

A complete characterization of paths that are m -step competition graphs

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Abstract

For any digraph D let the m -step competition graph $C^m(D)$ be the graph with the same vertices as D where x and y share an edge in $C^m(D)$ if in D there are m -step paths from x and y to a common vertex z . This paper builds on the work of G.T. Helleloid (2005) and J. Kuhl and B.C. Swan (2010), characterizing the paths that are m -step competition graphs of a digraph. We show that the n -step path P_n is an m -step competition graph if and only if either $m \mid n - 1$ or $m \mid n - 2$.

1 Introduction

Competition graphs were first defined by Cohen [2] in 1968 as a tool for studying ecological systems. Since then, this construct has been extensively studied and generalized; see for example [4] and [5]. We will focus on the m -step competition graph, a generalization defined in 2000 by Cho et al. [1].

Let D be a digraph. Think of D as a food web, where an arc from x to y implies that x is a predator of y . Fix $m \in \mathbb{N}$. If there is an m -step path from x to z , then we say that x is an m -step predator of z , and z is its m -step prey. The m -step competition graph $C^m(D)$ is a graph with the same vertices as D ; vertices x and y are joined by an edge in $C^m(D)$ if they share a common m -step prey in D . We say that x and y are in m -step competition if they share an edge in $C^m(D)$. In general, we ask the following question: which graphs G are representable as the m -step competition graph of a digraph? This paper focuses on the case where $G = P_n$, the path on n vertices. See Figure 1 for an example.

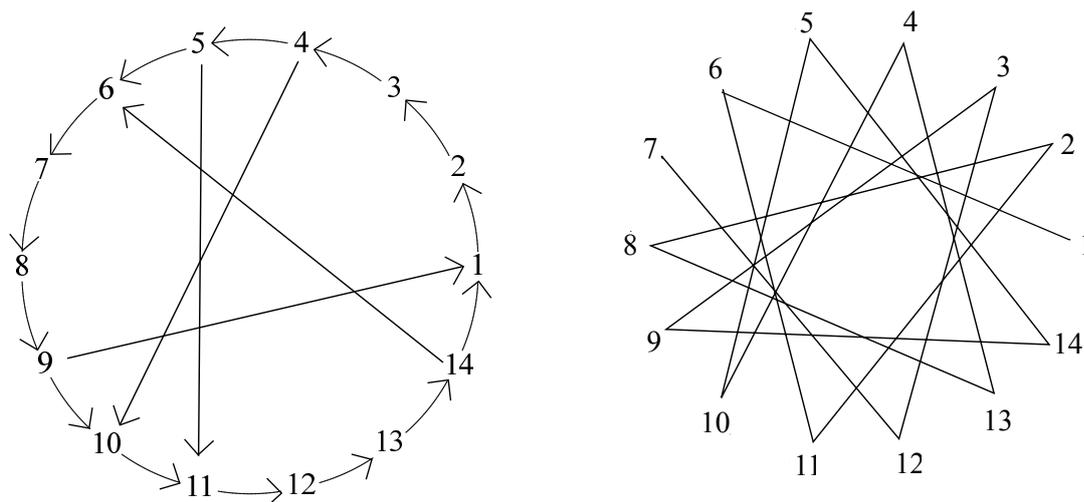


Figure 1: The path P_{14} [right] is the 4-step competition graph of the digraph on the left. This illustrates the construction introduced in [3] and generalized by [6] showing that every path of length $km + 1$ or $km + 2$ is an m -step competition graph.

Helleloid [3] showed that the star graph is the only connected triangle-free m -step competition graph on n vertices for $m \geq n$; in particular, P_n is not an m -step competition graph for

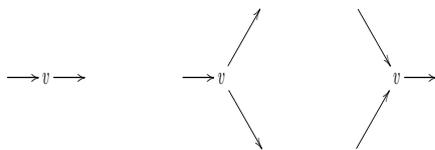
$m \geq n$. Kuhl and Swan [6] proved that P_n is not an m -step competition graph for $\frac{n}{2} \leq m \leq n-3$. In [6] it is also proven that P_n is an m -step competition graph if $m \mid n-1$ or $m \mid n-2$.

Our main result is that the aforementioned divisibility conditions are also necessary.

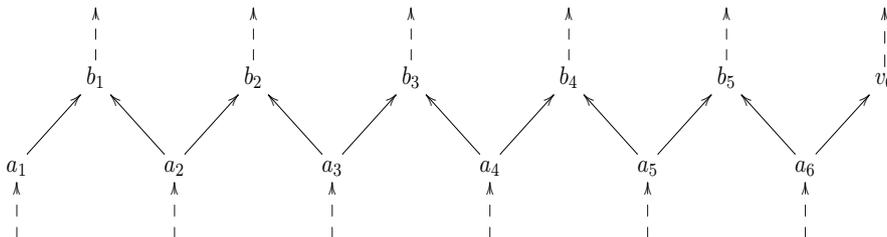
Theorem 1.1. *There exists a digraph D such that $C^m(D) = P_n$ if and only if $m \mid n-1$ or $m \mid n-2$.*

By the aforementioned results in [3] and [6] it suffices to prove that $C^m(D) = P_n$ implies $m \mid n-1$ or $m \mid n-2$, for $m < \frac{n}{2}$. For the rest of this paper $m < \frac{n}{2}$ and D will be a digraph such that $C^m(D) = P_n$. It suffices to consider $n \geq 4$ and $m \geq 3$.

The rest of this paper comprises the proof. In Section 2 we prove facts about the degree of vertices in D , culminating in Theorem 2.1, which states that every vertex v is one of the following three types



(Throughout this paper the “degree of v ” will denote the total degree, that is, the total number of incoming or outgoing arcs attached to v ; the *in-degree* and *out-degree* will denote the number of incoming, respectively outgoing, arcs.) We will show that D contains a structure



where each b_i is connected to some a_j by a directed path. In Section 3, we show that there are at most three vertices that do not fit this pattern, and describe their properties. In Section 4 we compute the lengths of the directed paths. In Section 5 we put all of this together to prove the main theorem.

Proposition 1.2. (*Helleloid*) *There is a bijection between the edges of $C^m(D)$ and $n-1$ of the n vertices of D .*

Proof. Map the edge between x_1 and x_2 to their shared m -step prey. If there are edges in $C^m(D)$ between the pair (x_1, x_2) and between the pair (y_1, y_2) , then these pairs cannot both have the same m -step prey z : otherwise $\{x_1, x_2, y_1, y_2\}$ would form a complete graph of order 4 in $C^m(D)$. There are $n-1$ edges in $C^m(D) = P_n$. So $n-1$ of the n vertices of D correspond to a unique edge in $C^m(D)$. \square

The n^{th} vertex not assigned in this way will henceforth be called the *anomaly*, and will be denoted by α . This vertex can either have no m -step predators, have one m -step predator, or be the second shared m -step prey of a pair of vertices. Note that, in the last case, the anomaly is not well-defined as there are two vertices with this property. If one of these vertices has fewer 1-step predators, define that to be the anomaly; otherwise, the choice can be made arbitrarily.

We represent this schematically in Figure 2. The top row is the desired competition graph P_n and the bottom row shows each vertex of D as the common m -step prey of an edge. (The dashed arrows represent m -step predation paths from the top row to the bottom row. The

dotted arrows represent such paths that may or may not exist, depending on the number of m -step predators of the anomaly.) So every vertex appears exactly once in the top row and exactly once in the bottom row.



Figure 2: *Heuristic for relating $C^m(D)$ [top] to D [bottom]*

We will make extensive use of the following simple observations, which mostly follow by considering Figure 2.

Proposition 1.3.

1. Every vertex of D has an m -step prey, and every vertex except possibly the anomaly has two m -step predators.
2. Every vertex has a 1-step prey, and every vertex except possibly the anomaly has a 1-step predator.
3. If a pair of vertices shares two m -step prey, one of these prey is the anomaly.
4. No vertex has three m -step predators.
5. If a vertex has three m -step prey, then one of these prey is the anomaly.
6. If x has only one m -step prey, then x is an endpoint of $C^m(D)$.
7. If every m -step prey of x is also an m -step prey of y , then x is an endpoint of $C^m(D)$.
8. If every m -step predator of x is also an m -step predator of y , then either x or y is the anomaly.
9. If two vertices share a k -step prey for $k \leq m$, then they have a common m -step prey.

Proof.

1. Every vertex is incident with an edge in $C^m(D)$, and hence shares an m -step prey with its neighbor. Every vertex except the anomaly is the common prey of a pair of vertices, by Proposition 1.2.
2. Follows from 1.
3. Edges correspond bijectively to $V(D) - \{\alpha\}$.
4. If three vertices shared a common m -step prey, then there would be a triangle in $C^m(D)$.
5. Every vertex shares an edge with at most two other vertices, producing at most two m -step common prey from the bijection. The remaining prey must be the anomaly.
6. Otherwise x would share an edge with two vertices, producing two (distinct non-anomaly) m -step common prey.
7. A pair of vertices cannot share more than one non-anomaly prey, so x has only one non-anomaly prey, and is an endpoint by 6.
8. If neither x nor y is the anomaly, then they are the m -step common prey of two distinct pairs of vertices in D which can have at most one vertex in common.
9. Let z be the k -step prey of x and y . By 1 z has an m -step prey, so it has an $(m - k)$ -step prey. This is the m -step prey of x and y .

□

2 The degree of vertices in D

In this section we give the following description of D (when such a digraph exists).

Theorem 2.1. *If $C^m(D) = P_n$ then every vertex in D has degree 2 or 3, and has at least one outgoing and one incoming arc.*

Notation 2.2. *Let x^+ denote a 1-step prey of x , and let x^- denote a 1-step predator of x .*

We need the following two lemmas modified from [6].

Lemma 2.3. *(Kuhl, Swan) If $1 < p < \min\{\frac{n}{2}, m\}$ then D contains no cycle of length p .*

Proof. The details will not be presented here, but the interested reader is referred to Lemma 3.2 in [6]. Though Kuhl and Swan state the lemma for $m \geq \frac{n}{2}$ the proof only uses the facts that $p < \frac{n}{2}$ and $p < m$. \square

Lemma 2.4. *(Kuhl, Swan) There can be at most one loop in D . If x has a loop, then $C^m(D - \{x\}) = P_{n-1}$ and the anomaly of $D - \{x\}$ (which is loopless) has no predator.*

Proof. Let x have a loop. By [6, Lemma ??] x has no 1-step prey $y \neq x$. We continue their argument, with slight modifications.

As it has no prey, x must have a 1-step predator y . Note that y can have no other 1-step predator because then x would have three m -step predators (a contradiction by Proposition 1.3.4). Also, y cannot have a loop, by Case 1a. Since x only has one m -step prey, it is an endpoint of $C^m(D)$. Thus $C^m(D - \{x\}) = P_{n-1}$ and y has no m -step predators. This shows that y is the anomaly.

Suppose x and z are both loops. By the above, x and z are both endpoints, and both 1-step prey of the (unique) anomaly y of D . There are edges in $C^m(D)$ between y and z , and between y and x , implying that $n = 3$. \square

In the next lemma we will show that this latter condition cannot occur: if α is the anomaly of a loopless digraph whose competition graph is a path, then α has at least one predator.

Lemma 2.5. *Suppose D is loopless. Then the anomaly has at least one m -step predator.*

Proof. Suppose not; that is, suppose that α has no m -step predator. Then there is some vertex without a 1-step predator. By Proposition 1.3.2, this vertex must be α . Note that, by Proposition 1.3.2, α has at least one 1-step prey.

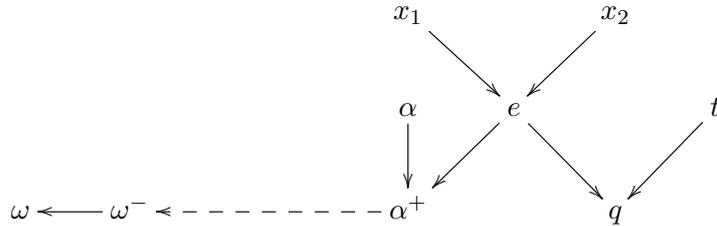


Figure 3: *Lemma 2.5 Case 1*

Case 1: The anomaly has only one 1-step prey. This 1-step prey $\alpha^+ \neq \alpha$ has an m -step predator (Proposition 1.3.1) so it has a 1-step predator e which has an $(m - 1)$ -step predator. Hence $e \neq \alpha$ (note by hypothesis there are no loops). By Proposition 1.3.9, α and e share an m -step prey ω . By Proposition 1.3.1, the penultimate step ω^- of the path to ω has two m -step predators x_1 and x_2 . (Note $\omega^- \neq \alpha$ since it has a 1-step predator.) The first vertices x_1^+ and x_2^+ along the m -step paths from x_1 and x_2 to ω^- are m -step predators of ω . Hence $x_1^+ = x_2^+ = e$ because α has no predators and α and e are the only m -step predators of ω (Proposition 1.3.4). So x_1 and x_2 are 1-step predators of e . By assumption α^+ is the first step

in any m -step path from α . Therefore, every prey of α is a prey of e , and α is an endpoint of $C^m(D)$ by Proposition 1.3.7. Since α and e share an m -step prey, they share an edge in $C^m(D)$. So e is not an endpoint, and thus has an m -step prey not shared by α . The first step q in this path must be different from α^+ . Since q has a 1-step predator e , we have $q \neq \alpha$. Furthermore, by Proposition 1.3.8 applied to α^+ and q , q must have an m -step predator not shared by α^+ . So the penultimate step in this path to q is a 1-step predator t of q such that $t \neq e$ and $t \neq \alpha$. If there is an $(m-2)$ -step path from q to ω^- , then α , e , and t share an m -step prey ω , contradicting Proposition 1.3.4. Otherwise, q has another $(m-2)$ -step prey, which means x_1 and x_2 share two m -step prey, contradicting Proposition 1.3.3 and concluding Case 1.

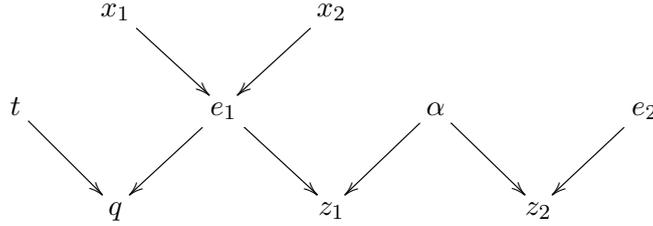


Figure 4: Lemma 2.5 Case 2

Case 2: The anomaly has two 1-step prey. Let z_1 and z_2 be 1-step prey of α ; since these have m -step predators, they have 1-step predators e_1 and e_2 , respectively, that are not the anomaly. First suppose that e_1 has another 1-step prey q . This situation is directly analogous to case 1. In particular, e_1 and α share an m -step prey ω , and the distinct m -step predators x_1 and x_2 of its 1-step predator ω^- are 1-step predators of e_1 . By Proposition 1.3.8 applied to q , q has another 1-step predator t . If q and z_1 share an $(m-1)$ -step prey, then t , e_1 , and α share an m -step prey, a contradiction. Otherwise, x_1 and x_2 share two common m -step prey.

Now assume e_1 or e_2 each have a single 1-step prey z_1 and z_2 , respectively. Every m -step prey of e_1 is a prey of α so e_1 is an endpoint; similarly e_2 is an endpoint. But there is an edge in $C^m(D)$ between e_1 and α , and between α and e_2 . But $e_1 \neq e_2$ (else e_1 has another 1-step prey) so $n = 3$, a contradiction. \square

Corollary 2.6. *No vertex in D has a loop.*

Proof. Apply lemma 2.5 to the digraph $D - \{x\}$ in Lemma 2.4. \square

Corollary 2.7. *Every vertex has an m -step prey and an m -step predator. In particular, every vertex has a 1-step prey and a 1-step predator.*

Lemma 2.8. *No vertex in D has three 1-step prey.*

Proof. We prove the result by contradiction. Suppose x has three 1-step prey a , b , and c . By Corollary 2.7, x has an $(m-1)$ -step predator z . Let z^+ be its 1-step prey along this path, and let x^- be the 1-step predator of x along this path. By applying Proposition 1.3.5 to z , we see that one of a , b , or c is the anomaly.

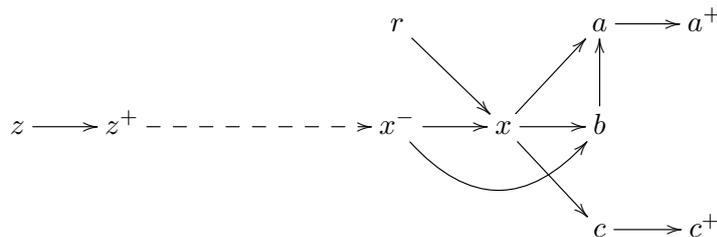


Figure 5: Lemma 2.8 Claim

Claim: The anomaly is not a 1-step prey of a vertex in $\{a, b, c\}$. Let a^+ , b^+ , and c^+ be arbitrary 1-step prey of their respective vertices. For the purposes of proving this claim, suppose (without loss of generality) that a is the anomaly and $b^+ = a$; we will obtain a contradiction. By Proposition 1.3.8, b has an m -step predator not shared with c ; the penultimate step along this path to b is not x , so b has a 1-step predator $q \neq x$. But $q = x^-$, since otherwise q , x , and x^- share a common prey on a path through a (Proposition 1.3.9). Since neither x nor b is the anomaly, x has an m -step predator not shared with b , so there exists a 1-step predator $r \neq x^-$ of x . The vertices r , x , and x^- are all distinct, and share a common m -step prey on a path through their common 2-step prey a . This contradicts Proposition 1.3.4, proving the claim.

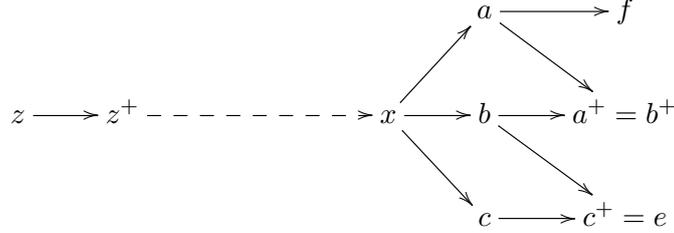


Figure 6: Lemma 2.8

Thus $\{a^+, b^+, c^+\}$ does not contain the anomaly. These vertices are all m -step prey of z^+ so by Proposition 1.3.5, they are not all distinct; without loss of generality suppose $a^+ = b^+$. There is an edge in $C^m(D)$ between a and b so at least one of them, say b , is not an endpoint. Then b has another m -step prey not shared with a , so b has another 1-step prey $e \neq b^+$. The m -step prey of z^+ are now b^+ , e , and c^+ ; as shown above none of these is the anomaly, so they cannot be all distinct. By design $b^+ \neq e$, and $b^+ \neq c^+$ because otherwise a , b , and c would all share an m -step prey. So $e = c^+$.

In $C^m(D)$ there is an edge between a and b and between b and c . Since $n > 3$, at least one of a and c is not an endpoint. Without loss of generality suppose a is not an endpoint. Then a has an m -step prey not shared with b ; as before, a has another 1-step prey $f \neq b^+$. If $f = e$ then a , b , and c all share a common prey. So b^+ , e , and f are all distinct. But they are also all m -step prey of z^+ , and none of them is the anomaly. This contradicts Proposition 1.3.5. \square

Corollary 2.9. *No two vertices share two 1-step prey.*

Proof. Suppose a and b share 1-step prey x and y . There is an edge between a and b in $C^m(D)$, so one of them, say a , is not an endpoint of $C^m(D)$. By Proposition 1.3.7, a has an m -step prey ω not shared by b . The first step of the path from a to ω cannot be x or y , else ω is also a prey of b . So a must have a third 1-step prey, contradicting Lemma 2.8. \square

Lemma 2.10. *There is no vertex in D with two 1-step predators and two 1-step prey.*

Proof. The proof is by contradiction. Let b be a vertex with 1-step predators d and e and 1-step prey f and g . Let q be an $(m - 2)$ -step prey of f , and let x be an $(m - 2)$ -step prey of g .

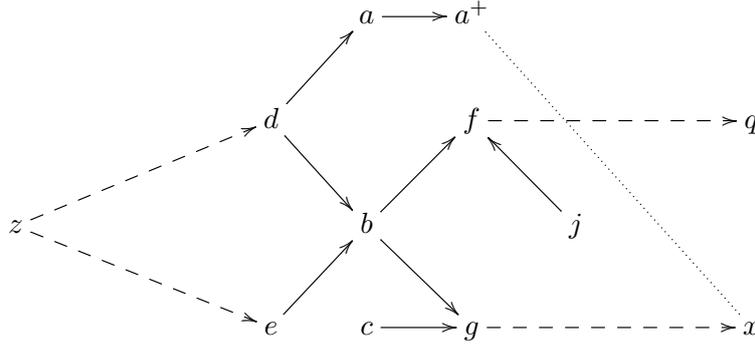


Figure 7: Lemma 2.10 Case 1

Case 1: Neither g nor f is the anomaly.

By Proposition 1.3.8, g has a predator not shared with f , and vice versa, so g and f have 1-step predators c and j , respectively, that are different from b (and each other). Note $q \neq x$ because otherwise j , b , and c have a common prey on a path through q . At least one of d and e is not an endpoint; without loss of generality assume this vertex is d . Let z be an $(m - 2)$ -step predator of d . Since d has an m -step prey t not shared with e , the first step a of the path to t must be different from b , and the second step a^+ must be different from f and g . Since a^+ , f , and g are all m -step prey of z , by Proposition 1.3.5 one of them (a^+) must be the anomaly. We show that this is impossible.

Since q and x are both m -step prey of d and e , one of q or x is the anomaly (Proposition 1.3.3). Without loss of generality assume x is the anomaly, so $a^+ = x$. The m -step predators of x are z (on a path through e , a , and $a^+ = x$), as well as d and e . The $(m - 2)$ -step predator of e is an m -step predator of f and g , so by Proposition 1.3.3, z is the only such predator. If $z = d$ or e then this vertex is its own $(m - 2)$ -step predator, and therefore there is a cycle of length at most $m - 2$, contradicting Lemma 2.3. Thus z , d , and e are distinct predators of x , contradicting Proposition 1.3.4 and concluding this case.

Case 2: The anomaly is in $\{f, g\}$.

Without loss of generality assume f is the anomaly.

Case 2a: There exist distinct vertices q and x . Since these are both m -step prey of d and e , one of them is the anomaly by Proposition 1.3.3. But q cannot be the anomaly f , since otherwise $q = f$ is its own $(m - 2)$ -step predator, contradicting Lemma 2.3. So $x = f$, and there is an $(m - 2)$ -step path from g to f . We show that this is impossible.

If $m = 3$ then g is a 1-step predator of f , and b , d , and e share a common prey on a path through f , contradicting Proposition 1.3.4. If $m = 4$ then g , d , and e share a common prey on a path through f , again a contradiction. Now assume $m \geq 5$ and let f^{--} be the 2-step predator of f taken along the path from g to f . Now d , e , and f^{--} share a common prey. These cannot be distinct (Proposition 1.3.4), so without loss of generality suppose $f^{--} = d$. Then there is an $(m - 4)$ -step path from g to d , and hence an $(m - 2)$ -cycle when the path $d \rightarrow b \rightarrow g$ is added. This contradicts Lemma 2.3.

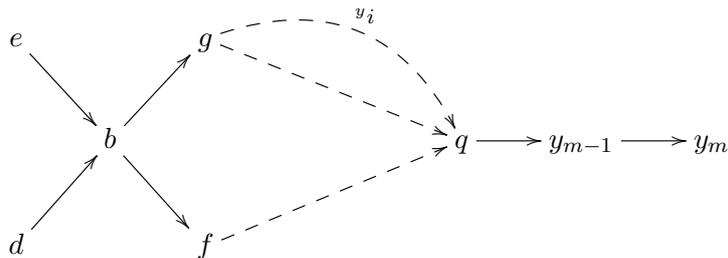


Figure 8: Lemma 2.10 Case 2b

Case 2b: $q = x$ is the only $(m - 2)$ -step prey of either f or g . In this case f and g share an m -step prey on a path through q , so at least one of them is not an endpoint. Suppose g is not an endpoint (the argument is analogous for f). Then g has two m -step prey, one of which is not shared with f . Let g, y_1, \dots, y_m be a walk such that y_m is not an m -step prey of f ; by the hypothesis of this case we can assume that $y_{m-2} = q$. But then $f, \dots, q, y_{m-1}, y_m$ is an m -step walk from f to y_m . This contradiction concludes the proof. \square

Proof of Theorem 2.1. By Corollary 2.7 the degree of each vertex is at least two. If a vertex had total degree at least four, then it would have either three incoming arcs, three outgoing arcs, or two incoming and two outgoing arcs. No vertex has three 1-step predators, else these predators would share an m -step prey. The other two cases are impossible by Lemma 2.8 and Lemma 2.10, respectively. In particular, if $C^m(D)$ is a path, then D is loopless (Lemma 2.4) and every vertex has total degree 2 or 3. \square

3 Properties of α and the endpoints of $C^m(D)$

Recall that if $|V(D)| = n$ then the goal is to show $m \mid n - 1$ or $m \mid n - 2$. We will partition D into sets containing m vertices. As implied by the desired divisibility conditions, there are one or two vertices that do not fit this pattern. The goal of this section is to describe those vertices. First we need some definitions.

Definition 3.1. *Let A be the set of vertices with two 1-step prey. Let B be the set of vertices with two 1-step predators.*

Definition 3.2. *Let \bar{A} be the set of vertices that are in A or have a 1-step prey in B . Let \bar{B} be the set of vertices that are in B or have a 1-step predator in A .*

Proposition 3.3. *If $a \in \bar{A}$, then all of its 1-step prey are in \bar{B} . If $b \in \bar{B}$, then all of its 1-step predators are in \bar{A} .*

Proof. Let $a \in \bar{A}$. If $a \in A$, then it has two 1-step prey which are in \bar{B} by definition. If $a \in \bar{A} - A$ then a has only one 1-step prey $a^+ \in B \subset \bar{B}$.

The other statement follows analogously. \square

Proposition 3.4.

1. $|\bar{A}| = |\bar{B}|$
2. $|A| = |B|$
3. $|\bar{A} - A| = |\bar{B} - B|$.

Proof.

1. Any vertex $x \notin \bar{A}$ has only one 1-step prey x_{prey} , for which x is the only 1-step predator. Any vertex $y \notin \bar{B}$ has only one 1-step predator y_{pred} , for which y is the only 1-step prey. The correspondence $x \mapsto x_{prey}$ (with inverse $y \mapsto y_{pred}$) gives a bijection between the complement of \bar{A} and the complement of \bar{B} , implying that $|\bar{A}| = |\bar{B}|$.
2. Let X be the set of vertices of total degree 2; by Theorem 2.1 these are the vertices that have exactly one 1-step prey and one 1-step predator. By these observations and Lemma 2.10, D is the disjoint union $X \cup A \cup B$. Since $\sum_{v \in D} out-degree(v) = \sum_{v \in D} in-degree(v)$ we have $|X| + 2|A| + |B| = |X| + |A| + 2|B|$, which implies $|A| = |B|$.
3. Since $A \subset \bar{A}$ and $B \subset \bar{B}$, this follows from the above statements.

□

Lemma 3.5. $\bar{B} - B = \{\alpha\}$ and $\bar{A} - A = \{e_1\}$ where e_1 is an endpoint of $C^m(D)$.

Proof. We separate this into two cases:

Case 1: $\bar{A} - A = \bar{B} - B = \emptyset$

By Proposition 3.4 $|A| = |B| = j$. Each vertex in A has two 1-step prey in $\bar{B} = B$. Since no vertex has three 1-step prey, each vertex of B is the common 1-step prey of a pair of vertices in A ; that is, each vertex in B corresponds to an edge in $C^m(D)$ between vertices of A . Hence among the j vertices of A there are j distinct edges in $C^m(D)$. This implies that $C^m(D)$ has a cycle, a contradiction.

Case 2: $|\bar{A} - A| = |\bar{B} - B| > 0$

Suppose $a_0 \in \bar{A} - A$. This vertex has a 1-step prey $b \in B$, which has another 1-step predator a_1 . Since $a_0 \notin A$, b is the first step of any path from a_0 to an m -step prey of a_0 . Hence every m -step prey of a_0 is also an m -step prey of a_1 , and a_0 is an endpoint of $C^m(D)$ by Proposition 1.3.7. Thus $\bar{A} - A \subset \{e_1, e_2\}$, where e_1 and e_2 are the two endpoints of $C^m(D)$.

Suppose $y_0 \in \bar{B} - B$. Then y_0 shares a 1-step predator $x \in A$ with y_1 . Every m -step predator of y_0 is an $(m-1)$ -step predator of x , and hence also an m -step predator of y_1 . By Proposition 1.3.8, either y_0 or y_1 is the anomaly. Furthermore, if y_1 is the anomaly, then y_0 has two m -step predators, both of which are shared by the anomaly y_1 ; that is, every vertex of $\bar{B} - B$ either is the anomaly or shares both m -step predators with the anomaly. There can be only one vertex q with this latter property (Proposition 1.3.3), so $\bar{B} - B \subset \{\alpha, q\}$.

First suppose $|\bar{B} - B| = 1$; let this element be y_0 as above. Note that $y_1 \in B$, since it is the 1-step prey of $x \in A$ and $y_1 \notin \bar{B} - B$. If y_0 has one m -step predator then it is the anomaly, as desired. Otherwise, y_0 has two m -step predators, which are exactly the predators of y_1 . In this case, y_0 has fewer 1-step predators than y_1 so by the convention stated earlier we declare y_0 to be the anomaly. This satisfies the lemma.

Now suppose $|\bar{B} - B| = 2$. As before let $j = |A| = |B|$. Let $\bar{B} - B = \{\alpha, q\}$; these vertices share a common 1-step predator $x \in A$. There are $2j + 2$ arcs leaving \bar{A} (by Proposition 3.3 ending inside \bar{B}): the two starting in x end inside $\bar{B} - B$, and the remaining $2j$ end in the j vertices of B . Since no vertex has more than two incoming arcs, there are j pairs of vertices in $\bar{A} - x$ that share a common 1-step prey in B (and hence an edge in $C^m(D)$). But $|\bar{A} - x| = j + 1$, so the vertices in $\bar{A} - x$ form a path of length $j + 1$ in $C^m(D)$. But since $\{e_1, e_2\} \subset \bar{A} - A \subset \bar{A} - x$, this path contains the endpoints e_1, e_2 of $C^m(D)$, implying that $C^m(D)$ has P_{j+1} as a connected component, for $j + 1 < |\bar{A}| < |D|$. □

Notation 3.6. In the following let e_1 continue to be the endpoint of $C^m(D)$ in $\bar{A} - A$.

Lemma 3.7. Either $\alpha \in A$ or $\alpha = e_1$.

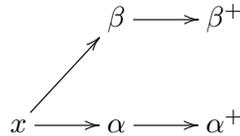


Figure 9: Lemma 3.7

Proof. Since $\alpha \in \bar{B} - B$, α has a 1-step predator x which has another 1-step prey β . Let α^+ be a 1-step prey of α and let β^+ be a 1-step prey of β . Suppose $\alpha^+ \notin B$. In particular, $\alpha^+ \neq \beta^+$, and all m -step predators of α^+ are $(m-2)$ -step predators of x and hence also m -step predators of β^+ . By Proposition 1.3.8 one of these is the anomaly. But the anomaly cannot be α^+ , and if $\beta^+ = \alpha$ then $\alpha \in B$, a contradiction. So $\alpha^+ \in B$, and $\alpha \in \bar{A}$. Either $\alpha \in A$ or $\alpha \in \bar{A} - A$, in which case α is an endpoint by Lemma 3.5. □

Lemma 3.8. *Either $e_1 \in B$ or $e_1 = \alpha$.*

Proof. Since $e_1 \in \bar{A} - A$ it has a 1-step prey $b \in B$, and b has another 1-step predator a . The endpoint e_1 has a 1-step predator y , which has a 1-step predator y^- . Also a has a 1-step predator x , which has a 1-step predator x^- . If $y \in A$, then $e_1 \in \bar{B}$; in this case either $e_1 \in B$ or (by Lemma 3.5) it is the anomaly.

So assume $y \notin A$; we will obtain a contradiction. Every m -step prey of y is also a m -step prey of x , so $y = e_2$. Note that if $y^- = e_1$, then there would be a 2-cycle, contradicting Lemma 2.3. So y^- is not an endpoint, and must be in A (otherwise $x^- \neq y^-$ and every m -step prey of y^- is also an m -step prey of x^-). Since $y \in \bar{B}$, either $y \in B$ or (by Lemma 3.5) $y \in \bar{B} - B$ is the anomaly. But if y were the anomaly, then by Lemma 3.7 either it would be in A or it would be the endpoint $e_1 \in \bar{A} - A$. Neither of these is the case, so $y \in B$. Let q be the other 1-step predator of y . If distinct, q , y^- , and x^- all share an m -step prey, so either $x^- = y^-$ or $q = x^-$. In either case $x^- \in A$, so $x \in \bar{B}$. Either $x \in B$ or x is the anomaly.

First suppose $x \in B$, and let t be a 1-step predator of x different from x^- . Then q , y^- , x^- , and t all share an m -step prey on a path through b . By Proposition 1.3.4, these can represent at most two distinct vertices. By construction $q \neq y^-$ and $t \neq x^-$, so without loss of generality assume $y^- = x^-$ and $q = t$. But these are distinct and share two 1-step prey x and y (which are distinct since $y \notin A$), contradicting Corollary 2.9.

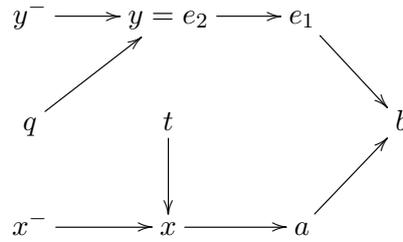


Figure 10: *Lemma 3.8*

Now suppose x is the anomaly. Note that x is not an endpoint: if $x = e_2$ then $y = e_2 \in A$, and if $x = e_1$ then $e_1 \in A$ (by definition $e_1 \in \bar{A} - A$). Thus by Lemma 3.7, $x \in A$, so $a \in \bar{B}$. Furthermore $b \in B$ so $a \in \bar{A}$. Since $a \in \bar{A} \cap \bar{B}$, either $a \in \bar{A} - A$ or $\bar{B} - B$ (because $a \notin A \cap B$ by Lemma 2.10), so by Lemma 3.5 either $a = e_1$ or $a = \alpha$. But $a \neq e_1$ by construction, and x , not a , is the anomaly. This is a contradiction. \square

Corollary 3.9. $\{\alpha, e_1\} = \bar{A} \cap \bar{B}$

Proof. By Lemma 3.5, $\alpha \in \bar{B}$ and $e_1 \in \bar{A}$. By applying Lemma 3.5 to Lemmas 3.8, and 3.7, respectively, $\alpha \in \bar{A}$ and $e_1 \in \bar{B}$. So α and e_1 are in $\bar{A} \cap \bar{B}$. Conversely, suppose $x \in \bar{A} \cap \bar{B}$. By Lemma 2.10, $x \notin A \cap B$, so x is in $\bar{A} - A$ or $\bar{B} - B$. By Lemma 3.5, x is α or e_1 . \square

4 Vertices outside $\bar{A} \cup \bar{B}$

In this section we will identify disjoint sets $S(a)$ mostly of cardinality $m - 1$, which are indexed by $a \in \bar{A}$.

Definition 4.1. *Let C be the set of vertices that are not in \bar{A} or \bar{B} . By Theorem 2.1 every vertex in C has exactly one 1-step predator and one 1-step prey. So every vertex c_i of C belongs to a path $x \rightarrow c_1 \rightarrow \dots \rightarrow c_\nu \rightarrow y$ where all c_i are in C and x and y are in $\bar{A} \cup \bar{B}$. Under these circumstances let $S(y)$ be the path $c_1 \rightarrow \dots \rightarrow c_\nu \rightarrow y$. Define the length $|S(y)| = \nu + 1$.*

Proposition 4.2. *$S(y)$ is well-defined and $y \in \bar{A}$.*

Proof. First suppose that there were two paths $c_1 \rightarrow \dots \rightarrow c_\nu \rightarrow y$ and $c'_1 \rightarrow \dots \rightarrow c'_\mu \rightarrow y$. Then $c_\nu = c'_\mu$, since otherwise $y \in B$ and c_ν and c'_μ would be in \bar{A} . But then $c_{\nu-1} = c'_{\mu-1}$ because c_ν and c'_μ each have only one 1-step predator. Suppose that $\mu \geq \nu$. Then by induction there is some i such that $c_1 = c'_i$. But now $i = 1$ since the 1-step predator of $c_1 = c'_i$ is not in C . Thus there can be only one path $S(y)$. Furthermore, by definition $y \in \bar{A} \cup \bar{B}$, and $y \notin \bar{B}$ because by Proposition 3.3 every 1-step predator would be in \bar{A} . \square

Definition 4.3. If $y \in \bar{A}$ has no 1-step predators in C then let $S(y) = \{y\}$. (In particular, this is the case when $y \in \bar{A} \cap \bar{B}$, since any 1-step predator of y would be in \bar{A} by Proposition 3.3.)

Remark 4.4. By definition every vertex of D is either in C or $\bar{A} \cup \bar{B}$, and every vertex of C belongs to some path $S(a)$ for $a \in \bar{A}$ as in 4.2. Thus $V(D) = \bigcup_{a \in \bar{A}} S(a) \cup \bar{B}$.

We will show that $|S(a)| = m - 1$ if $a \notin \{\alpha, e_1\}$.

Lemma 4.5. $|V(D)| = n = \sum_{a \in \bar{A}} (|S(a)| + 1) - |\bar{A} \cap \bar{B}|$

Proof. If $x_i \notin \bar{A} \cup \bar{B}$ then x_i appears in exactly one path $S(a)$. Thus the vertices that appear twice in the decomposition in Remark 4.4 are precisely $\bar{A} \cap \bar{B}$. So $n = |V(D)| = \sum_{a \in \bar{A}} |S(a)| + |\bar{B}| - |\bar{A} \cap \bar{B}| = \sum_{a \in \bar{A}} (|S(a)| + 1) - |\bar{A} \cap \bar{B}|$ since $|\bar{A}| = |\bar{B}|$. \square

Lemma 4.6. For $a \in \bar{A}$:

1. $|S(a)| \leq m - 1$
2. $|S(a)| = m - 1$ if $S(a) = \{c_1, \dots, c_k, a\}$ such that a has two 1-step prey in B and the 1-step predator c_1^- of c_1 is in B

Proof. 1. Let $k = |S(a)|$. To obtain a contradiction suppose $k \geq m$ with $S(a) = (c_1, \dots, c_{k-1}, a)$. All of the c_i have degree 2, and $a \notin B$ (or else $c_{k-1} \in \bar{A}$) so a has just one m -step predator c_{k-m} , or c_1^- if $k = m$. This is only possible if a is the anomaly. By Lemma 3.5, $\alpha \in \bar{B} - B$ so the unique 1-step predator c_{k-1} of a is in A . This is a contradiction, so $k = |S(a)| \leq m - 1$.

2. To obtain a contradiction suppose $|S(a)| \leq m - 2$ and let $c_1^- \in B$ have 1-step predators a_0 and a_1 . Let a have 1-step prey b_0 and b_1 in B . Now b_0 and b_1 have 1-step predators b_0^- and b_1^- , respectively, which are different from a . By Corollary 2.9, $b_0^- \neq b_1^-$. The $(m - (k + 2))$ -step prey y_0 and y_1 of b_0 and b_1 , respectively, are m -step prey of a_0 and a_1 . Either $y_0 = \alpha$ or $y_1 = \alpha$, else $y_0 = y_1$, in which case b_0^- , a , and b_1^- all share a common prey. So $y_0 \neq y_1$ and without loss of generality suppose y_0 is the anomaly. Since $y_0 \in \bar{B} - B$, the unique 1-step predator y_0^- of y_0 has another 1-step prey q . Also $y_0 \notin B$ so $y_0 \neq b_0$ or b_1 . Now a_0 and a_1 have two non-anomaly prey q and y_1 . (If $q = y_1$ then b_0^- , a , and b_1^- share a common prey.) This is a contradiction, so $|S(a)| \geq m - 1$, and by 1, $|S(a)| = m - 1$.

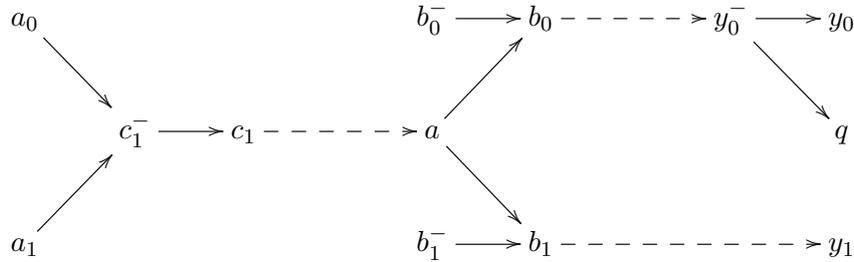


Figure 11: Lemma 4.6.2

\square

Lemma 4.7. If $x \notin \{e_1, \alpha\}$ then $|S(x)| = m - 1$.

Proof. Let $S(x) = (c_1, \dots, c_n, x)$ for $x \in \bar{A} - \{e_1, \alpha\} = A$. We claim that the 1-step predator $c_1^- \in \bar{A} \cup \bar{B}$ of c_1 is in \bar{B} . If instead $c_1^- \in \bar{A}$, then $c_1 \in \bar{B}$ so there are no c_i ; that is, c_1^- is the predator of x and $x \in \bar{A} \cap \bar{B}$. By Corollary 3.9 and Definition 4.3, $x \in \{e_1, \alpha\}$, a contradiction. Furthermore, we claim that $c_1^- \in B$. If not, then $c_1^- \in \bar{B} - B$ so $c_1^- = \alpha$ by Lemma 3.5. By Lemma 3.7, $c_1^- \in \bar{A}$ so $c_1 \in \bar{B}$, a contradiction as above.

Unless $x \in A$ is the 1-step predator of the anomaly, the 1-step prey of x are in B , and by Lemma 4.6.2, $|S(x)| = m - 1$. So assume x is the 1-step predator of the anomaly. Let $S(x)$ be the path from c_1^- to x , with $c_1^- \in B$ having 1-step predators a_0 and a_1 . The 1-step prey of x are α and β , and $\beta \in B$ since α is the only vertex of $\bar{B} - B$. Let $r \neq x$ be a 1-step predator of β . Let $k = |S(x)|$.

For the sake of contradiction suppose $k \leq m - 2$. The $(m - (k + 2))$ -step prey of β has m -step predators a_0, a_1 , and the $(k + 1)$ -step predator of r . These cannot be distinct so suppose this latter vertex is a_0 . Then there is a k -step path from some 1-step prey of a_0 to $r \in \bar{A}$, and hence there is a path $S(r) \neq S(x)$ of length at most k . This path must be $S(e_1)$ or $S(\alpha)$; that is, $r = e_1$ or $r = \alpha$.

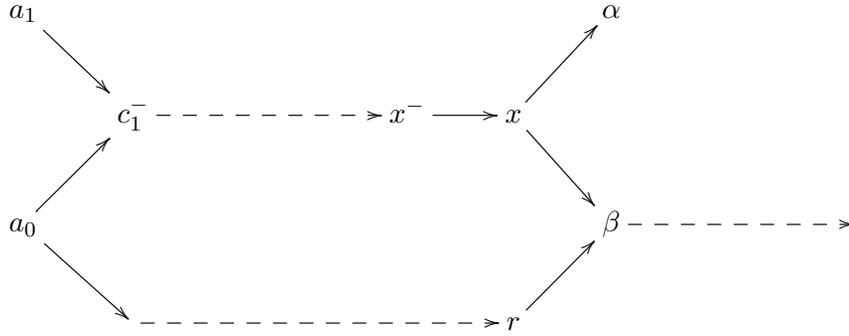


Figure 12: Lemma 4.7

There are two cases:

Case 1: $r = \alpha$. In this case β has three $(k + 2)$ -step predators: a_0, a_1 , and c_1^- (through the path $c_1^- \rightarrow \dots \rightarrow x \rightarrow \alpha = r \rightarrow \beta$). These are all distinct, and they share an m -step prey on a path through β , contradicting Proposition 1.3.4.

Case 2: $r = e_1 \neq \alpha$. Since $e_1 \neq \alpha$, $e_1 \in B$ by Lemma 3.8. Let e_1 have two 1-step predators z_1 and z_2 . If $z_1 = x^-$ then $x^- \in A$ and $x \in \bar{B}$. But $x \in A$ so $x \in \bar{B} - B$ by Lemma 2.10, implying $x = \alpha$, a contradiction. So x^-, z_1 , and z_2 are distinct, but they share an m -step predator (on paths through β), a contradiction. So $|S(x)| \geq m - 1$, and by Lemma 4.6.2, $|S(x)| = m - 1$. \square

We have shown that $|S(a)| = m - 1$ except possibly in two cases. Now we are ready to prove the main theorem.

5 Main theorem

Theorem 5.1. *Let $m < \frac{n}{2}$, where $n \geq 4$ and $m \geq 3$. If there exists a digraph D such that $C^m(D) = P_n$, then $m \mid n - 1$ or $m \mid n - 2$.*

Proof. By Corollary 3.9 and Definition 4.3, we have $|S(\alpha)| = 1 = |S(e_1)|$. By Lemma 3.8, there are two cases.

Case 1: $e_1 \in B$. Since $e_1 \neq \alpha$, Corollary 3.9 gives $|\bar{A} \cap \bar{B}| = 2$. By Lemma 4.7, $|S(x)| = m - 1$

for $x \in \bar{A} - \{e_1, \alpha\}$. By Lemma 4.5

$$n = (|S(e_1)| + 1) + (|S(\alpha)| + 1) + \sum_{a \in \bar{A} - \{e_1, \alpha\}} (|S(a)| + 1) - |\bar{A} \cap \bar{B}| = Nm + 2$$

where $N = |\bar{A} - \{e_1, \alpha\}|$. So $m \mid n - 2$.

Case 2: $e_1 = \alpha$.

By Corollary 3.9, $|\bar{A} \cap \bar{B}| = 1$. By Lemma 4.7, $|S(x)| = m - 1$ for $x \in \bar{A} - \{e_1\}$. By Lemma 4.5

$$n = (|S(e_1)| + 1) + \sum_{a \in \bar{A} - \{e_1\}} (|S(a)| + 1) - |\bar{A} \cap \bar{B}| = Nm + 1$$

where $N = |\bar{A} - \{e_1\}|$. So $m \mid n - 1$. □

Combining this with the previous results in [3] and [6], we obtain the following theorem.

Theorem 5.2. *The path P_n is an m -step competition graph if and only if $m \mid n - 1$ or $m \mid n - 2$.*

6 Conclusion

While this completes the characterization of paths as m -step competition graphs, for most graphs G it is not known whether $C^m(D) = G$ for any digraph D . A long-term goal would be to find general criteria for determining whether G is an m -step competition graph. At least at this point, however, this question seems hard, as techniques used so far have depended quite heavily on the precise structure of the goal graph G . Since it is not even clear which kinds of properties to consider in a potential m -step competition graph G , it is important to generalize known results in order to generate intuition on the nature of m -step competition graphs. In particular, one could consider:

1. *Disjoint collections of cycles and paths:* a study of these would complete the characterization of graphs of maximum total degree 2.
2. *Y-shaped graphs:* this would be an important first step towards classifying trees.
3. *Paths with pendant vertices:* This is one generalization of the notion of a simple path as a broken cycle. Cho et al. [1] showed that cycles with pendant vertices are never m -step competition graphs.
4. *Trees*

Another potentially promising approach would be to characterize the m -step competition graphs that arise from digraphs with certain desirable properties, such as those digraphs having maximum total degree 3 or minimum cycle length m .

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