta) = 0$.

Lemma 13.2. *(4.11)*

(1)
$$2\frac{(h_\beta, h_\alpha)}{(h_\alpha, h_\alpha)} \in \mathbb{Z};$$

(2)
$$4\frac{\sum_{\beta\in\Phi}(h_\beta, h_\alpha)^2}{(h_\alpha, h_\alpha)^2} = \frac{4}{(h_\alpha, h_\alpha)} \in \mathbb{Z};$$

(3) $(h_\alpha, h_\beta) \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Phi$*;*

(4) *For all* $\alpha, \beta \in \Phi$ *we have*
$$\beta - \left(\frac{2(h_\beta, h_\alpha)}{(h_\alpha, h_\alpha)}\right)\alpha \in \Phi.$$

Proof. (1) First, $(h_\alpha, h_\alpha) = \alpha(h_\alpha) \neq 0$ by **4.7**(2).
$$\frac{2(h_\beta, h_\alpha)}{(h_\alpha, h_\alpha)} = \frac{2\beta(h_\alpha)}{\alpha(h_\alpha)} = 2\frac{q-p}{2}$$
where $\beta - q\alpha, \cdots, \beta + p\alpha$ is the α-string through β.

(2) For $x, y \in H$ we have
$$(x, y) = \sum_{\beta\in\Phi} \beta(x)\beta(y)$$
by various things: **4.2**(1), **4.8**(1). So $(h_\alpha, h_\alpha) = \sum_{\beta\in\Phi}\beta(h_\alpha)^2 = \sum_\beta (h_\beta, h_\alpha)^2$. Adjust to get what we want.

(3) Follows from (1) and (2).

(4)
$$\beta - \left(\frac{2(h_\beta, h_\alpha)}{(h_\alpha, h_\alpha)}\right)\alpha = \beta + (p-q)\alpha$$
lies in the α-string through β. □

Define
$$\widetilde{H} = \left\{\sum \lambda_\alpha h_\alpha \,:\, \lambda_\alpha \in \mathbb{Q}\right\}$$
to be the rational span of the h_α's. Clearly $\widetilde{H} \subset H$. Since the h_α span H we can find a subset $h_1, \cdots, h_\ell$ which is a $\mathbb{C}$-basis of H.

Lemma 13.3. *(4.12) The Killing form restricted to* $\widetilde{H}$ *is an inner product, and* $\{h_1, \cdots, h_\ell\}$ *is a* $\mathbb{Q}$*-basis of* $\widetilde{H}$*.*

Proof. $B_L|_{\widetilde{H}}$ is symmetric, bilinear, rational-valued by **4.11**(3).

Let $x \in \widetilde{H}$. Then

$$(x,x) = \sum_{\alpha\in\Phi} \alpha(x)^2 = \sum_{\alpha\in\Phi} (h_\alpha, x)^2.$$

Each $(h_\alpha, x) \in \mathbb{Q}$ and $(x,x) \geq 0$ with equality only if each $(h_\alpha, x) = \alpha(x) = 0$ (and thus when $x = 0$).

It remains to show that each h_α is a rational linear combination of the h_i's. We have $h_\alpha = \sum \lambda_i h_i$, but $\lambda_i \in \mathbb{C}$. $(h_\alpha, h_j) = \sum \lambda_i (h_i, h_j)$ and $(h_i, h_j) \in \mathbb{Q}$ by **4.11**(3). The matrix (h_i, h_j) is an $\ell \times \ell$ matrix over $\mathbb{Q}$, non-singular as $B_L|_{\widetilde{H}}$ is non-degenerate.

So it is an invertible rational matrix. Multiplying by the inverse gives $\lambda_i \in \mathbb{Q}$. □

We can now regard $\widetilde{H}$ as embedded in a real ℓ-dimensional Euclidean space.

THEOREM 13.4. *Let $\Phi' = \{h_\alpha : \alpha \in \Phi\}$. Then*

(1) *Φ' spans E and $0 \notin \Phi'$.*
(2) *If $h \in \Phi'$ then $-h \in \Phi'$.*
(3) *If $h, k \in \Phi'$ then $\frac{2(k,h)}{(h,h)} h \in \mathbb{Z}$.*
(4) *If $h, k \in \Phi'$ then $k - \frac{2(k,h)}{(h,h)} h \in \Phi'$.*

*Thus Φ' is a reduced root system in E (it's reduced by **4.8**(1)).*

Now suppose we have a general root system Φ with inner product $(-,-)$.

DEFINITION 13.5. For a root system write $n(\beta,\alpha)$ for $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

REMARK 13.6. $n(-,-)$ is not necessarily symmetric. With respect to the Euclidean structure let $|\alpha| = (\alpha,\alpha)^{\frac{1}{2}}$. The angle θ between α and β is given by

$$(\alpha,\beta) = |\alpha||\beta|\cos\theta.$$

Note that $n(\beta,\alpha) = 2\frac{|\beta|}{|\alpha|}\cos\theta$.

PROPOSITION 13.7. *(4.14)*

$$n(\beta,\alpha)n(\alpha,\beta) = 4\cos^2\theta.$$

Since $n(\beta,\alpha)$ is an integer, $4\cos^2\theta$ is an integer, so it must be one of $0, 1, 2, 3$ or 4 (with 4 being when α and β are proportional).

For non-proportional roots there are 7 possibilities up to transposition of α and β:

$n(\alpha,\beta)$	$n(\beta,\alpha)$	θ	
0	0	$\frac{\pi}{2}$	
1	1	$\frac{\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
-1	-1	$\frac{2\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
1	2	$\frac{\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
-1	-2	$\frac{3\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
1	3	$\frac{\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$
-1	-3	$\frac{5\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$

Exercise 13.8. (4.15) Let α and β be two non-proportional roots. If $n(\beta,\alpha) > 0$ then $\alpha - \beta$ is a root.

Proof. Look at the table above: if $n(\beta,\alpha) > 0$ then $n(\alpha,\beta) = 1$, and so $s_\alpha(\beta) = \beta - n(\alpha,\beta)\alpha = \beta - \alpha$ is a root. Furthermore, $s_{\beta-\alpha}(\beta-\alpha) = \alpha - \beta$ so $\alpha - \beta$ is a root. $\square$

Definition 13.9. A subset Δ of a root system Φ in E is a *base* of Φ if

(1) Δ is a basis of the vector space E
(2) each $\beta \in \Phi$ can be written as a linear combination
$$\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$$
where all k_α's are integers of the same sign (i.e. all ≥ 0 or all ≤ 0)

Suppose we have a base Δ (we'll see they exist next time).

Definition 13.10. The *Cartan matrix* of the root system w.r.t. the given base Δ is the matrix $(n(\alpha,\beta))|_{\alpha,\beta\in\Delta}$.

Example 13.11. The Cartan matrix for G_2 is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ where $\{\alpha,\beta\}$ is a base.

Remark 13.12. $n(\alpha,\alpha) = 2$ for all α, and so we necessarily have 2's on the leading diagonal. We will see that the other terms have to be negative.

Definition 13.13. A *Coxeter graph* is a finite graph, where each pair of distinct vertices is joined by 0,1,2, or 3 edges. Given a root system with base Δ, the Coxeter graph of Φ (w.r.t. Δ) is defined by having vertices as elements of Δ, and vertex α is jointed to vertex β by 0, 1, 2, or 3 edges according to whether $n(\alpha,\beta)n(\beta,\alpha) = 0, 1, 2$ or 3.

Example 13.14. The Coxeter graph of the root systems of rank 2 are:

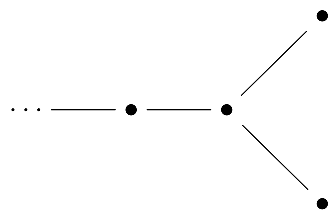
- Type $A_1 \times A_1$: $\bullet \qquad \bullet$
- Type A_2: $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$
- Type B_2: $\bullet\!=\!=\!\bullet$
- Type G_2: $\bullet\!\equiv\!\equiv\!\bullet$

Theorem 13.15. *Every connected non-empty Coxeter graph associated to a root system is isomorphic to one of the following:*

- *Type A_n (n vertices):*

- *Type B_n*

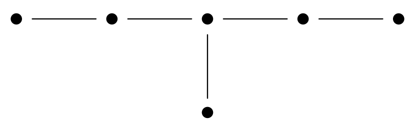
- D_n



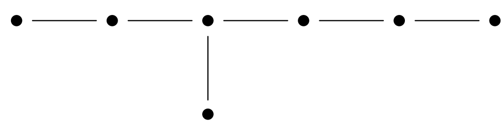
- G_2

- F_4

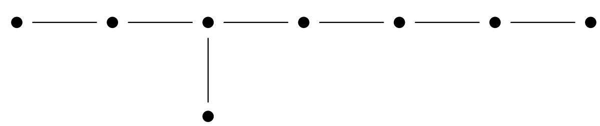
- E_6



- E_7



- E_8



Lecture 14: November 7

Coxeter graphs are not enough to distinguish between root systems. It gives the angles between two roots without saying anything about their relative lengths. In particular, B_n and C_n have the same Coxeter graph. There are two ways of fixing this problem: (1) put arrows on some of the edges; (2) label the vertices. Method (1) is illustrated below:

Type B_n:

Type C_n:

Type G_2:

$$\bullet \Rrightarrow \bullet$$

Type F_4:

$$\bullet \longrightarrow \bullet \Rightarrow \bullet \longrightarrow \bullet$$

Dynkin diagrams are the right way to do method (2).

DEFINITION 14.1. A *Dynkin diagram* is a labelled Coxeter graph, where the labels are proportional to the squares (α, α) of the lengths of the root α.

Specifying the Dynkin diagram is enough to determine the Cartan matrix of the root system (and in fact this is enough to determine the root system). We'll come back to constructing the root systems from the listed Dynkin diagrams in two lectures' time.

If $\alpha = \beta$ then $n(\alpha, \beta) = 2$. If $\alpha \neq \beta$, and if α and β are *not* joined by an edge, then $n(\alpha, \beta) = 0$. If $\alpha \neq \beta$ and if α and β *are* joined by an edge, AND the label of $\alpha \leq$ the label of β, then $n(\alpha, \beta) = -1$. If $\alpha \neq \beta$ and if α and β are joined by i edges $(1 \leq i \leq 3)$ and if the label of $\alpha \geq$ the label of β, then $n(\alpha, \beta) = -i$.

THEOREM 14.2. *(4.20) Each nonempty connected Dynkin diagram of an irreducible root system is isomorphic to one of the following:*

- *Type A_n*

$$1 \longrightarrow 1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 1$$

- *Type B_n*

$$2 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2 \longrightarrow 2 \longrightarrow 1$$

- *Type C_n*

$$1 \longrightarrow 1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 2$$

- *D_n*

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & \diagup & \\ \cdots & \longrightarrow & 1 & \longrightarrow & 1 & \\ & & & & \diagdown & \\ & & & & & 1 \end{array}$$

  for $n \geq 4$.

- *Type G_2*

$$1 \equiv 3$$

- *Type F_4 is*

$$1 \longrightarrow 1 = 2 \longrightarrow 2$$

- *Every vertex in the E_i graphs has label 1.*

EXERCISE 14.3. Given a root system Φ then the inverse roots (coroots) $\alpha^\vee$ for $\alpha \in \Phi$ form a dual or inverse root system. Show that if Φ is of type B_n then its dual will be of type C_n.

Remarks about the proof of **4.20**. The method is to associate a quadratic form to a graph G with ℓ vertices. Take $\mathbb{R}^\ell$, a real vector space. Then

$$Q((x_1, \cdots, x_\ell)) = \sum q_{ij} x_i x_j \text{ where } q_{ii} = 2, q_{ij} = q_{ji} = -\sqrt{s} \text{ for } i \neq j$$

where s is the number of edges connecting v_i with v_j. This ensures that the Coxeter graphs of a root system Φ are associated with positive definite forms. Check this, using the fact that $q_{ii} = 2$, $q_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_i||\alpha_j|}$ for $i \neq j$.

So we then classify the graphs whose quadratic form is positive definite.

Exercise 14.4. If G is positive definite and connected, then:

- The graphs obtained by deleting some vertices, and all the edges going into them, are also positive definite.
- If we replace multiple edges by a single edge, then we get a tree (i.e. there are no loops). The number of pairs (v_i, v_j) are connected by an edge is $< \ell$.
- There are no more than 3 edges from a given edge in the Coxeter graph.

Then show:

Theorem 14.5. *(4.22)[unproved here]*

The following is a complete list of positive definite graphs:

1. A_ℓ *($\ell \geq 1$)*
2. B_ℓ *($\ell \geq 2$)*
3. D_ℓ *($\ell \geq 4$)*
4. E_6, E_7, E_8, F_4, G_2.

Then you have to consider the possible labellings for the graphs to yield the Dynkin matrix/ Cartan matrix.

□

Digression 14.6. The classification of positive definite graphs appears elsewhere, e.g. in the complex representation theory of quivers. A quiver has vertices and oriented edges. A representation of Q is a vector space V_i associated with each vertex i, and a linear map $V_i \to V_j$ corresponding to each directed edge $i \to j$.

The representation theory of quivers is quite widespread in algebraic geometry.

An *indecomposable representation* is one which cannot be split as a direct sum of two non-trivial representations.

Question 14.7. Which quivers only have finitely many indecomposable complex representations, i.e. "of finite representation type"?

ANSWER 14.8 (Gabriel). It happens iff the underlying (undirected) graph is one of:

$$A_n, D_n, E_6, E_7, E_8$$

and the indecomposable representations are in 1-1 correspondence with the positive roots of the root system of the Coxeter graph, by setting $E = \mathbb{R}^\ell$ (where ℓ is the number of vertices) and consider the dimension vector $(\dim V_1, \dim V_2, \cdots, \dim V_\ell)$.

The modern proof of Gabriel's results involves looking at representation categories of the various quivers with the same underlying graph. There are "Coxeter functors" mapping from a representation of one such quiver to a representation of another such with the effect of reflecting the dimension vector.

OK, back to our unfinished business. We were assuming that we had a base, and therefore Cartan matrices, etc. etc. Now we prove that we actually do have a base.

Recall that a subset Δ of a root system Φ is a *base* if

(1) it is a basis of E, and
(2) each root can be written as $\sum_{\alpha\in\Delta} k_\alpha \alpha$, where the k_α are either all ≥ 0, or all ≤ 0.

Note that the expression $\beta = \sum k_\alpha \alpha$ is unique.

DEFINITION 14.9. (4.22*) β is a *positive root* if all $k_i \geq 0$, and a *negative root* if all the $k_i \leq 0$.

The elements of Φ are *simple roots* w.r.t. Δ if they cannot be written as the sum of two positive roots.

We can partially order Φ by writing $\gamma \geq \delta$ if $\gamma = \delta$ or $\gamma - \delta$ is a sum of simple roots with non-negative integer coefficients. Let Φ^+ and Φ^- denote the sets of positive roots and negative roots, respectively.

LECTURE 15: NOVEMBER 9

Recall we have hyperplanes P_α that are fixed by reflections. Then $E\backslash\bigcup_{\alpha\in\Phi} P_\alpha$ is nonempty. If $\gamma \in E\backslash\bigcup_{\alpha\in\Phi} P_\alpha$ then say γ is *regular* (note: this is not the same regular we met before). For $\gamma \in E$ define

$$\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}.$$

If γ is regular, then $\Phi = \Phi^+(\gamma) \cup (-\Phi^+(\gamma))$. Call $\alpha \in \Phi^+(\gamma)$ *decomposable* if $\alpha = \alpha_1 + \alpha_2$, with each $\alpha_1, \alpha_2 \in \Phi^+(\gamma)$; otherwise, α is *indecomposable*.

LEMMA 15.1. *(4.23) If $\gamma \in E$ is regular, then the set $\Delta(\gamma)$ of all the indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ. Every base has this form.*

Proof. Each $\alpha \in \Phi^+(\gamma)$ is a non-negative integral combination of the elements of $\Delta(\gamma)$. Otherwise, choose "bad" α with (γ, α) minimal; so α is not indecomposable. That is, we can decompose it as $\alpha = \alpha_1 + \alpha_2$ for $\alpha_i \in \Phi^+(\gamma)$, and $(\gamma, \alpha) = (\gamma, \alpha_1) + (\gamma, \alpha_2)$. By minimality, α_1 and α_2 are "good" and hence α is. This is a contradiction.

So $\Delta(\gamma)$ spans E and satisfies (2) of the definition of a base. To show linear independence, it's enough to show that each $(\alpha, \beta) \le 0$ when $\alpha \ne \beta$ in $\Delta(\gamma)$. *Why does this suffice? Suppose $\sum r_\alpha \alpha = 0$ where $r_\alpha \in \mathbb{R}$ and $\alpha \in \Delta(\gamma)$. Then rewrite this as $\sum s_\alpha \alpha = \sum t_\beta \beta$, where s_α and t_β are all positive. Define $\varepsilon = \sum s_\alpha \alpha = \sum t_\beta \beta$. Then*

$$\begin{aligned} 0 \le (\varepsilon, \varepsilon) &= (\textstyle\sum s_\alpha \alpha, \sum t_\beta \beta) \\ &= \sum \underbrace{s_\alpha t_\beta}_{\ge 0} \underbrace{(\alpha, \beta)}_{\le 0} \end{aligned}$$

so $s_\alpha = t_\beta = 0$.

Otherwise, $(\alpha, \beta) > 0$ and so $\alpha - \beta$ is a root by **4.15**. So $\alpha - \beta \in \Phi^+(\gamma)$ or $\beta - \alpha \in \Phi^+(\gamma)$. In the first case, $\alpha = (\alpha - \beta) + \beta$ and so α is not indecomposable and similarly for the second case.

Now we show that every base has this form. Suppose Δ is a given base. Choose γ such that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$ (check we can do this). So γ is regular; we'll show that $\Delta = \Delta(\gamma)$. Certainly $\Phi^+ \subset \Phi^+(\gamma)$. Hence $-\Phi^+ \subset -\Phi^+(\gamma)$. So $\Phi^+ = \Phi^+(\gamma)$. *Now I claim that $\Delta \subset \Delta(\gamma)$: if $\alpha \in \Delta$ could be written as $\alpha = \alpha_1 + \alpha_2$ for $\alpha_1, \alpha_2 \in \Phi^+(\gamma) = \Phi^+$, then write each α_i as a positive sum of roots in Δ; this is a contradiction to the unique expression of α as a sum of simple roots.* But $|\Delta| = |\Delta(\gamma)| = \dim E$. So $\Delta = \Delta(\gamma)$. ☐

Lemma 15.2. *(4.24)*

1. *$(\alpha, \beta) \le 0$ and $\alpha - \beta \notin \Phi$ for all distinct $\alpha, \beta \in \Delta$ (and hence the non-diagonal entries of the Cartan matrix w.r.t. Δ are ≤ 0).*
2. *If $\alpha \in \Phi^+$ and $\alpha \notin \Delta$ then $\alpha - \beta \in \Phi^+$ for some $\beta \in \Delta$.*
3. *Each $\alpha \in \Phi^+$ is of the form $\beta_1 + \cdots + \beta_k$ where each partial sum $\beta_1 + \cdots + \beta_i \in \Phi^+$ and where each $\beta_i \in \Delta$ (the terms in the sum are not necessarily distinct).*
4. *If α is simple then s_α (the reflection) permutes $\Phi^+ \backslash \{\alpha\}$. So if $\rho = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta$ then $s_\alpha(\rho) = \rho - \alpha$.*

Proof. (1) $\alpha - \beta \in \Phi$ would contradict part (2) of the definition of the base. "Obtuseness" (i.e. $(\alpha, \beta) \le 0$) follows from **4.15**.

(2) If $(\alpha, \beta) \le 0$ for all $\beta \in \Delta$ then $\Delta \cup \{\alpha\}$ would be linearly independent. *(Because $\alpha \in \Phi^+$ we can write $\alpha = \sum k_i \beta_i$ for $k_i \ge 0$. Then we have $0 < (\alpha, \alpha) = \sum \underbrace{k_i}_{\ge 0} \underbrace{(\beta_i, \alpha)}_{\le 0}$, which is a contradiction.)* So $(\alpha, \beta) > 0$ for some $\beta \in \Delta$, and so $\alpha - \beta \in \Phi$ by **4.15**. Write $\alpha = \sum k_{\beta_i} \beta_i$ where $k_{\beta_i} \in \mathbb{Z}^{\ge 0}$. If $k_\beta > 0$ then $\alpha - \beta$ still has all positive coefficients. *Since $\alpha \notin \Delta$, it can be written as a sum of at least two simple roots, and subtracting β still leaves one simple root with positive coefficient. Since $\alpha - \beta \in \Phi$, it is a* positive *root.*

(3) follows from (2) by induction.

(4) If $\beta = \sum k_\gamma \gamma \in \Phi^+ \backslash \{\alpha\}$ then there is some $k_\gamma > 0$ with $\gamma \neq \alpha$. But the coefficient of γ in $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is $k_\gamma > 0$. So $s_\alpha(\beta) \in \Phi^+ \backslash \{\alpha\}$. ☐

Remark 15.3. The half positive root sum ρ is a fundamental object in representation theory.

Now consider the Weyl group – it's generated by the reflections s_α for $\alpha \in \Phi$.

Lemma 15.4. *(4.25) Let Δ be a base with simple roots, in a root system Φ.*

(1) *If $\sigma \in GL(E)$ satisfies $\sigma(\Phi) = \Phi$, then $\sigma s_\alpha \sigma^{-1} = s_{\sigma(\alpha)}$.*
(2) *If $\alpha_1, \cdots, \alpha_t \in \Delta$ is a collection of (not necessarily distinct) roots, and if $s_{\alpha_t} \cdots s_{\alpha_2}(\alpha_1)$ is a negative root, then for some u with $1 \leq u \leq t$ we have $s_{\alpha_t} \cdots s_{\alpha_1} = s_{\alpha_t} \cdots s_{\alpha_{u+1}} s_{\alpha_{u-1}} \cdots s_{\alpha_2}$.*
(3) *If $\sigma = s_t \cdots s_1$ is an expression for σ in terms of simple reflections (corresponding to simple roots $\alpha_1, \ldots, \alpha_t$) with t minimal, then $\sigma(\alpha_1) < 0$.*

Proof. (1) Take $\alpha \in \Phi$, $\beta \in E$. Then

$$\begin{aligned}(\sigma s_\alpha \sigma^{-1})\sigma(\beta) &= \sigma s_\alpha(\beta) \\ &= \sigma(\beta - n(\beta,\alpha)\alpha) \\ &= \sigma(\beta) - n(\beta,\alpha)\sigma(\alpha)\end{aligned}$$

So $\sigma s_\alpha \sigma^{-1}$ fixes the hyperplane $\sigma(P_\alpha)$ elementwise, and sends $\sigma(\alpha) \mapsto -\sigma(\alpha)$.

Thus $\sigma s_\alpha \sigma^{-1} = s_{\sigma(\alpha)}$.

(2) Take u minimal with $s_u \cdots s_2(\alpha_1) < 0$. Then $1 < u \leq t$ and $\beta = s_{u-1} \cdots s_2(\alpha_1) > 0$ by minimality. By **4.24**(4) we have $\beta = \alpha_u$. Let $\sigma = s_{u-1} \cdots s_2$. Then $s_u = s_{\sigma(\alpha_1)} = \sigma s_{\alpha_1} \sigma^{-1}$ by (1). The result follows by rearranging.

(3) is immediate. ☐

Remark 15.5. Compare with the proof that an orthogonal map is a product of reflections. We're looking at a particular orthogonal map preserving the root system Φ, and showing that it is a product of well-chosen reflections.

Lecture 16: November 12

Lemma 16.1. *(4.26) Let $W = W(\Phi)$ be the Weyl group.*

(1) *If $\gamma \in E$ is regular, then there exists $\sigma \in W$ with $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$. Thus W permutes the bases transitively.*
(2) *For $\alpha \in \Phi$ then $\sigma(\alpha) \in \Delta$ for some $\sigma \in W$.*
(3) *W is generated by the simple reflections: $W = \langle s_\alpha : \alpha \in \Delta \rangle$*

(4) *W permutes the bases regularly: if $\sigma(\Delta) = \Delta$ for $\sigma \in W$ then $\sigma = 1$.*

Proof. Let $W' = \langle s_\alpha : \alpha \in \Delta \rangle \leq W$. It suffices to prove (1) and (2) for W' in place of W. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Choose $\sigma \in W'$ with $(\sigma(\gamma), \rho)$ as large as possible. Then for $\alpha \in \Delta$ we have $s_\alpha \sigma \in W'$. So

$$(\sigma(\gamma), \rho) \geq (s_\alpha \sigma(\gamma), \rho) = (\sigma(\gamma), s_\alpha(\rho))$$

where equality holds because reflections preserve inner products.

$$= (\sigma(\gamma), \rho) - (\sigma(\gamma), \alpha)$$

using **4.24**(4)

Hence each $(\sigma(\gamma), \alpha) \geq 0$; equality would imply $(\gamma, \sigma^{-1}(\alpha)) = 0$ and hence $\gamma \in P_{\sigma^{-1}(\alpha)}$ contradicting regularity. Also $\sigma^{-1}(\Delta)$ is a base with $(\gamma, \alpha) > 0$ for all $\alpha' \in \sigma^{-1}(\Delta)$. So the argument of **4.23**(3) shows that $\sigma^{-1}(\Delta) = \Delta(\gamma)$. Since any base is of the form $\Delta(\gamma)$ (by **4.23**), transitivity on bases follows.

(2) It suffices to show that each root α is in a base and then we use (1). Choose $\gamma_1 \in P_\alpha \backslash \bigcup(P_\beta : \beta \neq \pm\alpha)$. Let $\varepsilon = \frac{1}{2}\min(|(\gamma_1, \beta)| : \beta \neq \pm\alpha)$. Choose γ_2 with $0 < (\gamma_2, \alpha) < \varepsilon$ and $|(\gamma_2, \beta)| < \varepsilon$ for each $\beta \neq \pm\alpha$ (e.g. γ_2 may be a multiple of α *this works because there are finitely many β's, and there is a $\delta\alpha$ that satisfies this condition for each β; take the minimum δ*) . Define $\gamma = \gamma_1 + \gamma_2$. Then $0 < (\gamma, \alpha) < \varepsilon$ because γ_1 is orthogonal to α. Now I claim that $|(\gamma, \beta)| > \varepsilon$ for each $\beta \neq \pm\alpha$.

$$\begin{aligned} |(\gamma, \beta)| &= |(\gamma_1, \beta) + (\gamma_2, \beta)| \\ &\geq \underbrace{|(\gamma_1, \beta)|}_{\geq 2\varepsilon} - \underbrace{|(\gamma_2, \beta)|}_{<\varepsilon} > \varepsilon \end{aligned}$$

Suppose $\alpha = \beta_1 + \beta_2$ where $\beta_i \in \Phi^+(\gamma)$. By definition, $(\gamma, \beta_i) > 0$ so the assertion above that $|(\gamma, \beta_i)| > \varepsilon$ here says $(\gamma, \beta_i) > \varepsilon$. But then

$$(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2) > 2\varepsilon$$

contradicting the fact that $(\gamma, \alpha) < \varepsilon$. So α is an indecomposable element of $\Phi^+(\gamma)$ and so $\alpha \in \Delta(\gamma)$.

(3) It's enough to show $\alpha \in \Phi$ implies $s_\alpha \in W'$ (and so $W = W'$). By (2) find some $\sigma \in W'$ with $\sigma(\alpha) \in \Delta$. So $s_{\sigma(\alpha)} \in W'$. But $s_{\sigma(\alpha)} = \sigma s_\alpha \sigma^{-1}$ by **4.25**(1). σ and σ^{-1} are products of simple reflections, so s_α is as well.

(4) Suppose (4) is false: that $\sigma \neq 1$ and $\sigma(\Delta) = \Delta$. Write σ as a product of simple reflections in the shortest way possible. Use **4.25**(3) to obtain a contradiction. ☐

Definition 16.2. (4.27) The *Weyl chambers* of Φ are the connected components of $E \backslash \bigcup(P_\alpha : \alpha \in \Phi)$, the sets of regular γ giving the same $\Delta(\gamma)$.

Remark 16.3. The base $\Delta(\gamma)$ is the set of roots perpendicular to hyperplanes bounding the Weyl chamber C containing γ and pointing into C. So the bases are in 1-1 correspondence with the Weyl chambers.

EXERCISE 16.4. For $\sigma \in W$, define the length $\ell(\sigma) = \ell$ to be the minimal number of simple reflections required when expressing σ as a product of simple reflections.

Define $n(\sigma)$ to be the number of positive roots mapped to negative roots by σ. Show that $\ell(\sigma) = n(\sigma)$.

THEOREM 16.5. *(4.28) $W(\Phi)$ is presented as*

$$\left\langle s_\alpha \text{ for } \alpha \in \Delta \,:\, s_x^2 = 1, (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \right\rangle$$

where $m(\alpha, \beta) = 2, 3, 4$ or 6 according to whether the angle between α and β is $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$, or $\frac{5\pi}{6}$.

We will not prove this.

Irreducible root systems.

Type	# positive roots	W	$\dim L$
A_ℓ	$\frac{1}{2}\ell(\ell+1)$	$S_{\ell+1}$	$\ell(\ell+2)$
B_ℓ, C_ℓ	ℓ^2	$C_2^\ell \rtimes S_\ell$[1]	$\ell(2\ell+1)$
D_ℓ	$\ell^2 - \ell$	index 2 subgroup of above	$\ell(2\ell-1)$
E_6	36	order $72 \cdot 6!$	78
E_7	63	order $8 \cdot 9!$	133
E_8	120	order $2^6, 3, 10!$	248
F_4	24	order 1152	52
G_2	6	D_{12} (exercise)	14

Construction of irreducible roots systems. General strategy: Take an orthonormal basis $\{e_1, \cdots, e_n\}$ of Euclidean n-space. Then set

$$I = \left\{ \text{integral combinations of the } \frac{1}{2}e_i \right\}$$

Let J be a subgroup of I. Take fixed reals $x, y > 0$ with $\frac{x}{y} = 1, 2,$or 3. Given this setup we will define $\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = x \text{ or } y\}$ such that $E = \mathit{span}(\Phi)$. We need that each reflection preserves lengths and leaves Φ invariant and so ensure that $n(\beta, \alpha) \in \mathbb{Z}$. Note that if $J \leq \mathbb{Z}e_i$ (rather than the $\frac{1}{2}e_i$) and $\{x, y\} \subset \{1, 2\}$ then this is satisfied.

Let's do this for A_ℓ. $n = \ell + 1$; take

$$J = \left(\sum \mathbb{Z}e_i\right) \cap \left(\sum_{i=1}^{\ell+1} e_i\right)^\perp$$

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 2\} = \{e_i - e_j \,:\, i \neq j\}.$$

Then $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq \ell$ are linearly independent, and if $i < j$ then

$$e_i - e_j = \sum_{k=1}^{j-1} \alpha_k.$$

So $\{\alpha_i\}$ form a base of Φ.

Then $(\alpha_i, \alpha_j) = 0$ unless $j = i, i \pm 1$. $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_{i+1}) = -1$. So Φ has a Dynkin diagram of type A_ℓ. If you think about the Weyl group, each permutation $(1, \cdots, \ell+1)$ induces an automorphism of Φ. $W(\Phi) \cong S_{\ell+1}$ since s_{α_i} arises from the transposition of i and $i+1$, and the transpositions $(i, i+1)$ generate $S_{\ell+1}$. Note: this is the root system coming from $\mathfrak{sl}_{\ell+1}$.

Lecture 17: November 14

Last time we constructed a root system of type A_ℓ. We were trying to construct a root system as

$$\Phi = \left(\alpha \in J \leq \operatorname{span}\{\tfrac{1}{2}e_1, \cdots, \tfrac{1}{2}e_n\} \,:\, \|\alpha\|^2 = x \text{ or } y\right).$$

Type B_ℓ (for $\ell \geq 2$): Here $n = \ell$ (i.e. we're going to embed this in an ℓ-dimensional space), and take $J = \sum \mathbb{Z}e_i$. Set

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 1 \text{ or } 2\} = \{\pm e_i, \pm e_i \pm e_j, i \neq j\}$$

Take $\alpha_i = e_i - e_{i+1}$ for $i < \ell$, and $\alpha_\ell = e_\ell$. We have linear independence, and $e_i = \sum_{k=1}^{\ell} \alpha_k$, and $e_i - e_j = \sum_{k=1}^{j-i} \alpha_k$. So $\alpha_1, \cdots, \alpha_\ell$ form a base. This root system corresponds to B_ℓ. The Weyl group action has all permutations and sign changes, so it is:

$$\underbrace{C_2^\ell}_{\substack{sign\\changes}} \rtimes \underbrace{S_\ell}_{permutations}$$

This root system arises from $SO_{2\ell+1}(\mathbb{C})$, the skew-symmetric matrices of size $2\ell+1$.

Type C_ℓ**:** As before $n = \ell$. Set $J = \sum \mathbb{Z}e_i$ and

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 2 \text{ or } 4\} = \{\pm 2e_i, \pm e_i \pm e_j, i \neq j\}.$$

We have a base $e_1 - e_2, e_2 - e_3, \cdots, e_{\ell-1} - e_\ell, 2e_\ell$. This arises from $\mathfrak{sp}_{2\ell}(\mathbb{C})$, the symplectic Lie algebra. If

$$X = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

then this consists of the matrices A such that $AX + XA^t = 0$.

Type D_ℓ**:** Again set $n = \ell$ and $J = \sum \mathbb{Z}e_i$. Define

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 2\} = \{\pm e_i \pm e_j \,:\, i \neq j\}.$$

Set $\alpha_i = e_i - e_{i+1}$ for $i < \ell$, and $\alpha_\ell = e_{\ell-1} + e_\ell$. This gives a Dynkin diagram D_ℓ. The simple reflections cause permutations and an even number of sign changes. The Weyl group is a split extension (semidirect product) $C_2^{(\ell-1)}$ by S_ℓ, where $C_2^{(\ell-1)}$ is an index 2 subgroup in C_2^ℓ.

Type E_8: Here $n = 8$. Set $f = \frac{1}{2}\sum_{i=1}^8 e_i$ and define

$$\begin{aligned} J &= \{cf + \sum c_i e_i \,:\, c, c_i \in \mathbb{Z} \text{ and } c + \sum c_i \in 2\mathbb{Z}\} \\ \Phi &= \{\alpha \in J \,:\, \|\alpha\|^2 = 2\} \\ &= \{\pm e_i \pm e_j \,:\, i \neq j\} \cup \{\tfrac{1}{2}\sum_1^8 (-1)^k e_i \,:\, \sum k_i \text{ is even}\} \end{aligned}$$

The base is then:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(e_1 + e_8 - \sum_2^7 e_i) \\ \alpha_2 &= e_1 + e_2 \\ \alpha_i &= e_{i-1} - e_{i-2} \text{ for } i \geq 3 \end{aligned}$$

Too complicated to say much about the Weyl group.

Types E_6 and E_7: Take Φ from E_8 and intersect with subspaces $\Phi \cap \langle g \rangle^\perp$ (where $\langle g \rangle^\perp$ is the subspace of things orthogonal to g) and $\Phi \cap \langle g, h \rangle^\perp$ for suitable g, h. We obtain root systems E_7, E_6. The base is $\alpha_1, \cdots, \alpha_7$ in the E_7 case, and $\alpha_1, \cdots, \alpha_6$ in the E_6 case.

Type F_4: $n = 4$ here. Set $h = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$, and

$$J = \sum \mathbb{Z} e_i + \mathbb{Z} h$$

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 1 \text{ or } 2\} = \{\pm e_i, \pm e_i + e_j \ (i \neq j), \pm\tfrac{1}{2}e_1 \pm \tfrac{1}{2}e_2 \pm \tfrac{1}{2}e_3 \pm \tfrac{1}{2}e_4\}$$

The base is $e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$. Again the Weyl group is sort of complicated.

Type G_2: Set

$$J = \mathbb{Z} e_i \cap \langle e_1 + e_2 + e_3 \rangle^\perp$$

$$\Phi = \{\alpha \in J \,:\, \|\alpha\|^2 = 2 \text{ or } 6\} = (\pm\{e_i - e_j, 2e_i - e_j - e_k \,:\, i, j, k \text{ are distinct}\}).$$

The base is:

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= -2e_1 + e_2 + e_3 \end{aligned}$$

Exercise 17.1. Show that Φ can be described as the set of integers of norm 1 or 3, in the quadratic field generated by cube roots of 1.

Definition 17.2. Suppose we are given a root system Φ arising from a Lie algebra of base Δ, with positive roots Φ^+. Set

$$N = \sum_{\alpha \in \Phi^+} L_\alpha \qquad N^- = \sum_{\alpha \in \Phi^-} L_\alpha \qquad H = L_0 \qquad B = H \oplus N$$

We call B the *Borel subalgebra* corresponding to Δ and H. Note that

$$L = N^- \oplus H \oplus N = N^- \oplus B.$$

Recall that B is soluble, N and N^- are nilpotent, and $B^{(1)} = N$.

THEOREM 17.3. *(4.29) We can present the Lie algebra by generators X_i, Y_i, K_i. We have* Weyl relations*:*

$$\begin{aligned} [K_i, K_j] &= 0 \\ [X_i, Y_i] &= K_i \\ [X_i, Y_j] &= 0 \text{ if } i \neq j \\ [K_i X_j] &= n(i,j) X_j \\ [K_i, Y_j] &= -n(i,j) Y_j. \end{aligned}$$

There are also other relations:

$$\begin{aligned} (ad\ X_i)^{1-n(i,j)} X_j &= 0 \\ (ad\ Y_i)^{1-n(i,j)} Y_j &= 0. \end{aligned}$$

N is generated by the X_i, N^- is generated by the Y_i, and L is generated by the elements X_i, Y_i, and K_i.

(Not proven here, but I think you can do it by putting things together.)

This fits in with what we did before when we found $h_\alpha, e_\alpha, e_{-\alpha}$, where $h_\alpha \in H$, $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$. Set

$$K_i = 2\frac{h_{\alpha_i}}{(h_{\alpha_i}, h_{\alpha_i})}$$

(so we are basically just scaling h_α from before). With this scaling,

$$\alpha_i(K_j) = (h_{\alpha_i}, k_j) = n(i,j)$$

a Cartan matrix element. Fix $X_i \in L_{\alpha_i}, Y_i \in L_{-\alpha_i}$ with $[X_i, Y_i] = K_i$.

THEOREM 17.4. *(4.30) [Serre] For each reduced root system there exists a semisimple Lie algebra presented by generators and relations as in Theorem* **4.29**.

(Not proven here.)

Putting theorems **4.29** and **4.30** together shows that there is a 1-1 correspondence

$$\text{semisimple Lie algebras} \iff \text{root systems}$$

and we have a presentation for the Lie algebras. There is also a correspondence

$$\text{simple Lie algebras} \iff \text{irreducible root systems}.$$

EXERCISE 17.5. Look at type A_ℓ, B_ℓ, C_ℓ, D_ℓ and see that we do have presentations in terms of generators and relations as Theorem **4.29**.

Lecture 18: November 16

Next examples class on Thursday next week.

Chapter 5: Representation theory

Definition 18.1. The *universal enveloping algebra* $U(L)$ is an associative algebra generated by elements X of L subject to the relations

(1) $XY - YX - [X,Y] = 0$ *($[X,Y]$ is inherited from the Lie algebra structure on L, and XY is the product in the algebra);*
(2) ("linearity"/ *compatibility with structure of L*) $\lambda X + \mu Y$ as an element of $U(L)$ equals $\lambda X + \mu Y$ as an element of L.

If you want you can just take a generating set consisting of a basis $\{X_i\}$ for the Lie algebra.

Alternatively, you can view this as a quotient of the tensor algebra $T(L)$,

$$T(L) = k \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \cdots$$

which is a graded algebra. In this case, define $U(L) = T(L)/I$, where I is the ideal generated by $X \otimes Y - Y \otimes X - [X,Y]$.

There is a 1-1 correspondence

$$\left\{ \begin{matrix} \text{representations} \\ \rho : L \to \mathfrak{gl}(V) \end{matrix} \right\} \iff \left\{ \text{ left } U(L)\text{-modules } \right\}.$$

This should remind you of a similar correspondence in group theory:

$$\{ \text{ representations } \rho : G \to GL(V) \ \} \iff \{kG\text{-modules } V\}$$

where G is a finite group and kG is the group algebra.

For example, if L is abelian (and so $[X,Y] = 0$), then $U(L)$ is a commutative algebra. If L has basis $X_1, \cdots, X_n$ then $U(L) \cong k[X_1, \cdots, X_n]$ is the polynomial algebra in n variables.

However if L is non-abelian then $U(L)$ is a non-commutative algebra. But it's actually quite well-behaved. We can write $U(L) = \bigcup F_i$ where F_i is the span of products of $\leq i$ elements of L. We have an associated graded algebra

$$\operatorname{gr} U(L) = \bigoplus_{i \geq 0} F_i / F_{i-1}$$

that is commutative because of the relations $XY - YX = [X,Y]$ (if you define $F_{-1} = \{0\}$). (Note this is also an associative algebra: if $\alpha \in F_i$ and $\beta \in F_j$ then $(\alpha + F_{i-1})(\beta + F_{j-1}) = \alpha\beta + F_{i+j-1}$.)

$U(L)$ is said to be *almost commutative*, because the associated graded algebra is commutative.

THEOREM 18.2 (Poincaré-Birkhoff-Witt). *(5.2) If $X_1, \cdots, X_n$ is a basis of L then*

$$\{X_1^{m_1} \cdots X_n^{m_n} \,:\, m_i \geq 0\}$$

forms a basis for $U(L)$. In particular, $\operatorname{gr} U(L)$ is a (commutative) polynomial algebra in n variables.

(Proof later.)

COROLLARY 18.3. *(5.3) $U(L)$ is left-Noetherian (ascending chains of left ideals terminate).*

We're aiming for:

THEOREM 18.4 (Weyl). *(5.4) Suppose L is a semisimple, finite-dimensional Lie algebra over a field k of characteristic zero. Then all finite-dimensional representations are completely reducible. (That is, they may be expressed as a direct sum of irreducible representations.)*

Before proving this let's consider a special case.

PROPOSITION 18.5. *Suppose $\rho : L \to \mathfrak{gl}(V)$ is a representation of a semisimple Lie algebra L, and suppose there is a $\rho(L)$-invariant, irreducible subspace W such that $\dim V/W = 1$.*

Then there is a direct sum decomposition

$$V = W \oplus W'$$

for some 1-dimensional space W'.

Note that the induced representation $\overline{\rho} : L \to \mathfrak{gl}(V/W)$ is zero, because there are no nontrivial 1-dimensional representations of a semisimple Lie algebra.

Proof. Since quotients of semisimple Lie algebras are semisimple, we may *replace L with $L/\ker\rho$ and* assume that $\ker\rho = 0$; that is, ρ is a faithful representation.

Let β be the trace form of ρ, i.e. $\beta(x, y) = tr(\rho(x)\rho(y))$. This is an invariant symmetric bilinear form (i.e. $\beta(x, [y, z]) = \beta([x, y], z)$). (The Killing form is one example.)

In fact, it is non-degenerate – its radical R *(the set $\{x : \operatorname{tr}(\rho(x)\rho(y)) = 0\ \forall y\}$)* is an ideal of L which is necessarily soluble by Cartan's solubility criterion. Since we're dealing with a semisimple Lie algebra, we can't have any soluble ideals. Hence $R = 0$.

Take a basis $X_1, \cdots, X_n$ of L, and $Y_1, \cdots, Y_n$ of $L^\vee$ (dual of L) such that $\beta(X_i, Y_j) = \delta_{ij}$. Define the Casimir element $c = \sum_i \rho(X_i)\rho(Y_i) \in \mathfrak{gl}(V)$. Check that c commutes with $\rho(L)$ using the invariance of β; that is, $[c, \rho(L)] = 0$.

So $\ker c$, the kernel of multiplication by c, is invariant under $\rho(L)$. Since $\rho(L)(V) \leq W$ then $c(V) \leq W$. But we're supposing W is irreducible and so $c(W) = W$ or 0. But

$c(W) = 0$ implies $c^2 = 0$, which implies that

$$\begin{aligned} 0 &= \operatorname{tr} c && \text{(nilpotent endomorphisms have trace zero)} \\ &= \operatorname{tr}\left(\sum \rho(X_i)\rho(Y_i)\right) \\ &= \sum \beta(X_i, Y_i) = \dim V \end{aligned}$$

which is a contradiction since the characteristic is zero.

So $c(W) = W$ and hence $\ker c \cap W = 0$. But $c(V) \leq W$, and so $\ker c > \{0\}$. So $V = W \oplus \ker c$, and we've established that there is an invariant complementary subspace. □

The Casimir element appearing here is the image under ρ of an element of the enveloping algebra. $\mathrm{End}_k(L) \cong L \otimes L^* \cong L \otimes L$ using the isomorphism between L^* and L determined by β. Take the identity in $\mathrm{End}(L)$; its image in $L \otimes L$ is the Casimir element.

The invariance of our form ensures that the image is central in $U(L)$.

Lecture 19: November 19

Examples class: Thursday, 2-3:30.

We were proving Weyl's theorem (18.4). We've proved the case where $0 \leq W < V$, where V, W were $\rho(L)$-invariant, W was irreducible, and $\dim V/W = 1$. We showed that $V = W \oplus W'$, where W' was something 1-dimensional.

We start by redoing the last proposition in slightly more generality: removing the assumption that W is irreducible.

Proposition 19.1. *Suppose $\rho : L \to \mathfrak{gl}(V)$ is a representation of a semisimple Lie algebra L, and suppose there is a $\rho(L)$-invariant subspace W such that* $\dim V/W = 1$.

Then there is a direct sum decomposition

$$V = W \oplus W'$$

for some 1-dimensional space W'.

Proof. Suppose we've got $0 \leq W < V$ with $\dim V/W = 1$, and V/W has trivial representations.

Argue by induction on $\dim V$ *(induction hypothesis: if a n-dimensional space has a codimension-1 invariant subspace, then that's a direct summand).* If W is irreducible then we're done, by the special case from last time. If not, we have $0 < U < W$ with U invariant. By induction applied to V/U *with codimension-1 subspace W/U*, then

$V/U = W/U \oplus W'/U$, with W'/U being 1-dimensional. But then *(using induction again W')* $W' = U \oplus W''$. Then $V = W \oplus W''$. □

Proof of Weyl's theorem, **5.4**. Let $\rho : L \to \mathfrak{gl}(A)$ be a representation. If A is irreducible, we're done, so assume there is a $\rho(L)$-invariant subspace $B \leq A$ (i.e. $\rho(L)(B) \leq B$). Let $\mu : L \to \mathfrak{gl}(\mathfrak{gl}(A))$ be the map sending $x \mapsto (t \mapsto [\rho(x), t])$, where $[-,-]$ is the commutator (Lie bracket) in $\mathfrak{gl}(A)$. *Note that $\mu(\ell)(t) = 0$ means that t commutes with $\rho(\ell)$, and $\mu(L)(t)$ means that t commutes with all of $\rho(L)$.* Define

$$V = \{v \in \mathfrak{gl}(A) : v(A) \leq B, v|_B \text{ is a scalar map on } B\}$$

This has a subspace

$$W = \{v \in \mathfrak{gl}(A) : v(A) \leq B, v|_B = 0\}$$

Then $\dim V/W = 1$. Check that $\mu(L)(V) \leq W$; in particular, $\mu(L)W \leq W$. Applying the special case discussed at the beginning today, we have a complementary W_1 with

$$V = W \oplus W_1,$$

and $\mu(L)(W_1) \leq W_1$.

There is some $u \in W_1$ with $u|_B = Id_B$. *We'll see that this is a projection to B.* Also, for all $x \in L$ we have $\mu(x)(u) \in W \cap W_1 = \{0\}$. Thus u commutes with $\rho(L)$. So it is an L-endomorphism of A. Now $\rho(L)(\ker u) \leq \ker u$, i.e. $\ker u$ is invariant. But $\ker u \cap B = 0$ (since $u|_B = Id_B$). If $a \in A$ then $u(a) \in B$ and $u(1-u)(a) = 0$, and $a = u(a) + (1-u)(a)$. So $A = B \oplus \ker u$.

I claim that $A = B \oplus \ker u$, and that B and $\ker u$ are $\rho(L)$-invariant; by induction, each of these summands is a direct sum of irreducibles, and we're done. That is, we need to show the following:

(1) $B \cap \ker u = \{0\}$. *No nonzero element of B is killed by u because $u|_B = Id$.*
(2) $A = B + \ker u$. *Let $a \in A$. We need elements in $\ker u$ and in B. Our element of B is $u(a)$; this is in B because $u \in W_1 \subset V$ and elements of V send A into B. I claim $(1-u)(a) \in \ker u$, or equivalently $u(u(a)) = u(a)$, i.e. u is a projection to B. u maps A into B, and u is the identity on B. Now just notice that* $a = \underbrace{u(a)}_{\in B} + \underbrace{(1-u)a}_{\in \ker u}$.
(3) *B is $\rho(L)$-invariant. This is given information: we started by assuming $B \leq A$ was an invariant subspace.*
(4) $\ker u$ *is $\rho(L)$-invariant. We need $\rho(L)\ker u \leq \ker u$. If $a \in \ker u$, then we need $u(\rho(L)a) = 0$. It suffices to show that u commutes with $\rho(\ell)$. For that, it suffices to show that $0 = [\rho(\ell), u] = \mu(\ell)(u)$, and to show this, it suffices to show that $\mu(\ell)(u) \in W \cap W_1$ (this intersection is zero by definition of W_1).*
   - *$u \in W_1$ because $\mu(\ell)u \in W_1$ by invariance*
   - *To show $u \in W$, we need to show that it kills B:*

$$\begin{aligned}[\rho(\ell), u]b &= (\rho(\ell)u - u\rho(\ell))b \\ &= \rho(\ell)b - u\underbrace{\rho(\ell)b}_{\in B}\end{aligned}$$

$$= \rho(\ell)b - \rho(\ell)b = 0$$

□

Hard Exercise/ Theorem 19.2. Let $J \lhd L$ with L/J is semisimple, then there is a subalgebra L_1 of L with $L_1 \cap J = \{0\}$ and $L_1 + J = L$. Then $L_1 \cong L/J$ is semisimple.

Hint. We need to induct on $\dim L$. Reduce to the case where J is an abelian ideal. Look at

$$V = \{\varphi \in \operatorname{End} L \,:\, \varphi(L) \leq J, \varphi|_J \text{ is scalar}\}$$

which has a subspace

$$V = \{\varphi \in \operatorname{End} L \,:\, \varphi(L) \leq J, \varphi|_J = 0\}$$

so $\dim V/W = 1$. This is very similar to what we've just been doing. Try to fill in the details. □

Applying in the case where J is the radical of L, we get

Theorem 19.3 (Levi). *(5.5) If R is the radical of L then there exists a semisimple (or zero) Lie subalgebra L_1 such that $L = R + L_1$ and $R \cap L_1 = \{0\}$.*

Definition 19.4. L_1 in **5.5** is a *Levi subalgebra, or Levi factor* of L.

Remark 19.5. This can be phrased in terms of degree 2 cohomology groups of semisimple Lie algebras – the degree 2 cohomology is zero.

Recall: L is semisimple, H is a Cartan subalgebra with roots Φ, positive roots Φ^+, and base $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ of simple roots. We had $N = \sum_{\alpha \in \Phi^+} L_\alpha$, $N^- = \sum_{-\alpha \in \Phi^+} L_\alpha$, and $H = L_0$. We had a Borel subalgebra $B = H \oplus N$ (direct sum of vector spaces). We know that the derived subalgebra $B^{(1)} = N$, and

$$L = N^- \oplus H \oplus N$$

(This all came from the Cartan decomposition.)

Definition 19.6. (5.7) If $\rho : L \to \mathfrak{gl}(V)$ is a representation, then let

$$V^w = \{v \in V \,:\, \rho(h)v = w(h)v \ \forall h \in H\}$$

be the *weight space of weight w* in V. (Note we are using superscripts here, instead of subscripts as in the Chapter 4 stuff.) Before we were doing this with the adjoint representation; here we are using a general representation. The *multiplicity of w in V* is defined to be $\dim V^w$ if $\dim V^w \neq 0$. The set of such w are called the *weights of V*.

Lemma 19.7. *(5.8) We have the following:*

1. *$\rho(L_\alpha)V^w \subset V^{w+\alpha}$ if $w \in H^*$, $\alpha \in \Phi$;*
2. *the sum of the V^w (for weights w) is direct, and is invariant under $\rho(L)$.*

Proof. Exercise (similar to what was done before.) □

Definition 19.8. (5.9) v is a *primitive element of weight w* if it satisfies:

(1) $v \neq 0$ and has weight w
(2) $\rho(X_\alpha)(v) = 0$ for all $\alpha \in \Phi^+$. (Recall we have elements $X_\alpha \in L_\alpha, Y_\alpha \in L_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha \in L_0 = H$.)

Thus if v is primitive, $\rho(B)(v)$ is 1-dimensional, since $B = H \oplus \left(\bigoplus_{\alpha\in\Phi^+} L_\alpha\right)$. (All the X_α give zero, and we're left with the things in H.)

Conversely, if there is a 1-dimensional subspace of V invariant under $\rho(B)$ then each $b \in B^{(1)}$ acts like 0. But $N = B^{(1)}$, and so v satisfies (2) in the definition of primitive elements.

Switch to the language of modules: v is primitive $\iff$ $U(B)(v)$ is 1-dimensional.

If we restrict to finite-dimensional representations, then we have primitive elements. The converse is not true, however: there are some infinite-dimensional representations that also have primitive elements.

Lecture 20: November 21

We're thinking about $U(L)$-modules V (which are equivalent to representations of L), in particular ones containing a primitive element v. Recall N was the sum of the positive root spaces, generated by $X_{\beta_1}, \cdots, X_{\beta_s}$ where the β_i are positive roots. N^-, the sum of negative root spaces, is generated by $Y_{\beta_1}, \cdots, Y_{\beta_s}$. $H \oplus N$ is called the Borel subalgebra B, and we have a direct sum decomposition $L = N^- \oplus B = N^- \oplus H \oplus N$.

$$U(L) = \bigoplus Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s} U(B)$$

The $U(B)$-module generated by v is 1-dimensional.

Proposition 20.1. *(5.10) Let v be a primitive element of weight w. Then*

(1) *The $U(L)$-module W generated by v is spanned as a vector space by $Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s}(v)$.*
(2) *The weights of W have the form $w - \sum_{i=1}^{\ell} p_i\alpha_i$ for $p_i \in \mathbb{N}$ and simple roots α_i.*
(3) *w is a weight of W with multiplicity 1.*
(4) *W is indecomposable.*

Proof. (1) By the Poincaré-Birkhoff-Witt theorem,

$$U(L) = \bigoplus Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s} U(B).$$

So

$$\begin{aligned} W &= U(L)(v) \\ &= \sum Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s} U(B)(v) \end{aligned}$$

By the definition of primitivity, $U(B)(v)$ is one-dimensional. So W is spanned by $Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s}(v)$.

(2) By Lemma **5.8**(1), $Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_1}(v)$ has weight $w - \sum m_i\beta_i$. But each β_i is an integral combination of simple roots with coefficients ≥ 0. So the weight of $Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s}(v)$ is of the form $w - \sum p_i\alpha_i$ with $p_i \geq 0$. The multiplicities (i.e. the dimensions of the weight spaces) are finite.

(3) $-\sum m_j\beta_j$ can only be 0 if each $m_j = 0$. So the only $Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s}(v)$ that is of weight w is v. So the weight space of W of weight w is 1-dimensional.

(4) If $W = W_1 \oplus W_2$ is a direct sum of two nonzero $U(L)$-modules, then $W^\omega = W_1^\omega \oplus W_2^\omega$. But W^ω has dimension 1 – it's spanned by v.

So one of W_i^ω is zero, and the other is 1-dimensional, and $v \in W_i$ for some i. But v generates W by definition, and so $W = W_i$ for some i. This is a contradiction. □

THEOREM 20.2. *(5.11) Let V be a simple $U(L)$-module (one that comes from an irreducible representation), and suppose it contains a primitive element v of weight ω.*

(1) *v is the only primitive element of V up to scalar multiplication. Its weight is the* highest weight *of V.*
(2) *The weights appearing in V are of the form $\omega - \sum p_i\alpha_i$ for $p_i \in \mathbb{Z}_{\geq 0}$, where α_i are the simple roots. They have finite multiplicity. ω has multiplicity 1, and V is the sum of the weight spaces.*
(3) *For two simple modules V_1, V_2 with highest weight ω_1 and ω_2, respectively, then*
$$V_1 \cong V_2 \iff \omega_1 = \omega_2$$

Proof. Apply **5.10**. Since V is simple, $V = W$ and we get (2) immediately.

(1) Let v' be a primitive element of weight ω' in V. Then by (2),
$$\omega' = \omega - \sum p_j\alpha_j \text{ for } p_j \geq 0$$
$$\omega = \omega' - \sum p'_j\alpha_j \text{ for } p'_j \geq 0$$
This is only possible if all the p_j, p'_j are zero and $\omega = \omega'$. So v' must be a scalar multiple of v.

(3) For $V_1 \cong V_2$ one knows that $\omega_1 = \omega_2$.

Conversely, suppose $\omega_1 = \omega_2$. Set $V = V_1 \oplus V_2$ and take $v = v_1 + v_2$, where v_1 is the primitive element in V_1. v is a primitive element of V. Let $W = U(L)(v)$. The projection $\pi : V \to V_2$ induces a homomorphism $\pi|_W : W \to V_2$, and we know that $\pi(v) = v_2$. So π is surjective (since v_2 generates V_2). $\ker \pi|_W = V_1 \cap W \leq V_1$; however, the only elements of W of weight ω are scalar multiples of v (**5.10**), and $v \notin \ker \pi$. But V_1 is simple and so $\ker \pi = 0$, being a proper submodule. So $W \cong V_2$. But similarly, $W \cong V_1$. So $V_1 \cong V_2$. □

THEOREM 20.3. *(5.12) For each $\omega \in H^*$, there is a simple $U(L)$-module of highest weight ω (i.e. one that has a primitive element of that particular weight).*

Some of these might not be finite-dimensional.

Proof. For each $\omega \in H^*$, one can define a 1-dimensional $U(B)$-module spanned by v of weight ω by saying $X_\alpha(v) = 0$ for all $\alpha \in \Phi^+$. (Recall $B = H \oplus N$, and here N is acting like zero. So really, we have an action of the CSA H.) Then form

$$\bigoplus Y_{\beta_1}^{m_1} \cdots Y_{\beta_s}^{m_s} \langle v \rangle$$

a direct sum of subspaces induced by monomials. This is a $U(L)$-module – it's actually

$$U(L) \otimes_{U(B)} \langle v \rangle$$

This module has a primitive element v of weight ω. If V' is a $U(L)$-submodule then it is a direct sum of weight spaces and it is a proper submodule iff it lies in

$$\bigoplus_{\substack{not\ all\\ m_i = 0}} Y_{\beta_s}^{m_1} \cdots Y_{\beta_s}^{m_s}(v)$$

We can form the sum of all proper submodules – it will lie in this direct sum, and itself be proper *(if $\sum_{\substack{proper\\ S}} S$ were all of $U(L)v$, then v would be in it, hence in some S, a contradiction)*. If we sum all the submodules $\neq V$, we get something that is $\neq V$, and so we have a maximal proper submodule of our induced module. So we can form the quotient module, which is therefore simple, and the highest weight is ω. We can write down a primitive element, the image of our original primitive element inside the quotient. Thus we do have a simple module with highest weight ω. □

Remark 20.4. Sometimes the induced module is itself simple, and note that this would give us an infinite-dimensional simple module. Sometimes the simple module is finite-dimensional.

Lecture 21: November 23

Example 21.1. Let L be a simple, finite-dimensional Lie algebra. Then it is a simple $U(L)$-module via the adjoint action. The highest weight is the unique root that dominates the others. Note that in this case in our procedure from last time we are quotienting out by a nonzero maximal submodule of the induced module. For example, $\mathfrak{sl}_2 = N^- \oplus H \oplus N$, where N is the span of $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, H is the span of $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and N^- is the span of $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Recall $[H, X] = 2X$ and $[H, Y] = -2Y$. N is the highest weight space. We can get to the others by applying ad_Y.

Definition 21.2. (5.13) The induced module we used, $U(L) \otimes_{U(B)} U$ (for any a 1-dimensional $U(B)$-module U of weight w) is called the *Verma module of highest weight w*.

We showed that it has a unique simple quotient.

Proposition 21.3. *(5.14) Let V be a finite-dimensional $U(L)$-module, and L semisimple. Then*

(1) $V = \sum V^\lambda$ *(a sum of weight spaces)*
(2) *If* $V \neq 0$ *then* V *contains a primitive element.*
(3) *If* V *is generated by a primitive element then it is simple.*

Proof. (1) We've already seen that V is a sum of weight spaces, since the elements of H are semisimple, and thus are diagonalizable, and the weight spaces are the common eigenspaces.

(2) The Borel subalgebra is soluble and Lie's Theorem tells us that V contains a common eigenvector. This is a primitive element.

(3) We've seen (Weyl's theorem) that V is a direct sum of simple $U(L)$-modules. But if V is generated by a primitive element, then it is indecomposable (last time). Thus V is simple in this case. $\square$

So it remains for the finite dimensional representation theory to identify the weights giving rise to finite dimensional simple modules and to establish the multiplicities of the weights for such a module.

Back to $\mathfrak{sl}_2$: we have H, X, Y as before.

Proposition 21.4. (5.15) *Let* V *be a* $U(\mathfrak{sl}_2)$*-module with primitive element* v *of weight* ω*. Set* $e_n = \frac{Y^n}{n!}(v)$ *for* $n \geq 0$*, with* $e_{-1} = 0$*. Then*

(1) $H \cdot e_n = (\omega - 2n)e_n$
(2) $Y \cdot e_n = (n+1)e_{n+1}$
(3) $X \cdot e_n = (\omega - n + 1)e_{n-1}$ *for all* $n \geq 0$*.*

Proof. (2) is obvious.

(1) says e_n has weight $\omega - 2n$.

$$HY(v) = ([H,Y] + YH)(v) = (-2Y + YH)(v) = (\omega - 2)Y(v)$$

An easy induction yields that $Y^n(v)$ has weight $\omega - 2n$.

(3)

$$\begin{aligned} nXe_n &= XY(e_{n-1}) \\ &= ([X,Y] + YX)(e_{n-1}) \\ &= H(e_{n-1}) + (\omega - n + 2)Ye_{n-2} \\ &= (\omega - 2n + 2)e_{n-1} + (\omega - n + 2)(n-1)(e_{n-1}) \\ &= n(\omega - n + 1)e_{n-1} \end{aligned}$$

$\square$

Corollary 21.5. (5.16) *Either*

(1) $\{e_n : n \geq 0\}$ *are all linearly independent; or*

(2) *the weight ω of the primitive element v is an integer $m \geq 0$, and the elements $e_1, \cdots, e_m$ are linearly independent and $e_i = 0$ for $i > m$.*

Proof. From **5.15**, the elements e_i have distinct weights, and so the nonzero ones are linearly independent. If they are all nonzero we have case (1), and if not we have $e_0, \cdots, e_m$ nonzero and then $0 = e_{m+1} = e_{m+2} = \cdots$. By **5.15**(3) with $n = m+1$ we get $Xe_{m+1} = (\omega - m)e_m$. But $e_{m+1} = 0$, and $e_m \neq 0$. Deduce that $\omega = m$. This is case (2). $\square$

Corollary 21.6. *(5.17) If V is finite-dimensional then we have case (2). The subspace W of V with basis $e_0, \cdots, e_m$ is a $U(\mathfrak{sl}_2)$-submodule – it is generated by a primitive element, and thus is a simple $U(\mathfrak{sl}_2)$-module.*

Proof. Finite dimensionality implies we have a primitive element and we're in case (2). H has eigenvalues $m, m-2, \cdots, -m$, each with multiplicity 1 in the submodule W. $\square$

Remark 21.7. Note that in the case where the highest weight $m = 1$, the simple module we're generating has dimension 2 and is the canonical module for $\mathfrak{sl}_2$. In general for highest weight m the simple module generated is the m^{th} symmetric power of the canonical 2-dimensional one.

Theorem 21.8. *(5.18) For $\omega \in H^*$ and E_ω the simple $U(L)$-module of highest weight ω. (E_ω is often called $L(\omega)$.) Then*

$$E_\omega \text{ is finite-dimensional} \iff \text{for all } \alpha \in \Phi^+, \omega(H_\alpha) \in \mathbb{Z}_{\geq 0}$$

Proof. Recall notation: we have $X_\alpha, Y_\alpha, H_\alpha$ corresponding to each $\alpha \in \Phi^+$. We have that $X_\alpha, Y_\alpha, H_\alpha$ generate a subalgebra of L is isomorphic to $\mathfrak{sl}_2$.

($\implies$) If v is a primitive element (for L – if you take a Borel subalgebra, v is generating a 1-dimensional submodule) of E_ω, then it is also primitive for each $\mathfrak{sl}_2$-subalgebra $\langle H_\alpha, X_\alpha, Y_\alpha \rangle$ (since the Borel subalgebra of each $\mathfrak{sl}_2$-subalgebra is inside the Borel subalgebra of L). So we know $\omega(H_\alpha)$ must be an integer ≥ 0, from what we've just done.

Lecture 22: November 26

Proof of Theorem **5.18**. ($\implies$) (Already did this.) We have copies of $\mathfrak{sl}_2$ generated by $X_\alpha, Y_\alpha, H_\alpha$. Apply our knowledge of $U(\mathfrak{sl}_2)$-modules.

($\impliedby$) Assume $\omega(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta$. Take a primitive element of weight ω in E_ω. For each $\alpha \in \Delta$ let $m_\alpha = \omega(H_\alpha)$, $v_\alpha = Y_\alpha^{m_\alpha + 1}(v)$.

If $\alpha \neq \beta$, then $[X_\alpha, Y_\beta] = 0$ (i.e. X_α and Y_β commute), and so $X_\beta v_\alpha = Y_\alpha^{m_\alpha+1} X_\beta(v)$. But $X_\beta(v) = 0$ since v is a primitive element, and so this is zero. But we know that $X_\alpha(v_\alpha) = 0$ because of the definition of v_α. If $v_\alpha \neq 0$, then it would be primitive of weight $\omega - (m_\alpha + 1)\alpha$. Primitive elements in E_ω are just scalar multiples of v. But it has the wrong weight, so this is a contradiction, and $v_\alpha = 0$.

$$\{Y_\alpha^c(v) \,:\, 1 \leq c \leq m_\alpha\}$$

spans a finite-dimensional nonzero $U(\mathfrak{g}_\alpha)$-submodule of E_ω, where $\mathfrak{g}_\alpha$ is the copy of $\mathfrak{sl}_2$ spanned by $X_\alpha, Y_\alpha, H_\alpha$.

Now let

$$T_\alpha = \{ \text{ finite-dimensional } U(\mathfrak{g}_\alpha)\text{-submodules of } E_\omega \}$$

Let U_α be their sum. Then U_α is a nonzero $U(L)$-submodule of our simple E_ω. So $E_\omega = U_\alpha$ and we've shown that E_ω is a sum of finite-dimensional $U(\mathfrak{g}_\alpha)$-submodules, for each α. Now let P_ω be the set of weights of E_ω.

Claim 22.1. *P_ω is invariant under the simple reflection s_α.*

Proof. Let $\pi \in P_\omega$, and let $y \neq 0$ in E_ω^π. We know about the weights of E_ω: $\pi(H_\alpha)$ is an integer, call it p_α. Put

$$x = \begin{cases} Y_\alpha^{p_\alpha}(y) & \text{if } p_\alpha \geq 0 \\ -X_\alpha^{-p_\alpha}(y) & \text{if } p_\alpha \leq 0 \end{cases}$$

But then $x \neq 0$, since the weight of x is $\pi - p_\alpha\alpha = \pi - \pi(H_\alpha)\alpha = s_\alpha(\pi)$, P_ω is invariant under s_α. □

Claim 22.2. *P_ω is finite.*

Proof. If $\pi \in P_\omega$ we know $\pi = \omega - \sum_\alpha p_\alpha \alpha$ for integers $p_\alpha \geq 0$. We just have to show these coefficients p_α are bounded. But there is an element $g \in$ Weyl group that interchanges the set of positive and the set of negative roots, and we know that this is a product of simple reflections (everything in the Weyl group is a product of simple reflections). It follows that $g(\pi)$ also belongs to P_ω (by the previous claim).

$$g(\pi) = \omega - \sum q_\alpha \alpha \text{ for } q_\alpha \geq 0$$

Applying g^{-1},

$$\pi = g^{-1}(\omega) + \sum r_\alpha \alpha \text{ where } r_\alpha \geq 0$$

So $p_\alpha + r_\alpha$ is the coefficient c_α of α in $\omega - g^{-1}(\omega)$. Thus we have that $p_\alpha \leq c_\alpha$, and the coefficients are bounded. □

Thus there are only finitely many weights in E_ω. But we know that the multiplicities are finite. So E_ω is finite-dimensional. □

Definition 22.3. (5.19) The weights satisfying the condition in **5.18** (i.e. the ones that give finite-dimensional highest weight modules) are called *integral*. They are all non-negative integral combinations of the *fundamental weights* (with respect to our chosen base $\Delta \subset \Phi$). If the base is $\{\alpha_1, \cdots, \alpha_\ell\}$, then we can define

$$\omega_i(H_{\alpha_j}) = \delta_{ij}.$$

The simple $U(L)$-modules with highest weight being a fundamental weight give rise to *fundamental representations.*

For example, for $\mathfrak{sl}_n$, let H be the CSA of trace-zero diagonal matrices. The roots are the linear forms given by

$$\alpha_{i,j} : \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \mapsto \lambda_i - \lambda_j$$

for $i \neq j$, and $\sum \lambda_i = 0$. A base is given by the collection of $\alpha_i := \alpha_{i,i+1}$ for each i.

$$H_{\alpha_i} = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

$$X_{\alpha_i} = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 0 & & \\ & & & 1 & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

$$Y_{\alpha_i} = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

The fundamental weights are $\omega_i(H) = \lambda_1 + \cdots + \lambda_i$ where

$$H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

ω_1 is the highest weight of the canonical representation on $\mathbb{C}^n$. In fact, ω_i is the highest weight of the i^{th} symmetric power of the canonical n-dimensional representation. All the finite-dimensional representations of $\mathfrak{sl}_n$ are obtained by decomposing the tensor powers of the canonical representation.

Characters. We want to consider the abelian group P of integral weights – it's generated by the fundamental weights $\omega_1, \cdots, \omega_n$. But it's convenient to work with it multiplicatively instead of additively. So rather than using π we write e^π and have $e^\pi e^{\pi'} = e^{\pi+\pi'}$.

The integral group algebra $\mathbb{Z}P$ consisting of elements $\sum_{\pi \in P} m_\pi e^\pi$ (finite integral sums) with multiplication extended linearly from $e^\pi e^{\pi'} = e^{\pi+\pi'}$.

DEFINITION 22.4. (5.20) Let V be a finite-dimensional $U(L)$-module. Define the character

$$ch(V) = \sum_{\pi \in P} (\dim V^\pi) e^\pi \in \mathbb{Z}P$$

Note: since V is finite-dimensional, all the weights are integral.

LECTURE 23: NOVEMBER 28

Recall the definition of $ch(V)$.

REMARK 23.1. Note that the Weyl group acts on H^*, leaving the set of integral weights invariant, and so the Weyl group acts on P and hence on $\mathbb{Z}P$.

Observe that $ch(V)$ is invariant under the action of the Weyl group. The characters of the fundamental modules generate the subalgebra of $\mathbb{Z}P$ of invariants under the Weyl group

$$\mathbb{Z}P^W = \mathbb{Z}[T_1, \cdots, T_n]$$

where T_i are the characters of the fundamental modules.

LEMMA 23.2. *(5.21) Obvious things.*

1. *$ch(V \oplus V') = ch(V) + ch(V')$ and $ch(V \otimes V') = ch(V)ch(V')$. X acts on $V \otimes V'$ like $1 \otimes X + X \otimes 1$ (where X is the element of the enveloping algebra coming from the Lie algebra in the way we know)*
2. *Two finite-dimensional $U(L)$-modules V and V' are isomorphic iff $ch(V) = ch(V')$*

Proof. (1) Think about it.

(2) One way is obvious. So suppose V, V' are such that $ch(V) = ch(V')$. Also assume both V and V' are nonzero. Then V and V' have the same weights. Pick a weight ω so that $\omega + \alpha$ is not a weight for any $\alpha \in \Delta$. Then ω is a highest weight and a nonzero element of V^ω is a primitive element. It generates a simple $U(L)$-submodule V_1 of V by **5.14**. But V is completely reducible, by Weyl's theorem. So $V = V_1 \oplus V_2$. But similarly $V' = V_1' \oplus V_2'$. Since both V_1 and V_1' are simple with highest weight ω, they are isomorphic. So $V_1 \cong V_1'$. But $ch(V_1) = ch(V_1')$, and so $ch(V_2) = ch(V_2')$. Apply induction to V_2 and V_2' to get $V_2 \cong V_2'$. So $V \cong V'$. □

We will not prove the following theorem.

THEOREM 23.3. *(5.22) [Weyl's character formula] Let V be a finite-dimensional simple $U(L)$-module with highest weight λ. Then*

$$ch(V) = \frac{1}{D} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}$$

where

- $(-1)^{\ell(w)} = \pm 1$ *according to whether* w *is a product of an even or odd number of simple reflections.*
- $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$
- $D = \prod_{\alpha\in\Phi^+}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)$

In fact, $D \in \mathbb{Z}P$*: one can show that* $D = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$*. This enables us to write down the dimensions of* V*:*

$$\dim V = \prod_{\alpha\in\Phi^+}(\lambda+\rho)\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$$

where V *is the simple module of highest weight* λ*.*

The proof can be found in Bourbaki or Jacobsen.

Example 23.4 ($\mathfrak{sl}_2$). In $\mathfrak{sl}_2$ there is a unique positive root α, and $\rho = \frac{1}{2}\alpha$. P consists of the integral powers of e^ρ, ρ is the fundamental weight. A highest weight of a finite-dimensional simple module is $m\rho$, for $m \geq 0$. Weyl's character formula gives:

$$ch(V) = \frac{e^{(m+1)\rho} - e^{-(m+1)\rho}}{e^\rho - e^{-\rho}} = e^{m\rho} + e^{(m-2)\rho} + \cdots + e^{-m\rho}$$

Thus all the multiplicities are 1, and this fits with what we did before.

Closing remarks. In the 1970's a lot of work was done on the universal enveloping algebra. People also studied other related noncommutative algebras, like quantum algebras. Coalgebras arise when you play with how you regard the tensor product of modules. More recently, people have looked at algebras that are filtered downwards (by taking powers of a maximal ideal, or p-adic filtrations), which leads to noncommutative polynomial algebras.