A course in
Algebraic Geometry

Taught by C. Birkar

Michaelmas 2012

Last updated: May 30, 2013

Disclaimer

These are my notes from Caucher Birkar's Part III course on algebraic geometry, given at Cambridge University in Michaelmas term, 2012. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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My occasional comments are in *grey italics*.

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Lecture 1: October 5

Sheaves. Let X be a topological space.

Definition 1.1. A *presheaf* is a set of algebraic data on this space: For every open $U \subset X$ we associate an abelian group $\mathcal{F}(U)$, such that $\mathcal{F}(\emptyset) = 0$. In addition, for every $V \subset U$ we assign a "restriction map" $\mathcal{F}(U) \to \mathcal{F}(V)$, such that

- if $V = U$, then $\mathcal{F}(U) \to \mathcal{F}(V)$ is the identity;
- if $W \subset V \subset U$ then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d] \\ & \mathcal{F}(W) \end{tikzcd}$$

commutes.

(Think of $\mathcal{F}(U)$ as a set of functions on U.)

Definition 1.2. A *sheaf* is a presheaf $\mathcal{F}$ subject to the following condition. If $U = \bigcup U_i$ is an open cover, and we have a collection of $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Informally, a sheaf is a system in which global data is determined by local data. (If you have local data that is consistent, then that gives you global data.)

Usually there is another condition that says, if all the local sections are trivial, then the global section is also trivial. But this is given by the uniqueness of the preceding definition.

Example 1.3. Let X be a topological space, and define $\mathcal{U} = \{s : U \to \mathbb{R} \text{ continuous}\}$. You can check that $\mathcal{U}$ is a sheaf. (You could take $\mathbb{C}$ instead of $\mathbb{R}$, or any topological group.)

Stalks tell you what happens when you look at a sheaf near a point.

Definition 1.4. Let $\mathcal{F}$ be a presheaf on X, $x \in X$. We define the *stalk* of $\mathcal{F}$ at x to be

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

More explicitly, an element of $\mathcal{F}_x$ is represented by a pair (U, s) where $U \subset X$ and $s \in \mathcal{F}(U)$ is a section, with the relation that $(U, s) \sim (V, t)$ if there is some $W \subset U \cap V$ (containing x) such that $s|_W = t_W$.

Check that $\mathcal{F}_x$ is an abelian group.

Definition 1.5. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are presheaves [sheaves] on X. A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ *of presheaves [sheaves]* is a collection of homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$

for each open U, chosen consistently in the sense that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V) \end{tikzcd}$$

is commutative (where the vertical maps are restriction maps).

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then this is a morphism of sheaves.

Definition 1.6. φ is an *isomorphism* if it has an inverse morphism $\psi : \mathcal{G} \to \mathcal{F}$ such that $\varphi \circ \psi = Id$ and $\psi \circ \varphi = Id$.

For any $x \in X$, we get an induced homomorphism $\mathcal{F}_x \to \mathcal{G}_x$.

Given a presheaf, you can always construct an associated sheaf.

Definition 1.7 (Sheaf associated to a presheaf). Suppose $\mathcal{F}$ is a presheaf. The sheaf associated to $\mathcal{F}$ is a sheaf $\mathcal{F}^+$, along with a morphism $\mathcal{F} \to \mathcal{F}^+$, satisfying:

- given any morphism $\mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$, then there is a unique morphism $\mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{G} \\ & \mathcal{F}^+ \arrow[u] \end{tikzcd}$$

Construction 1.8. $\mathcal{F}^+(U)$ is the subset of sections $s : U \to \coprod_{x \in U} \mathcal{F}_x$ that satisfy the following condition:

- For every $x \in U$, there is some neighborhood $W \subset U$ of x, and a section $t \in \mathcal{F}(W)$, such that $s(y) = t_y$ for every $y \in W$. (Here t_y denotes the image of t in the stalk $\mathcal{F}_y$.)

Note that this means $s(x) \in \mathcal{F}_x$ for every x, and moreover, every point has a neighborhood W in which all of these elements t_y of the germ come from the *same* section $t \in \mathcal{F}(W)$.

The idea is that all the "bad" sections vanish in the stalk.

Exercise 1.9. Prove that there is a natural map $\mathcal{F} \to \mathcal{F}^+$ satisfying the conditions above, and that $\mathcal{F}^+$ is a sheaf.

It is "obvious" that if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

Remark 1.10. For every $x \in X$, then the map $\mathcal{F}_x \to \mathcal{F}^+_x$ is an isomorphism. This is not surprising, because $\mathcal{F}$ and $\mathcal{F}^+$ has the same local data. The proof of this is routine, and you should do this.

Definition 1.11 (Image and kernel of a morphism). Suppose that $\mathcal{F}$ and $\mathcal{G}$ are presheaves on some topological space X, and suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism.

- The presheaf kernel of φ is defined $\ker^{pre}(\varphi)(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$.
- The presheaf image of φ is similarly defined as $\mathrm{im}^{pre}(\varphi)(U) = \mathrm{im}(\mathcal{F}(U) \to \mathcal{G}(U))$.

Now assume that $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

- The kernel is $\ker^{pre}(\varphi)$ (i.e. you have to show that the presheaf kernel is actually a sheaf, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves).
- This does not work for the image. So define $\mathrm{im}(\varphi) = (\mathrm{im}^{pre}(\varphi))^+$.

Definition 1.12. We say that φ is *injective* if $\ker(\varphi) = 0$. We say that φ is *surjective* if $\mathrm{im}(\varphi) = \mathcal{G}$.

Remark 1.13. For φ to be surjective, all the $\varphi(U)$ do not need to be surjective. (They need to be "locally surjective".)

Example 1.14. Let $X = \{a, b\}$, $U = \{a\}$, $V = \{b\}$. (Together with the empty set, we have four open sets). Define a presheaf as follows:

$$\mathcal{F}(X) = \mathbb{Z}, \mathcal{F}(U) = \mathcal{F}(V) = 0$$

The stalks at both points vanish, so $\mathcal{F}^+ = 0$.

Example 1.15. Let X be the same as before. Now define a presheaf as follows:

$$\mathcal{G}(X) = 0, \mathcal{G}(U) = \mathcal{G}(V) = \mathbb{Z}$$

The smallest open set containing a is U, so $\mathcal{G}_a = \mathbb{Z}$, and similarly $\mathcal{G}_b = \mathbb{Z}$. Then $\mathcal{G}^+(X) = \mathbb{Z} \oplus \mathbb{Z}$ (the points are completely independent, hence functions have an independent choice of value on a or b), and $\mathcal{G}^+(U) = \mathcal{G}^+(V) = \mathbb{Z}$, and the restriction maps are just projections.

Lecture 2: October 8

Let X be a topological space, $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then

(1) φ is injective $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$
(2) φ is surjective $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$
(3) φ is an isomorphism $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$

Proof. Suppose φ is injective. Assume $\varphi_x(U, s) \to 0$. By definition, $\varphi_x(U, s) = (U, \varphi_U(s))$. In order for this to be zero on the stalk, it is locally zero: there is some $W \subset U$ such that $\varphi_U(s)|_W = 0$. By diagram chasing in

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(W) \arrow[r] & \mathcal{G}(W) \end{tikzcd}$$

$\varphi_W(s|_W) = 0$. Since φ is injective, $s|_W = 0$. Thus, $(U, s) = (W, s|_W) = 0$. Therefore, φ_x is injective.

Now suppose φ_x is injective at every x. Suppose $\varphi_U(s) = 0$ for some $s \in \mathcal{F}(U)$. Since φ_x is injective, and $\varphi_x(U, s) = 0$ in $\mathcal{F}_x$ for all x, then $(U, s) = 0$ in $\mathcal{F}_x$. That is, for all $x \in U$, there is some $W_x \subset U$ such that $s|_{W_x} = 0$. By definition of a sheaf, $s = 0$ in $\mathcal{F}(U)$. Thus φ_U is injective for all U.

Assume φ is surjective. Pick $(U, t) \in \mathcal{G}_x$. We can't necessarily get a preimage of t, but there is some small neighborhood in which $t|_W$ has a preimage. Let $s \in \mathcal{F}(W)$ be the preimage of $t|_W$. In particular, $\varphi_x(W, s) = (W, t|_W) = (U, t)$. So φ_x is surjective.

Now assume that φ_x is surjective for all x. *I don't understand why we can't just use the fact that* $\operatorname{im}\varphi$ *has the same stalks as the preimage version (the actual image), so showing* $\operatorname{im}\varphi = \mathcal{G}$ *is the same as showing* $\mathcal{G}_x = (\operatorname{im}^{pre}\varphi)_x$ *for all* x.

We have a factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[dr] & \mathcal{G} \\ & Im(\varphi) \arrow[u, hook] \end{tikzcd}$$

By replacing $\mathcal{F}$ by its image $Im(\varphi)$ we could assume that φ is injective. By the first part of the theorem, we can assume that φ_x is injective, hence an isomorphism for all x. We need to now show that (the new) φ is an isomorphism.

Pick $t \in \mathcal{G}(U)$. We will find local preimages and glue them. For each $x \in U$ there is some $W_x \subset U$ (containing x) and $s_x \in \mathcal{F}(W_x)$ such that $\varphi_{W_x}(s_x) = t|_W$. Since φ is injective, $s_x|_{W_x \cap W_y} = s_y|_{W_x \cap W_y}$. Now by the definition of sheaves, there is some

$$s \in \mathcal{F}(U)$$

such that $\varphi_U(s) = t$. This means that the φ_U are isomorphisms for all U, as we wanted.

Suppose that φ is an isomorphism. By (1) and (2) φ_x is an isomorphism. Now assume that φ_x are all isomorphisms. Use the same argument in (2).

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Definition 2.1. Suppose that X is a topological space. Then a *complex of sheaves on* X is a sequence

$$\cdots \overset{\varphi_{-1}}{\to} \mathcal{F}_{-1} \overset{\varphi_0}{\to} \mathcal{F}_0 \overset{\varphi_1}{\to} \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$$

such that $\operatorname{im}(\varphi_i) \subset \ker(\varphi_{i+1})$.

We say that this complex is an exact sequence if $\mathrm{im}(\varphi_i) \subset \ker(\varphi_{i+1})$ for all i. In particular, a short exact sequence is an exact sequence of the form

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

Exercise 2.2. A complex $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence iff the sequences $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is an exact sequence.

Example 2.3. Let X be a topological space, A an abelian group. Start by defining $\mathcal{F}(U) = A$ for every U. But this isn't a sheaf. Instead define the *constant sheaf at A* to be $\mathcal{F}^+$.

Proposition 2.4. *The constant sheaf at A is isomorphic to $\mathcal{G}$, defined as*

$$\mathcal{G}(U) = \{\alpha : U \to A \,:\, \alpha \textit{ is continuous}\}$$

where A is given the discrete topology.

Proof. $\mathcal{G}$ is a sheaf, and we have a natural morphism $\mathcal{F} \to \mathcal{G}$ where the function $\mathcal{F}(U) \to \mathcal{G}(U)$ is just the constant map $a \mapsto (U \xrightarrow{a} A)$. This uniquely determines a morphism $\varphi : \mathcal{F}^+ \to \mathcal{G}$.

I claim that this is an isomorphism. It suffices to show that φ_x is an isomorphism, and since $\mathcal{F}^+_x \cong \mathcal{F}_x$, it suffices to show that $\psi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.

ψ_x is injective: If $\psi_x(U, s) = 0 = (U, \psi_U(s)) = (U, \alpha_s)$ then $\alpha_s = 0$ on some point, and hence $s = 0$.

ψ_x is surjective: Pick $(U, t) \in \mathcal{G}_x$. ($t : U \to A$ where $x \mapsto t(x)$.) Put $W = t^{-1}\{t(x)\}$, which is open because A has the discrete topology. Now $(U, t) = (W, t|_W)$. Note that $t|_W$ is a constant function. If we put $s = t(x)$, then $\psi_x(W, s) = (W, t|_W)$. ○

Definition 2.5. Suppose that we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, then define a presheaf $f_*\mathcal{F}$ where $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$. This is called the *direct image of $\mathcal{F}$*.

Proposition 2.6. *$f_*\mathcal{F}$ is a sheaf.*

Proof. Assume $U = \bigcup U_i$ is an open cover of U, and there are sections $s_i \in (f_*\mathcal{F})(U_i)$ that agree on intersections. By definition, $(f_*\mathcal{F}(U_i)) = \mathcal{F}(f^{-1}U_i)$. Since $\mathcal{F}$ is a sheaf, there is some $s \in \mathcal{F}(f^{-1}U)$ such that $s|_{U_i} = s_i$. So $s \in (f_*\mathcal{F})$ is the thing you want. ○

Example 2.7. Let Y be a topological space, $X = \{x\} \subset Y$ and f is the inclusion map. Suppose A is an abelian group. A defines a sheaf $\mathcal{F}$ on X. We call $f_*\mathcal{F}$ the skyscraper sheaf on Y at X: this is

$$(f_*\mathcal{F})(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

Lecture 3: October 10

Schemes.

Conventions 3.1. Rings will always be commutative with unit, and ring homomorphisms take $1 \mapsto 1$. A local ring A is a commutative ring with a unique maximal ideal. A local homomorphism $A \to B$ of local rings is a homomorphism that takes the maximal ideal of A into the maximal ideal of B.

Definition 3.2. Suppose A is a commutative ring.

- The *spectrum of A* is
$$Spec(A) = \{P \,:\, P \lhd A \text{ is a prime ideal}\}$$
- If $I \subset A$ is any ideal, then define the vanishing set
$$V(I) = \{P \in Spec(A) \,:\, I \subset P\}$$

Lemma 3.3. *We have:*

(1) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$ *for all ideals* I, J
(2) $V(\sum I_\alpha) = \bigcap_\alpha V(I_\alpha)$ *for all families* I_α *of ideals*
(3) $V(I) \subset V(J) \iff \sqrt{I} \supset \sqrt{J}$

Recall that $\sqrt{I} = \{i^r \,:\, i \in I\} = \bigcap_{\mathfrak{P} \supset I} \mathfrak{P}$.

Proof. (1) $P \in V(IJ) \iff IJ \subset P \iff I \subset P$ or $J \subset P \iff P \in V(I) \cup V(J)$ where the second equivalence is by the definition of prime ideals. For the statement about $V(I \cap J)$ use the fact that $IJ \subset I \cap J$ so $I \subset P$ or $J \subset P$ implies $I \cap J \subset P \implies IJ \subset P$.

(2) $P \in V(\sum I_\alpha) \iff \sum_\alpha I_\alpha \subset P \iff I_\alpha \subset P$ for all $\alpha \iff P \in \cap_\alpha V(I_\alpha)$

(3) Use the commutative algebra fact that
$$\sqrt{I} = \bigcap_{P \supset I} P = \bigcap_{P \in V(I)} P$$
In particular, $V(I) = V(\sqrt{I})$, and (3) follows. ○

Definition 3.4. Let A be a commutative ring, $X = Spec(A)$. By taking the sets $V(I)$ to be closed in X we get a topology on X. This is called the *Zariski topology.*

Note that $Spec(A) - V(I) = \{\mathfrak{P}$ *that don't contain* $I\}$. *Define*
$$D(a) = \{\mathfrak{P} \,:\, a \notin \mathfrak{P}\}$$
for every $a \in I$*; then* $Spec(A) - V(I) = \bigcup_{a \in I} D(a)$*, and so the sets* $D(a)$ *form a basis for the Zariski topology.*

Definition 3.5. Define the sheaf $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \{s : U \to \bigsqcup_{P \in U} A_P \text{ such that } (*)\}$$

where the conditions (*) are:

(1) $s(P) \in A_P$ for all P
(2) For every $p \in U$ there is a neighborhood $W \ni p$ such that $s|_W$ is the constant function $\frac{a}{b}$ for some $a, b \in A$

The sections are elements that are locally given by fractions.

If $W = U$ in the definition, we also abuse notation to denote s by $\frac{a}{b}$. Note that $\mathcal{O}_X$ is a sheaf of rings: $\mathcal{O}_X(U)$ is a commutative ring and restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are ring homomorphisms.

Theorem 3.6. *If A is a commutative ring, $X = Spec(A)$, then*

(1) *The stalk $(\mathcal{O}_X)_P \cong A_P$ is an isomorphism of local rings, for every $P \in X$*
(2) *$\mathcal{O}_X(D(b)) \cong A_b$ where $D(b) = X \backslash V(\langle b \rangle)$ (set where b does not vanish) for all $b \in A$. Here $A_b = \{\frac{a}{b^n}\}$.*
(3) *$\mathcal{O}_X(X) \cong A$*

Proof. (1) Define $f : A_P \to (\mathcal{O}_X)_P$ by $f\left(\frac{a}{b}\right) = (D(b), s = \frac{a}{b})$. This is well-defined: if $\frac{a}{b} = \frac{a'}{b'}$ in A_P, then there is some $c \in A \backslash P$ such that

$$c(ab' - a'b) = 0$$

$$(D(b), \frac{a}{b}) = (D(b) \cap D(c) \cap D(b'), \frac{a}{b}) = (D(b) \cap D(c) \cap D(b'), \frac{a'}{b'}) = (D(b'), \frac{a'}{b'})$$

Injectivity: If $f(\frac{a}{b}) = 0$ then $s = \frac{a}{b} = 0$ on some $W \ni p$. So $\frac{a}{b} = s(p) = 0$ in A_p.

Surjectivity: Sections of $\mathcal{O}_X$ are locally defined as $\frac{a}{b}$, so this is trivial.

Exercise: show that f is a *local* homomorphism.

(2) Define $g : A_b \to \mathcal{O}_X(D(b))$ by $g(\frac{a}{b^n}) = \frac{a}{b^n}$. This is well defined, similar to (1).

Injective: assume $g(\frac{a}{b^n}) = \frac{a}{b^n} = 0$ in $\mathcal{O}_X(D(b))$. So for all $P \in D(b)$, $\frac{a}{b^n} = 0$, which means that there is some $c_P \in A \backslash P$ such that $c_P a = 0$. Let $\mathfrak{c}$ be the ideal spanned by all the c_P's. I claim $D(b) \cap V(\mathfrak{c}) = \emptyset$: if $P \in D(b)$ then $c_P \notin P$ so $P \not\supset \mathfrak{c}$ hence $P \notin V(\mathfrak{c})$. Equivalently, $V(\mathfrak{c}) \subset V(b)$. So $\sqrt{\mathfrak{c}} \supset \sqrt{(b)} \implies b^r = \sum_{\substack{finite \\ set\ of\ i}} t_i c_{p_i}$ for some $c_{p_1}, \cdots, c_{p_\alpha}$. Then $ab^r = \sum t_i a c_{p_i} = 0$ which shows that $\frac{a}{b^n} = 0$ in A_b.

Surjective: Pick $s \in \mathcal{O}_X(D(b))$. There is some open covering $D(b) = \bigcup U_i$ and $a_i, e_i \in A$ such that $s|_{U_i} = \frac{a_i}{e_i}$ for all i. The goal is to replace the U_i with something related to $D(b)$. For all $p \in U_i$, there is some $d \in A$ such that $p \in D(d) \subset U_i \subset D(e_i)$ *(since the $D(d)$ are* *a base for the Zariski topology)*. This implies $\sqrt{(d)} \subset \sqrt{(e_i)}$. Then there is some power r such that $d^r = c \cdot e_i$ for some $c \in A$. So $\frac{a_i}{e_i}|_{D(d^r)=D(d)} = \frac{ca_i}{ce_i} = \frac{ca_i}{d^r}$. By replacing the U_i's with appropriate $D(d^r)$, we could assume that $U_i = D(e_i)$ and $s|_{U_i} = \frac{a_i}{e_i}$.

We have a cover $D(b) = \bigcup_i D(e_i)$, which I claim can be taken to be finite. This follows from the fact that $V(\{e_\alpha\}_\alpha) \subset V(\langle b \rangle) \iff \sqrt{\{e_\alpha\}_\alpha} \supset \sqrt{\langle b \rangle} \iff b^n = \sum_\alpha \ell_\alpha e_\alpha$. We know all of this is true for *some* collection of e_α, but the last statement only makes sense for a finite collection, so we may assume the set $\{e_\alpha\}$ is finite.

As constant functions, $\frac{a_i}{e_i}$ were originally restrictions of a section s, so $\frac{a_i}{e_i} = \frac{a_j}{e_j}$ in $\mathcal{O}(D(e_i) \cap D(e_j)) = \mathcal{O}(D(e_i e_j))$.

By construction we have a commutative diagram:

$$\begin{tikzcd}
A_{e_i} \arrow[r] \arrow[d] & \mathcal{O}_x(D(e_i)) \arrow[d] & \\
A_{e_i e_j} \arrow[r] & \mathcal{O}_x(D(e_i e_j)) \arrow[r, equal] & \mathcal{O}_x(D(e_i) \cap D(e_j)) \\
A_{e_j} \arrow[r] \arrow[u] & \mathcal{O}_x(D(e_j)) \arrow[u] &
\end{tikzcd}$$

Since $A_{e_i e_j} \to \mathcal{O}_X(D(e_i e_j))$ is injective, $\frac{a_i}{e_i} = \frac{a_j}{e_j}$ in $A_{e_i e_j}$.

There is some m such that

$$(e_i e_j)^m (a_i e_j - a_j e_i) = 0$$

(by definition of how equality in the localized ring works). Equivalently, $a_i e_i^m e_j^{m+1} - a_j e_j^m e_i^{m+1} = 0$. Replace $\frac{a_i}{e_i}$ with $\frac{a_i e_i^m}{e_i^{m+1}}$ for each i; now we can assume that $a_i e_j - a_j e_i = 0$ for all i, j.

Now let $c = \sum \ell_i a_i$ (recall we had $b^n = \sum \ell_i e_i$). Then $e_j c = e_j(\sum \ell_i a_i) = \sum \ell_i a_i e_j = \sum \ell_i a_j e_i = a_j(\sum \ell_i e_i) = a_j b^n$. Therefore, $s|_{D(e_j)} = \frac{a_j}{e_j} = \frac{c}{b^n}$ for all j. This expression does not depend on j, and so $s = \frac{c}{b^n}$ is a constant section $\in A_b$.

(3) Simply put $b = 1$. Then $D(b) = X$ and $A_b = A$.

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Lecture 4: October 12

Definition 4.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ when X is a topological space and $\mathcal{O}_X$ is a sheaf of rings.

A morphism of ringed spaces $(Y, \mathcal{O}_Y) \to (X, O_X)$ is given by a continuous map $f : Y \to X$ and a morphism $\varphi : \mathcal{O}_X \to f_*\mathcal{O}_Y$. ("Functions on X pull back to functions on Y.")

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that $(\mathcal{O}_X)_P$ is a local ring.

A morphism $(Y, \mathcal{O}_Y) \overset{f,\varphi}{\to} (X, \mathcal{O}_X)$ between locally ringed spaces is a morphism of ringed spaces such that if $p \in Y$ and $q \in f(p)$ then the natural map $(\mathcal{O}_X)_q \to (\mathcal{O}_Y)_p$ is a local homomorphism.

An isomorphism is a morphism with an inverse.

THEOREM 4.2. *Suppose that A and B are rings*[1]*, and $(X = Spec(A), \mathcal{O}_X)$ and $(Y = Spec(B), \mathcal{O}_Y)$. Then*

(1) *$(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces.*
(2) *A homomorphism $\alpha : A \to B$ "naturally" induces a morphism of locally ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.*
(3) *Any morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces arises from a morphism $A \to B$ as in (2).*

Proof. (1) For every $q \in X$, $p \in Y$, the stalks $(\mathcal{O}_X)_q$ and $(\mathcal{O}_Y)_p$ are local rings.

(2) Define $f : Y \to X$ by $f(p) = \alpha^{-1}p$. We need to show that f is continuous. $f^{-1}(V(I)) = V(\alpha(I) \cdot B)$, the ideal generated by $\alpha(I)$. Define φ as follows: pick $s \in \mathcal{O}_X(U)$. s is a function $U \to \bigsqcup_{q \in U} A_q$ satisfying certain properties. We need to define a function $t \in (f_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(f^{-1}U)$, so $t : f^{-1}U \to \bigsqcup_{p \in f^{-1}U} B_p$. Remember, if $f(p) = q$ then we have a homomorphism $A_q \overset{\varphi_p}{\to} B_p$. So assign $t(p) = \varphi_p(s(q)) = \varphi_p(s(\alpha^{-1}p))$. If s is locally given by $\frac{a}{b}$ then t is locally given by $\frac{\alpha(a)}{\alpha(b)}$. So the morphism we're looking for just sends $s \mapsto t$; this gives a morphism $\mathcal{O}_X \overset{\varphi}{\to} f_*\mathcal{O}_Y$.

Note that the map $(\mathcal{O}_X)_q \to (\mathcal{O}_Y)_P$ is nothing but $\varphi_p : A_q \to B_p$. So, the morphism $(Y, \mathcal{O}_Y) \overset{f,\varphi}{\to} (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces.

(3) Suppose we are given a morphism of locally ringed spaces $(Y, \mathcal{O}_Y) \overset{g,\psi}{\to} (X, \mathcal{O}_X)$. We need to get a homomorphism $A \to B$. The morphism ψ gives a homomorphism $\mathcal{O}_X(X) \to (g_*\mathcal{O}_Y)(X) = \mathcal{O}_Y(Y)$. But from last lecture, $A = \mathcal{O}_X(X)$ and $B = \mathcal{O}_Y(Y)$. So this gives a homomorphism $\alpha : A \to B$.

[1]All our rings are commutative.

Let f, φ be those given in (2) induced by α. (So we need to show that $f = g$ and $\varphi = \psi$.) First we show that $f = g$, and moreover $\varphi_p = \psi_p$ for all $p \in Y$. We have a diagram

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[r, "\psi_X"] \arrow[d] & (f_*\mathcal{O}_Y)(X) \arrow[d] \\ (\mathcal{O}_X)_q \arrow[r, "\psi_q"] & (f_*\mathcal{O}_Y)_q \end{tikzcd}$$

If $q = g(p)$ then $(f_*\mathcal{O}_Y)_q = (\mathcal{O}_Y)_p$, and (using Theorem 3.6) we can rewrite the diagram as

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ A_q \arrow[r, "\psi_q"] & B_p \end{tikzcd}$$

Use the fact that ψ_q is a local homomorphism: the maximal ideal of $(\mathcal{O}_Y)_p$ pulls back to the maximal ideal of $(\mathcal{O}_X)_q$.

$$\begin{aligned} q &= \beta^{-1}(\text{max. ideal of } A_q) = \beta^{-1}\psi_p^{-1}(\text{maximal ideal of } B_p) \\ &= \alpha^{-1}\lambda^{-1}(\text{maximal ideal of } B_p) = \alpha^{-1}p = g(p) \end{aligned}$$

So $f(p) = q = g(p)$. The diagram also shows that $\psi_p = \varphi_p$.

Now we show that $\varphi = \psi$. For each open set $U \subset X$ we also have two commutative diagrams as above:

$$\begin{tikzcd} A = \mathcal{O}_X(U) \arrow[r, "\psi_U"] \arrow[d] & \mathcal{O}_Y(f^{-1}U) = B \arrow[d] \\ A_q = (\mathcal{O}_X)_q \arrow[r, "\psi_p"] & (\mathcal{O}_Y)_p = B_p \end{tikzcd}$$

$$\begin{tikzcd} A = \mathcal{O}_X(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{O}_Y(f^{-1}U) = B \arrow[d] \\ A_q = (\mathcal{O}_X)_q \arrow[r, "\varphi_p"] & (\mathcal{O}_Y)_p = B_p \end{tikzcd}$$

Pick $s \in \mathcal{O}_X(U)$. Then $\varphi_U(s) = \psi_U(s)$ in B_p for every $p \in f^{-1}U$. The bottom rows in both diagrams are the same. So you have two different sections coming from the top rows, that are eventually equal after looking in the stalks. But by the definition of sheaves, $\varphi_U(s) = \psi_U(s)$ in $\mathcal{O}_X(f^{-1}U)$, and so $\varphi = \psi$.

○

Definition 4.3. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that there is some ring A such that $(X, \mathcal{O}_X) = (Spec(A), \mathcal{O}_{Spec(A)})$.

A *scheme* is a locally ringed space $(Y, \mathcal{O}_Y)$ such that locally it is an affine scheme; that is, for all $p \in Y$ there is some $U \ni p$ such that $(U, \mathcal{O}_U)$ is an affine scheme.

Morphisms and isomorphisms are defined as locally ringed spaces.

Think of affine schemes as the disks in differential geometry: simple objects that you glue together to make manifolds (schemes).

EXAMPLE 4.4. $\mathbb{Z}$ is a commutative ring. Consider $X = Spec(\mathbb{Z}) = \{\text{prime ideals of } \mathbb{Z}\} \cong \{0, (2), (3), (5), \cdots\}$. If $p = 0$ then $(\mathcal{O}_X)_p = \mathbb{Z}_p = \mathbb{Q}$. If $p \neq 0$ then $(\mathcal{O}_X)_p = \mathbb{Z}_p$ is a local ring and $(\mathcal{O}_X)_p/\mathfrak{m} = \mathbb{F}_p$ (finite field).

LECTURE 5: OCTOBER 15

EXAMPLE 5.1. Suppose that A is a field. Then, $X = \mathrm{Spec}(A)$ has only one point. Then $\mathcal{O}_X = A$. A field extension $A \hookrightarrow B$ corresponds to a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. So Galois theory is a special case of algebraic geometry.

EXAMPLE 5.2. Suppose D is a DVR (Discrete Valuation Ring – regular Noetherian local ring of dimension 1). Then $\mathrm{Spec}(D)$ has two points, corresponding to (0) and $\mathfrak{m}$. (For example, take $D = \mathbb{Z}_p$ for some nonzero prime ideal (p).) If $p \neq 0$, then $(\mathcal{O}_X)_p = A_p = A$; if $p = 0$ then $(\mathcal{O}_X)_p = A_p = Frac(A)$.

EXAMPLE 5.3. Let R be a ring, $A = R[t_1, \cdots, t_n]$ a polynomial ring in n variables. Define $\mathbb{A}^n_R = \mathrm{Spec}(R[t_1, \cdots, t_n])$ which is called the n-affine space over R.

For any ideal $I \subset R[t_1, \cdots, t_n]$ we have the scheme $\mathrm{Spec}(R[t_1, \cdots, t_n]/I)$. The homomorphism $R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_n]/I$ gives a morphism $e : Y = \mathrm{Spec}(R[t_1, \cdots, t_n]/I) \to \mathrm{Spec}(R[t_1, \cdots, t_n]) = X$. It is easy to see that $e(Y) = V(I) \subset X$.

EXAMPLE 5.4. Suppose k is an algebraically closed field. (This is classical algebraic geometry.)

Suppose $I = \langle f_1, \cdots, f_r \rangle \subset k[t_1, \cdots, t_n]$. Define $S = \{(a_1, \cdots, a_n) : f_j(a_1, \cdots, a_n) = 0 \ \forall j\}$. There is a 1-1 correspondence

$$S \iff \{ \text{ maximal ideals of } k[t_1, \cdots, t_n]/I\} \subset \mathrm{Spec}(k[t_1, \cdots, t_n]/I)$$

where $(a_1, \cdots, a_n) \mapsto \langle t_1 - a_1, \cdots, t_n - a_n \rangle$.

EXAMPLE 5.5. Let $k = \mathbb{C}$. Then $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[t] = \{0, \langle t - a \rangle\}_{a \in \mathbb{C}}$. This is "$\mathbb{C}$ plus a point".

Take a point $P = 0$. Then $\overline{\{P\}}$ is the smallest closed set containing P. Since $0 = 0 \cdot (t - a)$ is in every prime ideal, this is the whole space! So P is called the *generic point*.

DEFINITION 5.6. Let X be a topological space, and $P \in X$. If $\overline{\{P\}} = X$, then we say that P is a *generic point*.

Take $I = \langle t \rangle$. Then $* = \operatorname{Spec} \mathbb{C} = \operatorname{Spec}(\mathbb{C}[t]/\langle t \rangle) \to \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}$ is just the inclusion of the point 0 (corresponding to the prime $\langle t \rangle$).

Take $J = \langle t^3 \rangle$. Then $Z = \operatorname{Spec}(\mathbb{C}[t]/J)$ is one point, with multiplicity 3. There is a natural morphism $Z \to \mathbb{A}^1_{\mathbb{C}}$ sending $* \to \langle t \rangle$.

Example 5.7. Take $I = \langle t_2 - t_1^2 \rangle \subset k[t_1, t_2]$. You can't draw this in $\mathbb{C}$, but it's a parabola in $\mathbb{R}$. X is "smooth", so it gives a complex manifold. If S is the complex solution of $t_2 - t_1^2$, then each $x \in X$ has an open neighborhood (in the complex topology, not the Zariski topology), which is biholomorphic to an open disc in $\mathbb{C}$.

Example 5.8. Set $R = \mathbb{Z}$. Let $\langle t_1^m + t_2^m - t_3^m \rangle \subset \mathbb{Z}[t_1, \cdots, t_3]$. What are the solutions of I over $\mathbb{Z}$? ("or, what planet have you been living on?")

$X = \operatorname{Spec}(\mathbb{Z}[t_1, t_2, t_3]/I)$ is not empty.

Example 5.9. Let $R = \mathbb{R}$. Take $I = \langle t_1^2 + t_2^2 + 1 \rangle \subset \mathbb{R}[t_1, t_2]$. The set of solutions over $\mathbb{R}$ is empty. However, $X = \operatorname{Spec} \mathbb{R}[t_1, t_2]/I$ is not trivial. The homomorphism $A = \mathbb{R}[t_1, t_2]/\langle t_1^2 + t_2^2 + 1 \rangle \to \mathbb{C}[t_1, t_2]/\langle t_1^2 + t_2^2 + 1 \rangle = B$ gives a morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$.

Example 5.10. First look at the induced map $\operatorname{Spec} \mathbb{C}[t] \to \operatorname{Spec} \mathbb{R}[t]$. This takes $\langle t - i \rangle$ and $\langle t + i \rangle$ to $\langle t^2 + 1 \rangle$. (So you get factorizations of real polynomials over $\mathbb{C}$.)

Let $I = \langle t^2 + 1 \rangle \subset \mathbb{R}[t]$. There are no solutions over $\mathbb{R}$ but the homomorphism $\mathbb{R}[t]/\langle t^2 + 1 \rangle \to \mathbb{C}[t]/\langle t^2 + 1 \rangle$ gives a morphism $\operatorname{Spec} \mathbb{C}[t]/\langle t^2 + 1 \rangle \to \operatorname{Spec} \mathbb{R}[t]/\langle t^2 + 1 \rangle$.

Example 5.11. Take $I = \langle t_2^2 - t_1(t_1^2 - 1) \rangle \subset \mathbb{C}[t_1, t_2]$. Then $X = \operatorname{Spec}(\mathbb{C}[t_1, t_2]/I)$ is an elliptic curve.

Example 5.12. Let $I = \langle t_1 t_2 - 1 \rangle \subset \mathbb{C}[t_1, t_2]$, and take $X = \operatorname{Spec} \mathbb{C}[t_1, t_2]/I$. We have a natural homomorphism $\mathbb{C}[t_1] \to \mathbb{C}[t_1, t_2]/I$. This gives a morphism $X \to Y = \mathbb{A}^1_C$. Over $\mathbb{R}$ this looks like $y = \frac{1}{x}$, and the morphism is just projection to the x-axis. In terms of ideals, $\langle t_1 - a_1, t_2 - a_2 \rangle \mapsto \langle t_1 - a_1 \rangle$. This is almost a 1-1 correspondence. But, there is nothing that maps to 0. If you just remove this point you get an isomorphism: there is an induced homomorphism

$$\mathbb{C}[t_1]_{t_1} = \mathbb{C}[t_1, t_2]/I$$

that is an isomorphism.

$D(t_1) = \mathbb{A}^1 - \{0\}$.

Lecture 6: October 18

RECALL, A graded ring is a ring of the form $\bigoplus_{d \geq 0} S_d$ such that, for every $d, e \geq 0$, $S_d \cdot S_e \subset S_{d+e}$. The elements of S_d are called *homogeneous elements of degree d*.

Homogeneous ideals are ideals I of S such that $I = \bigoplus_{d \geq 0}(I \cap S_d)$. Equivalently, I is generated by homogeneous elements.

Set $S_+ = \bigoplus_{d>0} S_d$.

If P is a homogeneous prime, then
$$S_{(P)} = \left\{ \frac{a}{b} \in S_P \,:\, a, b \text{ are homogeneous, } \deg(a) = \deg(b) \right\}.$$
If $b \in S_+ = \bigoplus_{d>0} S_d$ is homogeneous then
$$S_{(b)} = \left\{ \frac{a}{b^r} \in S_b \,:\, a \text{ is homogeneous, } \deg(a) = \deg(b^r) \right\}.$$
Definition 6.1. Let S be a graded ring. Define the projective space of S as
$$\mathrm{Proj}(S) = \{P \lhd S \,:\, P \text{ is a homogeneous prime }, P \not\supset S_+\}$$

If I is a homogeneous ideal, let $V_+(I) = \{P \in \mathrm{Proj}\, S \,:\, P \supset I\}$. Similarly, set $D_+(b) = \mathrm{Proj}\, S \backslash V_+((b))$.

Lemma 6.2. *Let S be a graded ring. Then*

(1) *If $I, J \lhd S$ are homogeneous, then*
$$V_+(IJ) = V_+(I \cap J) = V_+(I) \cup V_+(J)$$
(2) *If $I_\alpha \lhd S$ are homogeneous then $V_+(\sum I_\alpha) = \bigcap V_+(I_\alpha)$*

Proof. Same as in the affine case. ○

Definition 6.3. Let S be a graded ring. Put $X = \mathrm{Proj}(S)$. The lemma says that X is a topological space, where the closed subsets are the $V_+(I)$. (This is the *Zariski toplogy on* $\mathrm{Proj}(S)$.)

We define a sheaf of rings $\mathcal{O}_X$ on X as:
$$\mathcal{O}_X(U) = \{s : U \to \bigsqcup_{p \in U} S_{(p)} \,:\, (*)\}$$
satisfying the conditions:

(1) $s(p) \in S_{(p)}$ for all $p \in U$
(2) There exists some $W \subset U$ and $a, b \in S$ of the same degree, such that for all $q \in W$, we have $s(q) = \frac{a}{b}$

Theorem 6.4. *Let S be a graded ring. Then*

(1) $(\mathcal{O}_X)_p \cong S_{(p)}$ *for all $p \in X$ (is an isomorphism of local rings)*
(2) *For each homogeneous element $b \in S_+$, we have a natural isomorphism of locally ringed spaces*
$$D_+(b) \cong \mathrm{Spec}\, S_{(b)}$$
where $D_+(b) = X \backslash V_+(\langle b \rangle)$
(3) $(X, \mathcal{O}_X)$ *is a scheme*

Sketch of proof. The proof looks a lot like the affine proof.

(1) Similar to the affine case (i.e. $Y = \operatorname{Spec} A$, $(\mathcal{O}_Y)_q \cong A_q$.)
(2) *Define a map $f : D_+(b) \to \operatorname{Spec} S_{(b)}$ by sending $P \mapsto P \cdot S_{(b)}$. (If M is a multiplicative subset of a ring A (like $\{b_i\}$) then there is a bijection between primes disjoint from M and primes of $S^{-1}A$.)* One shows that f is a homeomorphism of topological spaces. Because we're using the subspace topology, every closed subset looks like $V_+(I) \cap D_+(b)$. Then $f(V_+(I) \cap D_+(b)) = V(I_b \cap S_{(b)})$ where $I \lhd S$ is homogeneous. Then one shows that, for all $p \in D_+(b)$, we have $(\mathcal{O}_X)_P = S_{(P)} \cong (S_{(b)})_{f(p)} \cong (\mathcal{O}_Z)_{f(p)}$ where $Z = \operatorname{Spec} S_{(b)}$. Finally, for each open set $U \subset D_+(b)$ we get an isomorphism $\mathcal{O}_X(U) \cong \mathcal{O}_Z(f(U))$. (Use the fact that there's a homeomorphism with $f(U)$...)
(3) Follows from (1) and (2). For every point $p \in X$ there exists some b such that $p \in D_+(b)$ (where $P \not\supset S_+$).

○

Example 6.5. R is a ring, $S = R[t_0, \dots, t_n]$. The scheme $\operatorname{Proj}(S)$ is called the n-projective space over R which is denoted by $\mathbb{P}^n_R$.

By the theorem $D_+(t_i) \cong \operatorname{Spec} S_{(t_i)}$, but this is affine n-space. For example, $S_{(t_0)} \cong R[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}]$. This is isomorphic to $R[u_1, \dots, u_n]$ where the u_i are new variables. Then

$$\operatorname{Spec} S_{(t_0)} \cong \mathbb{A}^n_R$$

So the projective scheme is locally affine space. In particular, $\mathbb{P}^n_R$ is covered by $n+1$ copies of $\mathbb{A}^n_R$. For example, look at the case when $R = \mathbb{C}$, and $n = 1$. Then $\mathbb{P}^1_{\mathbb{C}}$ is the scheme associated to the Riemann sphere: if you remove the south pole (or the north pole), you get affine space.

Definition 6.6. Suppose that X is a scheme.

(1) We say that X is *irreducible* if it is irreducible as a topological space: if $U, V \subset X$ are nonempty, then $U \cap V \neq \emptyset$. (This is sort of the opposite of being Hausdorff.)
(2) We say that X is *reduced* if $\mathcal{O}_X(U)$ is a reduced ring for every $U \subset X$: this means that $\operatorname{nil}(\mathcal{O}_X(U)) := \sqrt{0} = 0$. (So the ring has no nilpotent elements.) For exmple, $\operatorname{Spec}\left(\mathbb{C}[t]/\langle t^4\rangle\right)$ is not reduced.
(3) We say that X is *integral* if $\mathcal{O}_X(U)$ is an integral domain, for all open $U \subset X$.

Example 6.7. Suppose $X = \operatorname{Spec} A$.

X is irreducible $\iff$ $\operatorname{nil}(A)$ is prime: X is irreducible iff whenever $X = V(I) \cap V(J)$, then $X = V(I)$ or $V(J)$. But $X = V(I) \cup V(J)$ iff $V(0) = \operatorname{Spec} A = V(IJ)$. The condition $X = V(I)$ or $V(J)$ can be reformulated as follows: $\sqrt{0} \supset \sqrt{IJ} \implies \sqrt{0} \supset I$ or $\sqrt{0} \supset J$. Equivalently, $\sqrt{0} = \operatorname{nil}(A)$ is a prime ideal.

X is reduced iff A is reduced, i.e. $\text{nil}(A) = 0$. If X is reduced, then A is reduced by definition. Conversely, if A is reduced and $s \in \mathcal{O}_X(U)$ is nilpotent, then the image of s in A_P is nilpotent for every $P \in U$. This implies that $s = 0$ in such A_P because A_P is reduced. By the sheaf condition, this implies that $s = 0$ in $\mathcal{O}_X(U)$.

Lecture 7: October 19

Let $X = \operatorname{Spec} A$. A is integral $\iff$ 0 is a prime ideal $\iff$ $nil(A) = 0$ and it is prime $\iff$ X is reduced and irreducible.

Theorem 7.1. *Let X be a scheme. Then, X is integral $\iff$ X is reduced and irreducible.*

Note that we have already done the $\Longleftarrow$ direction in the affine case: this is Example 6.7.

Proof. ($\implies$) Assume X is integral. Then all the sections are integral domains, and so X is obviously reduced.

X is irreducible: otherwise, there are nonempty opens $U, V \subset X$ such that $U \cap V = \emptyset$. Without loss of generality assume U, V are affine. But the sheaf condition implies

$$\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$$

But the direct sum of two rings can never be integral; this is a contradiction.

Now assume that X is reduced and irreducible.

Claim 7.2. *For every $U \subset X$ open and $p \in U$, there is an open affine $W \subset U$ containing p.*

Proof of claim. By definition of schemes, p belongs to an affine $V = \operatorname{Spec} A \subset X$. Now $p \in U \cap V$ and $U \cap V$ is open in V. Since sets of the form $D(b)$ form a base for the open sets of the Zariski topology, we can find some $D(b) \ni p$ contained in $U \cap V$. Furthermore, $D(b)$ is affine because $D(b) \cong \operatorname{Spec} A_b$. ○

Assume $s, t \in \mathcal{O}_X(U)$ such that $st = 0$ and $s \neq 0$. We need to show that $t = 0$. By the above claim, we can write $U = \bigcup V_i$ where the $V_i = \operatorname{Spec} A_i$ are open and affine. Note that V_i inherits irreducibility and reduced-ness from X; in particular, by Example 6.7, A_i is integral. Since $s \neq 0$, there is some V_i such that $s|_{V_i} \neq 0$. But $st|_{V_i} = 0$, and by the integrality of A_i, we have $t|_{V_i} = 0$.

Now, by irreducibility, for all j we have $V_i \cap V_j \neq \emptyset$. So $t = 0$ on $V_i \cap V_j$, which is an open subspace of the affine space V_j. On affine spaces, all opens are dense, so $t = 0$ on all of V_j.

So $t|_{V_j} = 0$ for all j; since the V_j's form an open cover of U, the sheaf condition says that $t|_U = 0$.

○

Remark 7.3. Let X be an integral scheme. X has a unique generic point, i.e. there is a unique point $\eta \in X$ such that $\overline{\{\eta\}} = X$.

Existence: pick any affine $\operatorname{Spec} A = U \subset X$, and let $\eta = 0$ in $\operatorname{Spec} A = U$.

Uniqueness: if η, η' are generic points, then for any open affine $\operatorname{Spec} A = U \subset X$, $\eta, \eta' \in U$ then $\eta = \eta' = 0$ in $\operatorname{Spec} A = U$.

For example, if $X = \mathbb{A}^n_{\mathbb{C}}$ then $\eta = 0$ in $\operatorname{Spec} \mathbb{C}[t_1, \cdots, t_n]$ is the generic point. If $X = \operatorname{Spec} \mathbb{C}[t]/\left\langle t^\ell \right\rangle$ then $\eta = \langle t \rangle$.

In general, there does not have to be a unique generic point, e.g. when the space is not irreducible.

Definition 7.4. If X is an integral scheme and η is the generic point, then $(\mathcal{O}_X)_\eta$ is a field called the *function field* of X and denoted by $K(X)$. (If $\operatorname{Spec} A = U \subset X$ is affine, then $K(X) = A_\eta$ is the fraction field of A.)

Proposition 7.5. *For all $U \subset X$, the natural map*

$$\mathcal{O}_X(U) \xrightarrow{\alpha} K(X)$$

is injective.

Proof. If $\alpha(s) = 0$ then $s|_V = 0$ for all open affine sets $V \subset U$. By the sheaf condition, $s = 0$. ○

Example 7.6. Let $X = \mathbb{P}^n_k$, where k is a field. Then, since $\mathbb{P}_n$ is covered by copies of $\mathbb{P}^n$, we have $K(X) = K(\mathbb{A}^n_k)$, which is the fraction field of $k[t_1, \cdots, t_n]$, a.k.a. $k(t_1, \cdots, t_n)$.

Definition 7.7. A morphism $f : Y \to X$ is an open immersion if $f(Y)$ is open in X and f gives an isomorphism (as locally ringed spaces) between Y and $U = f(Y)$ (where U is given a sheaf structure by restriction of X). (If $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all $V \subset U$.)

We call Y an *open subscheme* of X. (For example, $X = \operatorname{Spec} A$, $Y = D(b) = \operatorname{Spec} A_b$.)

Definition 7.8. A morphism $g : Z \to X$ is a *closed immersion* if $g(Z)$ is a closed subscheme of X such that g gives a homeomorphism between Z and $g(Z)$, AND such that $\mathcal{O}_X \to g_*\mathcal{O}_Z$ is surjective.

A *closed subscheme* of X is an equivalence class of closed immersions: $g : Z \to X$ and $g' : Z' \to X$ are equivalent if there is an isomorphism $h : Z \to Z'$ such that $g = g'h$.

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\cong"'] & X \\ Z' \arrow[ur, "g'"'] & \end{tikzcd}$$

EXAMPLE 7.9. Let $X = \operatorname{Spec} A$. For any $I \triangleleft A$, the homomorphism $A \to A/I$ induces a morphism $Z = \operatorname{Spec} A/I \xrightarrow{f} X = \operatorname{Spec} A$.

This is a closed immersion: f is a homeomorphism between Z and $V(I) \subset X$.

The morphism $\mathcal{O}_X \xrightarrow{\varphi} f_*\mathcal{O}_Z$ is surjective: (1) if $P \notin V(I)$ then $(\mathcal{O}_X)_P \to (f_*\mathcal{O}_Z)_P = 0$; (2) if $P \in V(I)$ then $(\mathcal{O}_X)_P = A_P \to (f_*\mathcal{O}_X)_P = (A/I)_{P/I}$ is surjective. So the maps on the stalks are all surjective, which implies that φ is surjective.

For any $n \in \mathbb{N}$, I^n also defines a scheme structure on $V(I) = V(I^n)$, by $\operatorname{Spec} A/I^n$.

HARD EXERCISE 7.10. Every closed subscheme of $X = \operatorname{Spec} A$ is $\operatorname{Spec} A/I$ for some $I \triangleleft A$.

EXAMPLE 7.11. Let $X = \operatorname{Spec} \mathbb{C}[t_1, t_2] = \mathbb{A}^2_{\mathbb{C}}$. For each $a \in \mathbb{C}$ let $Z_a = \operatorname{Spec} \mathbb{C}[t_1, t_2]/\langle t_1t_2 - a\rangle$. Over $\mathbb{R}$ this looks like a hyperbola. Taking "the limit as $a \to 0$" we get Z_0, the union of the two axes: $V(t_1t_2) = V(t_1) \cup V(t_2)$.

Let $Z'_a = \operatorname{Spec} \mathbb{C}[t_1, t_2]/\langle at_2 - t_1^2\rangle$. Again drawing pictures in $\mathbb{R}^2$, we get a series of parabolas that approach $\{x = 0, y \geq 0\}$ as $a \to 0$. So we think "the limit" of the Z'_a's as $a \to 0$ is $Z_0 = \operatorname{Spec} \mathbb{C}[t_1, t_2]/\langle -t_1^2\rangle$. Even though $V(t_1^2) = V(t_1)$, the scheme remembers the fact that there seems to be multiplicity going on here: you have the two halves of the parabola that are collapsed together.

LECTURE 8: OCTOBER 22

DEFINITION 8.1. X, Y are schemes over a scheme S, i.e. we have two morphisms $X, Y \to S$. The *fibred product* $X \times_S Y$ is a scheme sitting in a commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

such that $X \times_S Y$ satisfies the following universal property: given a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

then there is a unique morphism $Z \to X \times_S Y$ giving a commutative diagram

$$\begin{tikzcd} Z \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

THEOREM 8.2. *The fibred product exists and is unique up to unique isomorphism.*

First, do the affine case. Suppose $S \subset \operatorname{Spec} A$, $X \subset \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, and there are morphisms $X, Y \to S$. Put $X \times_S Y = \operatorname{Spec} B \otimes_A C$. The morphisms $X, Y \to S$ give morphisms $A \to B, C$ and by basic properties of tensor products we get a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C \end{tikzcd}$$

which satisfies: if there is any other commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & R \end{tikzcd}$$

then there is a unique homomorphism $B \otimes_A C \to R$ such that the above diagram factors through this homomorphism. Now just take the Spec of the rings in this diagram, so we can prove existence of fibred products in the affine case.

For the general case, cover each scheme with affine schemes appropriately; construct the fiber product in the affine case, and glue.

DEFINITION 8.3. Suppose $X \xrightarrow{f} Y$ is a morphism, and $y \in Y$. We have natural morphisms

$$\operatorname{Spec} K(y) \to \operatorname{Spec}(\mathcal{O}_Y)_y \to Y$$

where $K(y) := (\mathcal{O}_Y)_y/\mathfrak{m}$ is called the residue field at y. The first morphism is induced by the quotient map $\mathcal{O}(Y)_y \to (\mathcal{O}_Y)_y/\mathfrak{m}$. For the second morphism: take any affine open $U \ni y$, and take the morphism $\mathcal{O}_Y(U) \to (\mathcal{O}_Y)_y$ which gives $\operatorname{Spec}(\mathcal{O}_Y)_y \to \operatorname{Spec} \mathcal{O}_Y(U) \cong U \subset Y$.

We define the *fiber* over y as $X_y = \operatorname{Spec} K(y) \times_Y X$:

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} K(y) \arrow[r] & Y \end{tikzcd}$$

EXAMPLE 8.4. Suppose we are given a morphism $X \to Y = \operatorname{Spec} \mathbb{Z}$ (in fact every scheme has a natural morphism into $\operatorname{Spec} \mathbb{Z}$). For example, take $X = \operatorname{Spec} \mathbb{Z}[t_1, t_2, t_3]/\left\langle t_1^2 + t_2^2 - 3t_3 \right\rangle$. If $y = 0$ in Y, then $(\mathcal{O}_Y)_y = \mathbb{Q}$ (localizing at zero gives the fraction field), and so $K(y) = \mathbb{Q}$. Then

$$X_y = \operatorname{Spec} K(y) \times_Y X = \operatorname{Spec} \mathbb{Q}[t_1, t_2, t_3]/\left\langle t_1^2 + t_2^2 - 3t_3 \right\rangle$$

If $y = \langle p \rangle \in Y$ for p prime, then $K(y) = \mathbb{F}_p$ (the finite field of p elements). Then

$$X_y = \operatorname{Spec} \mathbb{F}_p \times_Y X = \operatorname{Spec} \mathbb{F}_p[t_1, t_2, t_3]/\left\langle t_1^2 + t_2^2 - 3t_3 \right\rangle$$

This indicates that we should be looking at solutions over $\mathbb{F}_p$ for various primes, which is exactly what you do in number theory.

Example 8.5. Suppose X is a scheme over a field K (i.e. we have a morphism $X \to \operatorname{Spec} K$), and $K \hookrightarrow L$ is a field extension. We get a diagram

$$\begin{tikzcd} & X \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

We can consider the product $\operatorname{Spec} L \times_{\operatorname{Spec} K} X$ which is a scheme over L.

For example, if $X = \mathbb{A}^n_k = \operatorname{Spec} K[t_1, \cdots, t_n]$ then the product is

$$\mathbb{A}^n_L = \operatorname{Spec} L[t_1, \ldots, t_n] = \operatorname{Spec}(L \otimes_K K[t_1, \cdots, t_n])$$

Example 8.6. Let $X = \operatorname{Spec} \mathbb{C}[t_1, \cdots, t_3]/\left\langle t_2t_3 - t_1^2 \right\rangle$. The homomorphism $\mathbb{C}[u] \to \mathbb{C}[t_1, t_2, t_3]/\left\langle t_2t_3 - t_1^2 \right\rangle$ sending $u \mapsto t_3$ induces a dual morphism of schemes $X \to \operatorname{Spec} \mathbb{C}[u] = \mathbb{A}^1_{\mathbb{C}}$. (Geometrically, $(a_1, a_2, a_3) \mapsto a_3 \ldots$ but not all the points are of that form (only the classical ones).)

Let $y = \langle u - a \rangle$ for $a \in \mathbb{C}$. Then

$$K(y) = (\mathcal{O}_Y)_y/\mathfrak{m} = K[u]_{\langle u-a \rangle}/\left\langle u - a \right\rangle_{\langle u-a \rangle} \cong K[u]/\langle u - a \rangle \cong \mathbb{C}$$

Then the fiber is

$$X_y = \operatorname{Spec} K(y) \times_Y X = \operatorname{Spec}\left(\mathbb{C}[u]/\langle u - a \rangle \otimes_{K[u]} \mathbb{C}[t_1, t_2, t_3]/\left\langle t_2t_3 - t_1^2 \right\rangle\right).$$

Fact 8.7.

$$A/I \otimes_A M \cong M/IM$$

So X_y above can be rewritten as $\operatorname{Spec}[t_1, t_2]/\left\langle at_2 - t_1^2 \right\rangle$. In particular,

$$X_{y=\langle u \rangle} = \operatorname{Spec} \mathbb{C}[t_1, t_2]/\left\langle t_1^2 \right\rangle$$

which is not a reduced scheme (it's a double line).

Example 8.8. Suppose $X \xrightarrow{f} Y$ is a morphism where Y is an integral scheme. You can pick one fiber that reflects the properties of all the other fibers: the fibre over the generic point η_y is called the *generic fiber*.

Definition 8.9. Y is a scheme. Define the n-projective space over Y as $\mathbb{P}^n_Y = Y \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$. (Recall that there are natural morphisms from any scheme to $\operatorname{Spec} \mathbb{Z}$, in particular a morphism $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$.)

A morphism $X \xrightarrow{f} Y$ is called *projective* if it factors as

$$\begin{tikzcd} X \arrow[r, "\exists g"] \arrow[dr] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

where g is a closed immersion. This is the analogue of proper morphisms in topology (inverse image of a compact set is compact).

A morphism $Z \xrightarrow{h} Y$ is quasiprojective if it factors as

$$\begin{tikzcd} Z \arrow[r, "\exists e"] \arrow[dr] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

if e is an open immersion and f is a projection.

EXAMPLE 8.10. $\mathbb{P}^n_Y \to Y$ is projective for obvious reasons.

If $X \to \mathbb{P}^n_Y$ is a closed immersion, then composing this morphism with $\mathbb{P}^n_Y \to Y$ gives a projective morphism.

If $Y = \operatorname{Spec} R$, and $I \triangleleft R[t_0, \cdots, t_n]$ is a homogeneous ideal, then the homomorphism

$$S := R[t_0, \cdots, t_n] \to R[t_0, \cdots, t_n]/I$$

induces a morphism on schemes

$$Proj(S/I) \to Proj(S) = \mathbb{P}^n_R$$

which is a closed immersion.

(But, not every morphism of graded rings gives a closed immersion of schemes.)

LECTURE 9: OCTOBER 24

DEFINITION 9.1. Suppose $(X, \mathcal{O}_X)$ is a ringed space. An *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ on X such that each $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and the restriction maps are compatible with the module structure: if $s \in \mathcal{F}(U)$ and $a \in \mathcal{O}_X(U)$ then $(as)|_V = a|_V \cdot s|_V$.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves that is compatible with the module structure: i.e. $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

The kernel and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module.

If $\mathcal{F}_i$ is a family of $\mathcal{O}_X$-modules, then $\bigoplus \mathcal{F}_i$ is also an $\mathcal{O}_X$-module if the sum is finite, or if X is Noetherian in the infinite case. (Recall the direct sum of sheaves is defined as $(\bigoplus \mathcal{F}_i)(U) = \bigoplus \mathcal{F}_i(U)$.)

If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, then we define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, and $\mathcal{F}$ an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module (we had to use the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$).

DEFINITION 9.2. Let $X = \operatorname{Spec} A$ and M an A-module. We define $\widetilde{M}$ as follows:

$$\widetilde{M}(U) = \{s : U \to \bigsqcup_{p \in U} M_p \,:\, (*)\}$$

where the conditions (*) are:

(1) For all p, $s(p) \in M_p$
(2) For all p, there is some neighborhood $W \subset U$ containing p, and $m \in M$ and $a \in A$ such that $s(q) = \frac{m}{a}$ for all $q \in W$.

By taking restriction maps to be restriction of functions, $\widetilde{M}$ is a sheaf. Note that if $M = A$, then $\widetilde{M} = \mathcal{O}_X$.

THEOREM 9.3. *If $X = \operatorname{Spec} A$ and M is an A-module, then:*

(1) *$\widetilde{M}$ is an $\mathcal{O}_X$-module*
(2) $(\widetilde{M})_P \cong M_P$
(3) $\widetilde{M}(D(a)) = M_a$
(4) $\widetilde{M}(X) = M$

Proof. (1) Trivial from the definition.

(2), (3) and (4): Almost identical to the case where $M = A$. ○

REMARK 9.4. Let $X = \operatorname{Spec} A$. Any homomorphism $M \to N$ of A-modules gives a morphism $\widetilde{M} \to \widetilde{N}$ of $\mathcal{O}_X$-modules. If

$$0 \to M \to N \to L \to 0 \tag{9.1}$$

is a complex of A-modules, then we get a complex of sheaves

$$0 \to \widetilde{M} \to \widetilde{N} \to \widetilde{L} \to 0 \tag{9.2}$$

of $\mathcal{O}_X$-modules. Then (9.1) is an exact sequence iff (9.2) is an exact sequence: (9.1) is exact iff

$$0 \to M_P \to N_P \to L_P \to 0$$

are exact for all $P \in X$, iff

$$0 \to (\widetilde{M})_P \to (\widetilde{N})_P \to (\widetilde{L})_P \to 0$$

is exact for all $P \in X$, iff (9.2) is exact.

The same holds for longer complexes.

DEFINITION 9.5. Suppose $f : X \to Y$ is a continuous map of topological spaces, and assume $\mathcal{G}$ is a sheaf on Y. We define the *inverse image* $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \to \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, and $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then the previous inverse image is not an $\mathcal{O}_X$-module (but – see below – it is an $f^{-1}\mathcal{O}_Y$-module). So define the inverse image $f^*\mathcal{G}$ by $f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

We used the fact:

Claim 9.6. *$f^{-1}\mathcal{G}$ has a natural module structure over $f^{-1}\mathcal{O}_Y$.*

Proof. From the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ we get a morphism $f^{-1}\mathcal{O}_Y \to f^{-1}f_*\mathcal{O}_X$ and we also have a natural morphism $f^{-1}f_*\mathcal{O}_X \to \mathcal{O}_X$. So we get a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. ○

Theorem 9.7. *Suppose $\alpha : A \to B$ is a homomorphism of rings, $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. So, we have a morphism $X \xrightarrow{f} Y$. Then,*

1. *If M, N are A-modules, then $(\widetilde{M \otimes_A N}) = \widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N}$.*
2. *If M, N are A-modules, then $(\widetilde{M \oplus N}) \cong \widetilde{M} \oplus \widetilde{N}$*
3. *If L is a B-module, then $f_*\widetilde{L} \cong \widetilde{{}_AL}$ where ${}_AL$ is just L considered as an A-module.*
4. *If M is an A-module, then $f^*\widetilde{M} \cong \widetilde{M \otimes_A B}$.*

Proof. Define a map

$$P = \widetilde{M}(U) \otimes_{\mathcal{O}_Y(U)} \widetilde{N}(U) \to (\widetilde{M \otimes_A N})(U) \text{ where } s \otimes t \mapsto r = (p \mapsto s(p) \otimes t(p))$$

If s (respectively t) is locally given by $\frac{m}{a}$ (respectively $\frac{n}{b}$) then r is given by $\frac{m \otimes n}{ab}$. We get a morphism of presheaves $\varphi : P \to \widetilde{M \otimes_A N}$ from which we get a morphism of sheaves $\varphi^+ : P^+ \to \widetilde{M \otimes_A N}$ (P^+ means sheafification). Note $P^+ = \widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N}$.

It is enough to show that φ^+ is an isomorphism. On the stalks, we have $\varphi^+_P : P^+_P = P_P = \lim_{P \in U} \widetilde{M}(U) \otimes_{\mathcal{O}_Y(U)} \widetilde{N}(U) \to (\widetilde{M \otimes_A N})_P \cong (M \otimes_A N)_P \cong M_P \otimes_{A_P} N_P$. But this limit is isomorphic to $M_P \otimes_{A_P} N_P$.

So φ^+_P is an isomorphism, and hence φ^+ is an isomorphism.

(2) More or less the same as (1) only easier.

(3) We will define a morphism $\varphi : {}_A\widetilde{L} \to f_*\widetilde{L}$. For each open $U \subset Y$ we define ${}_A\widetilde{L}(U) \to (f_*\widetilde{L})(U) \cong \widetilde{L}(f^{-1}U)$ as follows: $s \in {}_A\widetilde{L}(U)$ is a map $s : U \to \coprod_{Q \in U} {}_AL_Q$, and we want to get a map $t : f^{-1}U \to \coprod_{P \in f^{-1}U} L_P$. If $f(P) = Q$ then we have a homomorphism ${}_AL_Q \xrightarrow{\beta_P} L_P$. So send $s \mapsto t(P) = \beta_P(s(Q))$, where $Q = f(P)$.

If s is locally given by $\frac{\ell}{a}$ then t is locally given by $\frac{\ell}{\alpha(a)}$. Recall, $f^{-1}(D(a)) = D(\alpha(a))$. Now we have a homomorphism

$$_A\widetilde{L}(D(a)) \to (f_*\widetilde{L})(D(a)) = \widetilde{L}(D(\alpha(a))) = L_{\alpha(a)}$$

and the goal is to prove it's an isomorphism. L is a module over B; $_AL$ is a module over A, but only via this homomorphism α. So $_AL_a \cong L_{\alpha(a)}$ just by going through the definition of how you define the localisation. φ_Q is an isomorphism for all $Q \in Y$ because for every open set U containing Q, there exists a such that $a \in D(a) \subset U$. (Stalks can be calculated using only open sets of the form $D(a)$.)

(4) Exercise (not examinable).

$\bigcirc$

Lecture 10: October 26

Definition 10.1. Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$ module. We say that $\mathcal{F}$ is *quasicoherent* if, for all $x \in X$, there is an open affine neighborhood $U = \operatorname{Spec} A \subset X$ (containing x), and an A-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. We say that $\mathcal{F}$ is coherent if we can choose M to be finitely-generated as an A-module.

Example 10.2. Suppose X is a scheme. Then $\mathcal{O}_X$ is coherent because, for all $x \in X$, there is some open affine $x \in U = \operatorname{Spec} A \subset X$ and $\mathcal{O}_X|_U = \widetilde{A}$.

Example 10.3. Let A be a DVR *(so there are only two primes, 0 and $\mathfrak{m}$, and one nontrivial closed set $\mathfrak{m}$)*, and $\mathcal{F}$ the $\mathcal{O}_X$-module defined as:

$$\begin{cases} \mathcal{F}(X) & = 0 \\ \mathcal{F}(U) & = K = Frac(A) \end{cases}$$

where $U = X - \mathfrak{m}$ if $\mathfrak{m}$ is the maximal ideal of A. *Since the only open cover is X itself, there's nothing to check for the sheaf condition.*

Then $\mathcal{F}$ is not quasicoherent: otherwise, there would be some open affine $V \subset X$ around $\mathfrak{m}$ such that $\mathcal{F}|_V$ is the sheaf of some module. But then the generic point also belongs to V, so $V = X$. Thus, there must be some A-module M such that $\mathcal{F} = \widetilde{M}$. However, $0 = \mathcal{F}(X) = M$, so $M = 0$, a contradiction.

Lemma 10.4. *Let $X = \operatorname{Spec} A$, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $M = \mathcal{F}(X)$. Then, there is a natural morphism $\widetilde{M} \to \mathcal{F}$.*

Proof. For any $b \in A$, we define a homomorphism $M_b = \widetilde{M}(D(b)) \to \mathcal{F}(D(b))$ by $\frac{m}{b^r} \mapsto \frac{1}{b^r} \cdot m|_{D(b)}$. Since each open $U \subset X$ is covered by open sets of the form $D(b)$, the above maps determine a morphism $\widetilde{M} \to \mathcal{F}$. $\bigcirc$

Corollary 10.5. *Let $X = \operatorname{Spec} A$, and M be an A-module. Then, $\widetilde{M}|_{D(b)} \cong \widetilde{M_b}$. (Moral: Quasicoherent sheaves are covered by arbitrarily small quasicoherent pieces.)*

Proof. By the lemma, there is a morphism $\widetilde{M_b} \to \widetilde{M}|_{D(b)}$. Now $\psi_p : (\widetilde{M_b})_p \to (\widetilde{M}|_{D(b)})_p$ is an isomorphism for each $p \in D(b)$ *(as both these rings are just M_p)*, so ψ is an isomorphism. ○

Definition 10.6. We say that a scheme X is *Noetherian* if we can cover X by finitely many open affine sets $U_1, \cdots, U_r$ such that $U_i = \operatorname{Spec} A_i$ where A_i is Noetherian.

Exercise 10.7. If X is Noetherian then for every open affine $U = \operatorname{Spec} A \subset X$, A is Noetherian.

Theorem 10.8. *Suppose X is a scheme. If $\mathcal{F}$ is quasicoherent, then for every open affine $U = \operatorname{Spec} A \subset X$, we have $\mathcal{F}|_U = \widetilde{M}$ for some A-module M.*

If $\mathcal{F}$ is coherent, then for every open affine $U = \operatorname{Spec} A \subset X$, we have $\mathcal{F}|_U = \widetilde{M}$ for some finitely-generated A-module M.

(In other words, it doesn't matter what kind of open affine covering you take.)

Proof. Pick an open affine $U = \operatorname{Spec} A \subset X$. For every point $x \in U$, there is some open affine subscheme $V = \operatorname{Spec} B$ such that $\mathcal{F}|_V \cong \widetilde{N}$ for some B-module N. The problem is that maybe $V \not\subset U$. But by Corollary 10.5, we can replace V with some $\operatorname{Spec} B_b$. Thus, start by assuming $V \subset U$. *The $D(b)$ form a basis for the topology, so you can find some $D(b) \subset U \cap V$. See Problem #15 of Example sheet 1.* Replace X by U and hence assume $X = \operatorname{Spec} A$.

Arguing similarly to the above, we can assume there exist $b_1, \cdots, b_r$ such that $X = \bigcup D(b_i)$ and such that $\mathcal{F}|_{D(b_i)} = \widetilde{M_i}$ for some A_{b_i}-module. Let $f_i : D(b_i) \to X$ be the inclusion map. Then $D(b_i) \cap D(b_j) = D(b_i b_j)$ and there is an inclusion, say f_{ij} into X. We have an exact sequence

$$0 \to \mathcal{F} \to \underbrace{\bigoplus_i (f_i)_* \mathcal{F}|_{D(b_i)}}_{\mathcal{G}} \to \underbrace{\bigoplus_{i,j} (f_{ij})_* \mathcal{F}|_{D(b_i b_j)}}_{\mathcal{N}}$$

For any open set U, we send $s \in \mathcal{F}(U)$ to $(s|_{D(b_i)\cap U})$, and we send $(t_i) \in \mathcal{G}(U)$ to

$$t_i|_{D(b_i b_j)\cap U} - t_j|_{D(b_i b_j)\cap U}$$

This sequence is exact because of the sheaf property of $\mathcal{F}$: if $t_i|_{D(b_i b_j)\cap U} = t_j|_{D(b_i b_j)\cap U}$ for all i, j then the t_i's are a consistent collection of sections on a covering $D(b_i) \cap U$ of U, and therefore come from a global section on U.

By Theorem 9.7(2) and (3), the last two sheaves in the sequence are given by

$$\mathcal{G} = \widetilde{\bigoplus{}_A M_i} \text{ for some } A_{b_i}\text{-module } M_i$$

$$\mathcal{N} = \widetilde{\bigoplus{}_A M_{ij}} \text{ for some } A_{b_i b_j}\text{-module } M_{ij}.$$

We have an exact sequence

$$0 = \mathcal{F}(X) \to \bigoplus{}_A M_i \to \bigoplus{}_A M_{ij}$$

which suggests defining $M := \mathcal{F}(X)$. Then we get a sequence

$$0 \to \widetilde{M} \to \widetilde{\bigoplus{}_A M_i} = \mathcal{G} \to \widetilde{\bigoplus{}_A M_{ij}} = \mathcal{N}$$

Since the kernel of a morphism is unique, $\mathcal{F} = \widetilde{M}$.

Note that taking sections is an exact functor because of the sheaf property.

The Noetherian case follows by the same argument. ○

THEOREM 10.9. *Suppose X is a scheme, $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then $\ker(\varphi)$ and $\operatorname{im}(\varphi)$ are again quasicoherent.*

If X is Noetherian, the same is true for coherent sheaves.

Proof. This is a local statement, so we can replace X by any open affine subscheme Spec A. $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent so say $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ for some A-modules M and N. The morphism $\varphi : \mathcal{F} \to \mathcal{G}$ determines a homomorphism of modules $M \xrightarrow{\alpha} N$. Let $K = \ker(\alpha)$. Then we get an exact sequence

$$0 \to K \to M \xrightarrow{\alpha} N$$

from which we get an exact sequence of sheaves

$$0 \to \widetilde{K} \to \widetilde{M} \xrightarrow{\varphi} \widetilde{N}.$$

(You have to check that you get the same morphism φ out: see problem on second example sheet).

So $\ker(\varphi) = \widetilde{K}$ is quasicoherent. For $\operatorname{im}(\varphi)$, apply a similar argument.

The same proof works in the Noetherian case. ○

LECTURE 11: OCTOBER 29

THEOREM 11.1. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module, $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then,*

1. *If $\mathcal{G}$ is quasicoherent, then $f^*\mathcal{G}$ is also quasicoherent.*
2. *If $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.*
3. *If $\mathcal{F}$ is quasicoherent, then $f_*\mathcal{F}$ is quasicoherent if:*
   - *For all $y \in Y$, there is some open affine $W \subset Y$ around y such that $f^{-1}W$ is a finite union $\bigcup U_i$ of open affines, and*
   - *$U_i \cap U_j$ is a finite union $\bigcup U_{ij}$ of open affines.*

(These conditions are satisfied when X is Noetherian.)

Proof. (1) We can assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Then $\mathcal{G} = \widetilde{M}$, and so $f^*\mathcal{G} = \widetilde{M \otimes_A B}$ by Theorem 9.7(4).

(2) Same as (1): we can take M to be finitely generated, which makes the tensor product also finitely generated.

(3) This is a local property, so we can assume that Y is affine. Let $f_i : U_i \to X$ and $f_{ij} : U_{ijk} \to X$ be the inclusions. We have an exact sequence

$$0 \to \mathcal{F} \to \bigoplus (f_i)_*(\mathcal{F}|_{U_i}) \to \bigoplus (f_{ij})_*(\mathcal{F}|_{U_{ijk}})$$

where the first nontrivial map is just restriction $s \mapsto (s|_{U_i})$ and the second map is

$$(s_i) \mapsto s_i|_{U_{ijk}} - s_j|_{U_{ijk}}.$$

So, we get an exact sequence

$$0 \to f_*\mathcal{F} \to \underbrace{\bigoplus f_*(f_i)_*\mathcal{F}|_{U_i}}_{\mathcal{L}} \to \underbrace{\bigoplus f_*(f_{ijk})_*\mathcal{F}|_{U_{ijk}}}_{\mathcal{M}}$$

Now $\mathcal{L}$ is quasicoherent and $\mathcal{M}$ is quasicoherent. Since $f_*\mathcal{F} = \ker(\mathcal{L} \to \mathcal{M})$, $f_*\mathcal{F}$ is quasicoherent by Theorem 10.9. ○

Example 11.2. Find an example of $f : X \to Y$, $\mathcal{F}$ coherent, X and Y Noetherian, such that $f_*\mathcal{F}$ is not coherent. (Remember: if f is projective, then it is known that $f_*\mathcal{F}$ is coherent.)

Definition 11.3. An *ideal sheaf* $\mathcal{I}$ on a scheme X is an $\mathcal{O}_X$-module such that $\mathcal{I} \subset \mathcal{O}_X$ (i.e. $\mathcal{I}(U) \subset \mathcal{O}_X(U)$, i.e. there is an injective morphism $\mathcal{I} \to \mathcal{O}_X$).

Theorem 11.4. *Suppose X is a scheme. Then there is a 1-1 correspondence between the following sets:*

$$\{ \text{ closed subschemes of } X \text{ } \} \iff \{ \text{ quasicoherent ideal sheaves } \}$$

If X is Noetherian you can replace quasicoherent with coherent.

Proof. Suppose that Y is a closed subscheme, and $f : Y \to X$ is its closed immersion. Remember we have a morphism $\varphi : \mathcal{O}_X \to f_*\mathcal{O}_Y$ which is surjective. Now take $\mathcal{I} = \ker \varphi$. This is clearly an ideal sheaf, but we have to show that it's quasicoherent. It is enough to show that $f_*\mathcal{O}_Y$ is quasicoherent. This is a local property, so we can assume $X = \operatorname{Spec} A$ is affine.

We show that f satisfies the conditions in the previous theorem (beware the notation is not the same). Take an open affine cover $Y = \bigcup U_i$ and let W_i be an open set in X such that $U_i = f^{-1}W_i$ (because the topology is a restriction of the topology of X). We can find a family of $b_\alpha \in A$ such that $D(b_\alpha) \subset W_i$ for some i, or $D(b_\alpha) \subset X \backslash Y$ and such that $\{D(b_\alpha)\}$ gives a covering of the W_i.

So $X = \bigcup D(b_\alpha)$, and $\langle \{b_\alpha\} \rangle = A$ – *otherwise, $\langle \{b_\alpha\} \rangle$ is contained in some maximal ideal, which is prime, and is in no $D(b_\alpha)$*. But when you have an ideal generating the ring, you only need finitely many generators *because 1 is some finite linear combination of generators $b_1, \cdots, b_r$, which therefore generate all of A.* So we can assume $X = D(b_1) \cup \cdots \cup D(b_r)$. For all α with $1 \leq \alpha \leq r$, $f^{-1}D(b_\alpha)$ is an open affine subscheme of some U_i, hence of Y_i. (If $g : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism induced by $h : A \to B$, then $g^{-1}D(a) = D(h(a))$.) Now, $Y = f^{-1}D(b_1) \cup \cdots \cup f^{-1}D(b_r)$.

Also note that $D(b_\alpha) \cap D(b_\beta) = D(b_\alpha \cdot b_\beta)$, which is affine, hence $f^{-1}D(b_\alpha) \cap f^{-1}(b_\beta)$ is affine for all α, β. So this condition is easy to satisfy, and you don't even need to take unions. Then, by the previous theorem, $f_*\mathcal{O}_Y$ is quasicoherent, so $\mathcal{I}$ is quasicoherent.

Conversely, assume that we are given a quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $X = \operatorname{Spec} A$ then $\mathcal{I}$ is given by an ideal (it's a module, but submodules of rings are ideals.) We construct the corresponding closed subscheme Y locally. First we define Y topologically. For each open affine $U = \operatorname{Spec} A \subset X$, define $Y_U = V(I) \subset U$, and $\mathcal{I}|_U = \widetilde{I}$. Then Y is given by the union of all the Y_U. But we need to verify for each $U = \operatorname{Spec} A$ and $U' = \operatorname{Spec} A'$, that $Y_U \cap U' = Y_{U'} \cap U$. Everything is local, so we can work with the corresponding rings.

Pick $p \in Y_U \cap U'$, so $p \in U'$. There is some $b' \in A'$ such that $p \in D(b') \subset U$ (i.e. you're surrounding p by a little affine open that sits in the intersection). We also have homomorphisms

$$A \to A'_{b'} \leftarrow A'.$$

We have an ideal $I' \subset A'$ such that $\mathcal{I}|_{U'} = \widetilde{I'}$, and $I \subset A$ similarly. So $\mathcal{I}|_{D(b')} = \widetilde{I'_{b'}}$, which corresponds to the ideal $I'_{b'} \subset A'_{b'}$. Think about this and see that $I \cdot A'_{b'} = I'_{b'}$. Abuse notation so p refers to the image of the prime in any of the above rings. $I \subset p \implies I'_{b'} \subset p \implies I' \subset p$. This last statement is exactly saying $p \in Y_{U'} \cap U$.

So we have shown that $Y_U \cap U' \subset Y_{U'} \cap U$. The same argument does the reverse direction. So, Y is well-defined.

Denote by $\mathcal{G}$ the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)/\mathcal{I}(U)$. Then we have an exact sequence

$$\mathcal{I} \to \mathcal{O}_X \to \mathcal{G} \to 0$$

(i.e. $\mathcal{G}$ is the quotient of sheaves). Now note that $\mathcal{G}|_{X \setminus Y} = 0$. This implies that $\mathcal{G} = f_*\mathcal{O}_Y$ for some sheaf $\mathcal{O}_Y$ on Y.

We can more directly define $\mathcal{O}_Y$ as $\mathcal{O}_Y(W) = \mathcal{G}(U)$ for any open U such that $U \cap Y = W$. This is well-defined, exactly because $\mathcal{G}|_{X \setminus Y} = 0$. Rewrite the exact sequence as

$$0 \to S \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

Taking $\mathcal{O}_Y$ as the structure sheaf on Y, $f : Y \to X$ is a closed immersion. In particular, Y is a closed subscheme.

○

Corollary 11.5. *If $X = \operatorname{Spec} A$, then we have a 1-1 correspondence*

$$\{ \text{ closed subschemes of } X \} \iff \{ \text{ ideals of } A \}.$$

Proof. Just apply the previous theorem. ○

Lecture 12: October 31

Suppose $S = \bigoplus_{d \geq 0} S_d$ is a graded ring, and $M = \bigoplus_{\ell \in Z} M_\ell$ a graded S-module (an S-module such that $S_d \cdot M_\ell \subset M_{d+\ell}$ – note that negative degrees exist). Define $\widetilde{M}$ on $X = \operatorname{Proj} S$ as

$$\widetilde{M}(U) = \{s : U \to \bigsqcup_{p \in U} M_{(p)} \, : \, (*)\}$$

where the conditions (*) are, as before,

- For every p, $s(p) \in M_{(p)}$.
- For every p, there is some neighborhood $W \subset U$ of p and some $m \in M$, $a \in S$ that are homogeneous of the same degree, such that $s(q) = \frac{m}{a} \in M_{(q)}$ for all $q \in W$.

Then $\widetilde{M}$ is an $\mathcal{O}_X$-module. In particular, $\widetilde{S} = \mathcal{O}_X$.

Theorem 12.1. *We have the following properties:*

1. $\widetilde{M}_P \cong M_{(P)}$
2. *For all $b \in S$ homogeneous of positive degree, then $\widetilde{M}|_{D_+(b)} \cong \widetilde{M_{(b)}}$.*
3. $\widetilde{M}$ *is quasicoherent.*

Proof. (1) and (2) are the same as the case $M = S$.

(3) follows from (2), because the open sets $D_+(b)$ form a cover of X. ○

Definition 12.2 ($M(n); \mathcal{O}_X(n); \mathcal{F}(n)$). Let S be a graded ring. If M is a graded S-module, define $M(n)$ to be the module whose degree-ℓ elements are $M_{n+\ell}$.

We define $\mathcal{O}_X(n) = \widetilde{S(n)}$.

We define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

It is easy to see that $M(n) = M \otimes_S S(n)$.

Definition 12.3. An $\mathcal{O}_X$-module $\mathcal{L}$ is *invertible* if for all $p \in X$, there is some open $U \ni p$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. (This is the analogue of line bundles.)

Theorem 12.4. *Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring which is generated by S_1 as an S_0-algebra (for example, a polynomial ring). Then,*

(1) *$\mathcal{O}_X(n)$ are invertible sheaves;*
(2) *If M, N are S-modules, then $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_S N}$.*
(3) *If M is an S-module, then $\widetilde{M}(n) \cong \widetilde{M(n)}$ and*
$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$$
.
(4) *If $\alpha : S \to T$ is a surjective graded homomorphism (one that preserves degrees) of graded rings, and $f : Y = \operatorname{Proj} T \to X = \operatorname{Proj} S$ is the corresponding closed immersion, then*
   - $f^*\widetilde{L} = \widetilde{L \otimes_S T}$
   - $f_*\widetilde{L} = \widetilde{{}_S L}$
   - $f^*\mathcal{O}_X(n) = \mathcal{O}_Y(n)$
   - $f_*\mathcal{O}_Y(n) = f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

Proof. (1) If $b \in S_1$ then
$$\mathcal{O}_X(n)|_{D_+(b)} \cong \widetilde{S(n)}|_{D_+(b)} \cong \widetilde{S(n)_{(b)}}$$
But we have an isomorphism $S(n)_{(b)} \to S_{(b)}$, that sends $\frac{a}{b^r} \mapsto \frac{a}{b^{r+n}}$ (because a and b^r have the same degree). Check using commutative algebra that this is an isomorphism (using the fact that $\deg b = 1$). So,
$$\mathcal{O}_X(n)|_{D_+(b)} \cong \widetilde{S|_{(b)}} = \mathcal{O}_{D_+(b)}$$
Since S is generated by elements of S, such $D_+(b)$ cover all of X.

(2) We define a homomorphism
$$\widetilde{M}(D_+(b)) \otimes_{\mathcal{O}_X(D_+(b))} \widetilde{N}(D_+(b)) \to \widetilde{M \otimes_S N}(D_+(b))$$
where $b \in S_1$. This is the same as a homomorphism
$$M_{(b)} \otimes_{S_{(b)}} N_{(b)} \to (M \otimes_S N)_{(b)}$$
Define this by sending
$$\frac{m}{b^r} \otimes \frac{n}{b^{r'}} \mapsto \frac{m \otimes n}{b^{r+r'}}$$
Since such $D_+(b)$ give a covering of X, the above homomorphism determines a homomorphism
$$\widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U) \to \widetilde{M \otimes_S N}(U)$$
for all open $U \subset X$. You have to show that this is compatible on restrictions. The homomorphism just defined (for $D_+(b)$) is an isomorphism of $S_{(b)}$-modules. This means that the morphism
$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}$$
determined by the above homomorphism is an isomorphism.

(3) Apply (2):

$$\widetilde{M}(n) := \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \cong M \widetilde{\otimes_S} S(n) \cong \widetilde{M(n)}$$

Moreover,

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{S(m)} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \cong S(m) \widetilde{\otimes_S} S(n) \cong S(\widetilde{m+n}) \cong \mathcal{O}_X(m+n)$$

where the penultimate isomorphism comes from the map $S(m) \otimes_S S(n) \to S(m+n)$ given by $a \otimes a' \mapsto a \cdot a'$.

(4) We define a homomorphism

$$(f^*\widetilde{L})(D_+(c)) \to L \widetilde{\otimes_S} T(D_+(c))$$

($c \in T, \deg c = 1$. Also take $b \in S_1$, with $b \to c$.) That is,

$$(f^*\widetilde{L})(D_+(c)) = (f^*\widetilde{L})|_{D_+(c)}(D_+(c)) = (f^*(\widetilde{L|_{D_+(b)}}))(D_+(c))$$

noting that $f^{-1}D_+(b) = D_+(c)$, and that we have a map $D_+(c) \xrightarrow{f} D_+(b)$ induced from $f : Y \to X$.

$$\cdots \cong L_{(b)} \widetilde{\otimes_{S(b)}} T_{(c)}(D_+(c)) \cong L_{(b)} \otimes_{S(b)} T_{(c)}$$

and

$$(L \widetilde{\otimes_S} T)(D_+(c)) \cong (L \otimes_S T)_{(c)}$$

and we define

$$L_{(b)} \otimes_{S(b)} T(c) \to (L \otimes_S T)_{(c)}$$

by sending

$$\frac{\ell}{b^r} \otimes \frac{t}{c^{r'}} \mapsto \frac{\ell \otimes t}{c^{r+r'}}$$

which is an isomorphism. Gluing together all such homomorphisms determines an isomorphism $f^*\widetilde{L} \cong L \widetilde{\otimes_S} T$.

Similarly, prove $f_*\widetilde{K} \cong \widetilde{{}_SK}$.

Now,

$$f^*\mathcal{O}_X(n) = f^*\widetilde{S(n)} \cong S(n) \widetilde{\otimes_S} T \cong \widetilde{T(n)} \cong \mathcal{O}_Y(n)$$

Similarly,

$$f_*\mathcal{O}_Y(n) = f_*\widetilde{T(n)} = \widetilde{{}_ST(n)} \cong {}_ST \widetilde{\otimes_S} S(n) \cong f_*\mathcal{O}_Y \otimes \mathcal{O}_X(n)$$

○

Definition 12.5. Let $X = \operatorname{Proj} S$, $\mathcal{F}$ an $\mathcal{O}_X$-module. Define $\Phi_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)(X)$ which is an S-module via the maps

$$\mathcal{O}_X(d)(X) \otimes \mathcal{F}(n)(X) \to \mathcal{F}(d+n)(X)$$

$$S_d \to \mathcal{O}_X(d)(X)$$

REMARK 12.6. Suppose S is a graded ring generated as an S_0-algebra by finitely many elements of degree 1. (The most important example is the polynomial ring $S_0[t_0, \cdots, t_n]$, or a quotient of it.)

(1) If $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, then $\mathcal{F} \cong \widetilde{\Gamma_*(\mathcal{F})}$. (But there could be many different modules that define the same sheaf.)
(2) If $S = S_0[t_0, \cdots, t_n]$, and $\mathcal{I}$ a quasicoherent ideal sheaf, then one can show that
$$I := \Gamma_*(\mathcal{I}) \subset \Gamma_*(\mathcal{O}_X) = S$$
Then $\mathcal{I} = \widetilde{I}$. Every closed subscheme is defined by some homogeneous ideal.
(3) If S_0 is Noetherian, then for each coherent sheaf $\mathcal{F}$, there is a morphism
$$\bigoplus_{finite} \mathcal{O}_X \twoheadrightarrow \mathcal{F}(n)$$
for some n. Equivalently,
$$\bigoplus_{finite} \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}$$

LECTURE 13: NOVEMBER 2

DEFINITION 13.1. Suppose $(X, \mathcal{O}_X)$ is a ringed space, $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally free of rank* n if, for every $x \in X$, there is some neighborhood $U \subset X$ of x such that
$$\mathcal{F}|_U \cong \underbrace{\mathcal{O}_U \oplus \cdots \oplus \mathcal{O}_U}_{n \text{ times}}$$
These are the analogues of vector bundles in differential geometry.

If $\mathcal{F}$ is locally free of rank $n = 1$, we call $\mathcal{F}$ invertible. These are the analogues of line bundles.

We define $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ as
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \left\{ \varphi : \mathcal{F}|_U \to \mathcal{G}|_U \; : \; \begin{array}{c} \varphi \text{ is a morphism} \\ \text{of } \mathcal{O}_X\text{-modules} \end{array} \right\}$$

DEFINITION/ THEOREM 13.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Then, the set of invertible sheaves (up to isomorphism) forms an abelian group called the *Picard group* of X and denoted by $\mathrm{Pic}(X)$.

Proof. Defining the group operation: If $\mathcal{L}$, $\mathcal{M}$ are invertible, define the product to be $\mathcal{L} \otimes \mathcal{M}$. Note that, for all $x \in X$, there is some neighborhood U of x such that $\mathcal{L}|_U \cong \mathcal{O}_U \cong \mathcal{M}|_U$. Thus
$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}|_U \cong \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{M}|_U \cong \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U \cong \mathcal{O}_U$$
So the tensor product is an invertible sheaf as well. The identity element is just $\mathcal{O}_X$: for any $\mathcal{O}_X$-module $\mathcal{F}$, we have $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$.

Defining inverses: For any invertible sheaf $\mathcal{L}$, define $\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. You should think of this as the dual of a vector space. First define a homomorphism

$$\mathcal{L}^{-1}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}(U) \to \mathcal{O}_X(U)$$

for each U, to be the evaluation map: $\varphi \otimes s \mapsto \varphi_U(s)$. This gives a morphism $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\Lambda} \mathcal{O}_X$. Now Λ is an isomorphism as we can check locally: for all $x \in X$, there is some $W \ni x$ such that $\mathcal{L}|_W \cong \mathcal{O}_W$; then

$$\mathcal{L}^{-1}|_W \otimes_{\mathcal{O}_W} \mathcal{L}|_W \xrightarrow{\Lambda|_W} \mathcal{O}_W \text{ is the same as } \mathcal{O}_W \otimes_{\mathcal{O}_W} \mathcal{O}_W \cong \mathcal{O}_W \to \mathcal{O}_W$$

that is an isomorphism.

Finally, it is easy to check that if $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are invertible, then $\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \cong (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{N}$ and $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \otimes \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$. ○

Example: remember that $\mathcal{O}_X(n)$ from the previous lecture are invertible sheaves.

Definition 13.3 (Cartier divisors and divisor class group). Let X be an integral scheme, $\eta \in X$ is the generic point, $K = \mathcal{O}_\eta$ is the function field. (For ease of notation, instead of $(\mathcal{O}_X)_P$ we just write $\mathcal{O}_P$.)

Recall, for any open $U \subset X$, the natural map $\mathcal{O}_X(U) \to K$ is injective. (This comes from the fact that the map from an integral domain to its field of fractions is injective.) So just regard the elements of $\mathcal{O}_X(U)$ as elements of K.

A *Cartier divisor* D is a system (U_i, f_i) where $X = \bigcup U_i$, the U_i are open, and $0 \neq f_i \in K$, such that $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i} \in \mathcal{O}_X(U_i \cap U_j)$. (This is analogous to meromorphic functions in complex analysis: on a neighborhood, two of them have the same zeroes and poles.) Two systems (U_i, f_i) and (V_α, g_α) define the same Cartier divisor if for all i, α, $\frac{f_i}{g_\alpha}$ and $\frac{g_\alpha}{f_i}$ are in $\mathcal{O}_X(U_i \cap V_\alpha)$.

We say that a Cartier divisor $D = (U_i, f_i)$ is equivalent to 0 (written $D \sim 0$) if $D = (X, f)$ for some $f \in K$.

If $D = (U_i, f_i)$ and $E = (V_\alpha, g_\alpha)$, then define $D + E = (U_i \cap V_\alpha, f_i g_\alpha)$.

For a Cartier divisor $D = (U_i, f_i)$, define $-D = (U_i, \frac{1}{f_i})$.

We say $D \sim E$ if $D - E = D + (-E) \sim 0$.

Define the divisor class group to be

$$Div(X) = \text{ abelian group generated by Cartier divisors} \Big/ \sim$$

where $\sim$ is the equivalence relation defined above.

Definition/ Theorem 13.4. Let X be an integral scheme. Then,

(1) For each Cartier divisor $D = (U_i, f_i)$, we define an $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ as:
$$\mathcal{O}_X(D)(U) = \{h \in K \,:\, hf_i \in \mathcal{O}_X(U \cap U_i)\ \forall i\}$$
and $\mathcal{O}_X(D)$ is invertible. (Actually, every invertible sheaf can be constructed this way, but we won't prove this here.)
(2) If D, E are Cartier divisors, $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \cong \mathcal{O}_X(D+E)$.
(3) $\mathcal{O}_X(D)^{-1} \cong \mathcal{O}_X(-D)$.
(4) $D \cong E \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(E)$

For (3), show that $\mathcal{O}_X(-D)$ satisfies the property of being an inverse. For (4), replace D with $D - E$ and E with 0, so wlog $D \cong 0$.

Proof. (1) It is easy to see that $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-module (the restriction maps are just induced by K).

$\mathcal{O}_X(D)$ is invertible: this is a local statement, so we can replace X by some U_i and hence assume that $D = (X, f)$. Now define
$$\mathcal{O}_X(D)(U) \to \mathcal{O}_X(U) \text{ where } h \mapsto hf$$
for all open sets $U \subset X$; it is easy to see that this is an isomorphism.

(2) For each open $U \subset X$, define
$$\mathcal{O}_X(D)(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(E)(U) \to \mathcal{O}_X(D+E)(U) \text{ where } h \otimes e \mapsto he$$
If $D = (U_i, f_i)$ and $E = (V_\alpha, g_\alpha)$, then $hf_i \in \mathcal{O}_X(U_i \cap U)$ and $eg_\alpha \in \mathcal{O}_X(U \cap V_\alpha)$ so $hef_ig_\alpha \in \mathcal{O}_X(U \cap U_i \cap V_\alpha)$ for all i, α, so $he \in \mathcal{O}_X(D+E)(U)$.

So we get a morphism $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D+E)$ which we can check to be an isomorphism locally, hence globally.

(3) $\mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \cong \mathcal{O}_X(0) \cong \mathcal{O}_X$. So $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{-1}$ because inverses are unique in $Pic(X)$.

(4) The claim is the same as showing that $D \sim 0 \iff \mathcal{O}_X(D) \cong \mathcal{O}_X$. ($D \cong E \iff D - E \cong 0$, and by the above, $\mathcal{O}_X(D) \otimes \mathcal{O}_X(-E) \cong Id \iff \mathcal{O}_X(D-E) \cong Id$, so replace D with $D - E$ and E with 0).

($\implies$) $D = (X, f)$. Then define a homomorphism $\mathcal{O}_X(D)(U) \to \mathcal{O}_X(U)$ sending $h \mapsto hf$, which is an isomorphism, hence giving an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

($\impliedby$) Conversely, assume $\mathcal{O}_X(D) \cong \mathcal{O}_X$, and say $D = (U_i, f_i)$. Pick an isomorphism $\varphi : \mathcal{O}_X \to \mathcal{O}_X(D)$. Let $f \in \mathcal{O}_X(D)(X)$ be the image of $1 \in \mathcal{O}_X(X)$. Then, $\frac{f_i}{f^{-1}} = ff_i \in \mathcal{O}_X(U_i)$ for all i. On the other hand, φ gives an isomorphism $\mathcal{O}_X(U_i) \to \mathcal{O}_X(D)(U_i)$. Now $\frac{1}{f_i} \in \mathcal{O}_X(D)(U_i)$. There must be some g_i that maps to $\frac{1}{f_i}$. So $\frac{1}{f_i} = g_if$, and so $\frac{f^{-1}}{f_i} = \frac{1}{f_if} \in \mathcal{O}_X(U_i)$. That is, $D = (U_i, f_i) = (X, f^{-1})$ and so $D \sim 0$.

Lecture 14: November 5

Sheaf of differential forms.

Definition 14.1. Suppose B is an A-algebra. For each $b \in B$ let db be a symbol. Define

$$\Omega_{B/A} = \text{ Free } B\text{-module generated by all } db \Big/ N$$

where N is generated by the following elements:

- da (for all $a \in A$)
- $d(b+b') - d(b) - d(b')$ (for all $b, b' \in B$)
- $d(bb') - bd(b') - b'd(b)$ (for all $b, b' \in B$)

We call $\Omega_{B/A}$ the module of differentials of B over A. The module $\Omega_{B/A}$ satisfies the following universal property: if $\alpha : B \to M$ is any map from B to a B-module M such that $\alpha(a) = 0$ for all $a \in A$, $\alpha(b+b') = \alpha(b) + \alpha(b')$, and $\alpha(bb') = \alpha(b)b' + b\alpha(b')$, then there is a unique B-homomorphism $\Omega_{B/A} \xrightarrow{\beta} M$ such that we have a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "b \mapsto db"] \arrow[dr, "\alpha"'] & & \Omega_{B/A} \arrow[dl, "\beta"] \\ & M & \end{tikzcd}$$

Example 14.2. Suppose $B = A[t_1, \cdots, t_n]$. Then $\Omega_{B/A}$ is free, generated by $dt_1, \cdots, dt_n$. To see this, define

$$\alpha : B \to M = \underbrace{\sum_{1}^{n} B \cdot dt_i}_{\textit{free module}} \text{ where } \alpha(f) = \sum \frac{\partial f}{\partial t_i} dt_i$$

where $\frac{\partial}{\partial t_i}$ is taken as in calculus. Then $\alpha(t_i) = dt_i$. Then, by the universal property of $\Omega_{B/A}$ we have a diagram

$$\begin{tikzcd} B \arrow[rr, "t_i \mapsto dt_i"] \arrow[dr, "\alpha"'] & & \Omega_{B/A} \arrow[dl, "\beta"] \\ & M & \end{tikzcd}$$

It is easy to see from the definition of $\Omega_{B/A}$ that $\Omega_{B/A}$ is generated as a B-module by $dt_1, \cdots, dt_n$. Now β is a surjective homomorphism, and M is free, so β is an isomorphism, and $\Omega_{B/A}$ is free.

Definition 14.3. Assume $f : X \to Y$ is a morphism of schemes. If $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, then we define $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ (a sheaf on X).

In general, when X and Y are not affine, cover Y by open affine schemes U_i, and cover each $f^{-1}U_i$ by open affine schemes $V_{i,\alpha}$. Now for each i, α we have $\Omega_{V_{i,\alpha}/U_i}$ and we glue them all together to form a sheaf $\Omega_{X/Y}$ on X. (One has to check that $\Omega_{V_{i,\alpha}/U_i}$ are compatible intersections.)

We call $\Omega_{X/Y}$ the sheaf of relative differential forms of X over Y.

Example 14.4. Let $X = \mathbb{A}^n_A$, $Y = \operatorname{Spec} A$. Then $\Omega_{X/Y} = \widetilde{\Omega_{A[t_1,\cdots,t_n]/A}} = \bigoplus_1^n \mathcal{O}_X$. (This is a free sheaf.)

Example 14.5. Let $X = \mathbb{P}^n_A := \operatorname{Proj} A[t_0, \cdots, t_n]$, $Y = \operatorname{Spec} A$. Cover X by $D_+(T_0), \cdots, D_+(t_n)$. For each i, $D_+(t_i) \cong \mathbb{A}^n_A$, so $\Omega_{D_+(t_i)/Y}$ is free.

We glue all the sheaves $\Omega_{T_+(t_i)/Y}$ together to get $\Omega_{X/Y}$. This doesn't say much: do you have global sections? Who knows.

Theorem 14.6. *Let $Y = \operatorname{Spec} A$, $X = \mathbb{P}^n_Y = \mathbb{P}^n_A$. Then, we have an exact sequence*

$$0 \to \Omega_{X/Y} \to \bigoplus_1^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$$

The proof of this theorem is not examinable. Let $S = A[t_0, \cdots, t_n]$; $X = \operatorname{Proj} S$. Define

$$L = \bigoplus_1^{n+1} S(-1)$$

Note that if $e_0 = (1, 0, 0, \cdots)$, and $e_1 = (0, 1, 0, \cdots)$, etc., are of degree 1 ($e_i \in L$), then $L = \sum_0^n Se_i$. Now $\widetilde{L} = \bigoplus_1^{n+1} \mathcal{O}_X(-1)$, $\widetilde{S} = \mathcal{O}_X$. Define $\alpha : L \to S$ by $\alpha(\sum g_i e_i) = \sum g_i t_i$ (note that this preserves degrees). Put $M = \ker \alpha$. So, we have an exact sequence $0 \to M \to L \xrightarrow{\alpha} S$ Then, we get an exact sequence

$$0 \to \widetilde{M} \to \widetilde{L} = \bigoplus_1^{n+1} \mathcal{O}_X(-1) \to \widetilde{S} = \mathcal{O}_X \to 0$$

That is, $L \to S$ is not surjective, but $\widetilde{L} \to \widetilde{S}$ is; note that this can never happen in the affine case. But it's true here, because $L_{(t_i)} \to S_{(t_i)}$ is surjective for all t_i (and surjectivity is a local statement).

It is enough to show that $\Omega_{X/Y} \cong \widetilde{M}$. For each t_i, $\Omega_{X/Y}|_{D_+(t_i)}$ is free, and $\widetilde{M}|_{D_+(t_i)} \cong \widetilde{M_{(t_i)}}$. Write $B = A[\frac{t_1}{t_0}, \cdots, \frac{t_n}{t_0}]$. We will define a morphism

$$\Omega_{X/Y}|_{D_+(t_0)} \to \widetilde{M}|_{D_+(t_0)}$$

(one can define similar morphisms for each other t_i). This is the same as

$$\widetilde{Bd\frac{t_1}{t_0}} + \cdots + Bd\frac{t_n}{t_0} \to \widetilde{M_{(t_0)}}$$

and this we define by

$$Bd\frac{t_n}{t_0} + Bd\frac{t_n}{t_0} \xrightarrow{\lambda_0} M_{(t_0)} \text{ where } \sum_1^n f_i d\frac{t_i}{t_0} \mapsto \left(-\frac{t_1}{t_0^2} f_1 - \cdots - \frac{t_n}{t_0^2 f_n}\right) e_0 + \frac{f_1}{t_0} e_1 + \cdots + \frac{f_n}{t_0} e_n$$

If the RHS=0, then all the f_i's are zero; so this is surjective. I claim it's also surjective. Suppose $\frac{g_0 t_0 + \cdots + g_n e_n}{t_0^r} \in M_{(t_0)}$ (assume $\deg(g_i e_i) = r$). Since $M_{(t_0)} = \ker\left(\bigoplus_1^{n+1} S(-1)_{(t_0)}\right) \to S_{(t_0)}$,

$$\frac{g_0 t_0 + \cdots + g_n t_n}{t_0^r} \in S_{(t_0)}.$$

Now, by some magic, we can check that

$$\begin{aligned}
&\lambda_0\left(\frac{g_1}{t_0^{r-1}} d\frac{t_1}{t_0} + \cdots + \frac{g_n}{t_0^{r-1}} d\frac{t_n}{t_0}\right) \\
&= \left(-\frac{t_1}{t_0^2}\frac{g_1}{t_0^{r-1}} + \cdots + \frac{t_n}{t_0^2}\frac{g_n}{t_0^{r-1}}\right) e_0 + \frac{g_1}{t_0^r} e_1 + \cdots + \frac{g_n}{t_0^r} e_n \\
&= \frac{1}{t_0}\left(-\frac{t_1 g_1}{t_0^r} - \cdots - \frac{t_n g_n}{t_0^r}\right) e_0 + \cdots \\
&= \frac{1}{t_0}\frac{t_0 g_0}{t_0^r} e_0 + \cdots = \frac{g_0 e_0}{t_0^r} + \cdots + \frac{g_n e_n}{t_0^r} = \omega.
\end{aligned}$$

The way the λ_i's are defined gives a compatible way of gluing all the isomorphisms $\Omega_{X/Y}|_{D_+(t_i)} \to \widetilde{M}|_{D_+(t_i)}$ into a global isomorphism $\Omega_{X/Y} \to \widetilde{M}$. ○

Corollary 14.7. *Under the same assumptions as in the theorem, we have $\Omega_{X/Y}(X)$; i.e. there is no global differential on $\mathbb{P}^n_A$.*

Proof. By the theorem,

$$\Omega_{X/Y}(X) \subset \bigoplus_1^{n+1} \mathcal{O}_X(-1)(X).$$

But I claim the second sheaf doesn't have any global sections (second example sheet). ○

Before next lecture: go learn some category theory.

Lecture 15: November 7

RECALL: a category has a collection of "objects" and for each two objects A, B, it has a set of "morphisms from A to B".

An abelian category, in this course, means one of the following:

- the category of abelian groups (denoted **Ab**), where the objects are abelian groups, and the morphisms are homomorphisms of abelian groups;
- the category of modules over a ring A (denoted $\mathbf{M}(A)$);

- the category of sheaves on a topological space X (denoted by $\mathbf{Sh}(X)$), where the objects are sheaves, and morphisms are morphisms of sheaves.
- the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (denoted $\mathbf{M}(X)$);
- the category of quasicoherent sheaves on a scheme X (denoted by $\mathbf{Qcoh}(X)$).

Definition 15.1. Suppose $\mathcal{A}$ is an abelian category. Then:

- A *complex* is a sequence
$$A^\bullet : \cdots \to A^{-1} \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$$
such that $d^{i+1} \circ d^i = 0$.
- For a complex $A^\bullet$ we define the i^{th} *cohomology* of $A^\bullet$:
$$h^i(A^\bullet) = \ker d^i / \operatorname{im}^{i-1}$$
(i.e. this measures how far your sequence is from being exact: $A^\bullet$ is exact at A_i $\iff$ $h^i(A^\bullet) = 0$.).
- A *chain map* between two complexes $A^\bullet$ and $B^\bullet$ is a commutative diagram
$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & A^i \arrow[r] \arrow[d] & A^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$
- A chain map $A^\bullet \to B^\bullet$ induces maps $h^i(A^\bullet) \to h^i(B^\bullet)$. (check this)
- An *exact sequence of complexes* is a sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ such that, for every i, the sequence $0 \to A^i \to B^i \to C^i \to 0$ is exact.
- For an exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ we get a long exact sequence (in a natural way)
$$\cdots \to h^{i-1}(C^\bullet) \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to h^{i+1}(B^\bullet) \to \cdots$$
(Check this by diagram chase.)

Definition 15.2. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor between abelian categories. (For example, if $\mathcal{A} = \mathbf{Sh}(X)$, and $\mathcal{B} = \mathbf{Ab}$, then there is a functor $F : \mathcal{A} \to \mathcal{B}$ defined by $F(\mathcal{F}) = \mathcal{F}(X)$.) We say that F is *additive* if the induced map $\operatorname{Hom}(A, A') \to \operatorname{Hom}(F(A), F(A'))$ is a homomorphism of abelian groups.

We say that F is left exact if it is additive and such that, for every exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence
$$0 \to F(A') \to F(A) \to F(A'')$$
is exact.

Similarly, we can define a right exact functor. A functor is *exact* if it is both right and left exact. (For example, the global sections functor is left exact but not right exact.)

Definition 15.3. Suppose that $\mathcal{A}$ is an abelian category. An object $I \in \mathcal{A}$ is called *injective* if, for every diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

of solid arrows that is exact (i.e. $A' \to A$ is injective), there is a map $A \to I$ that makes the diagram commute. (These are good for proving theorems, but are useless for computing anything.)

Example 15.4. If $\mathcal{A} = \mathbf{Ab}$, then $\mathbb{Q}$ is injective.

Definition 15.5. For an object A, an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is called an injective resolution if all the I^i are injective objects.

We say that $\mathcal{A}$ has *enough injectives* if every object in $\mathcal{A}$ has an injective resolution.

Definition 15.6. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant left exact functor (you should be thinking of the global sections functor $\mathbf{Sh} \to \mathbf{Ab}$). Suppose $\mathcal{A}$ has enough injectives (we will show in the next lecture that $\mathbf{Sh}$ does). For each object A in $\mathcal{A}$, we take an injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots$$

and apply F to get a complex

$$0 \xrightarrow{d^{-1}} F(I^0) \xrightarrow{d^0} F(I^1) \xrightarrow{d^1} \cdots \tag{15.1}$$

(This will not, in general, be exact.)

Define $R^iF(A) = h^i((15.1)) = \ker d^i / \operatorname{im} d^{i-1}$. Then $R^iF : \mathcal{A} \to \mathcal{B}$ are functors that are called the *right derived functors*. (You can check that this does not depend on the injective resolution chosen.)

Remark 15.7.

(1) Any morphism $A \to A'$ induces a morphism $R^iF(A) \to R^iF(A')$.
(2) For any exact sequence $0 \to A' \to A \to A'' \to 0$ we get a long exact sequence

$$0 \to \cdots \to R^{i-1}F(A'') \to R^iF(A') \to R^iF(A) \to R^iF(A'') \to R^{i+1}F(A') \to \cdots$$

For any diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

where the top and bottom are exact, we get a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & R^iF(A') & \longrightarrow & R^iF(A) & \longrightarrow & R^iF(A'') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & R^iF(B') & \longrightarrow & R^iF(B) & \longrightarrow & R^iF(B'') & \longrightarrow & \cdots \end{array}$$

(3) If A is injective, then $R^iF(A) = 0$ for all $i > 0$ (because you can just take the trivial resolution).
(4) $R^0F(A) = F(A)$

DEFINITION 15.8. We say that J is F-*acyclic* if all $R^iF(J) = 0$ for all $i > 0$ (e.g. if J is injective).

REMARK 15.9. For any A and resolution

$$0 \to A \to J^0 \to J^1 \to \cdots$$

where the J^i are F-acyclic, then

$$R^iF(A) = h^i(0 \to F(J^0) \to F(J') \to \cdots)$$

(i.e. we can use F-acyclic resolutions instead of injective resolutions to compute cohomology).

LECTURE 16: NOVEMBER 9

Let X be a topological space, and $\{\mathcal{F}_i\}_i$ be a family of sheaves on X. Then we can define the product $\prod \mathcal{F}_i$ by $(\prod \mathcal{F}_i)(U) = \prod \mathcal{F}_i(U)$. This is a sheaf, and for each j we have a natural projection map $\prod \mathcal{F}_i \to \mathcal{F}_j$ given by the projection $\prod \mathcal{F}_i(U) \to \mathcal{F}_j(U)$. For any sheaf $\mathcal{G}$, we have

$$\mathrm{Hom}(\mathcal{G}, \prod \mathcal{F}_i) = \prod \mathrm{Hom}(\mathcal{G}, \mathcal{F}_i).$$

THEOREM 16.1. *Suppose $(X, \mathcal{O}_X)$ is a ringed space, and $\boldsymbol{Mod}_{\mathcal{O}_X}(X)$ is the category of $\mathcal{O}_X$-modules. Then, $\boldsymbol{Mod}_{\mathcal{O}_X}(X)$ has enough injectives (every object has an injective resolution).*

Proof. Pick an $\mathcal{O}_X$-module $\mathcal{F}$. We will find an injective resolution for $\mathcal{F}$. For every $x \in X$, $\mathcal{F}_x$ is an $\mathcal{O}_x$-module (recall notation $\mathcal{O}_x = (\mathcal{O}_X)_x$). Use the following commutative algebra fact:

FACT 16.2. *If A is a ring, and M an A-module, then there is an injective homomorphism $M \to I$ where I is an injective module in the category of A-modules.*

So, we can find an injective $\mathcal{O}_x$-module I_x and an injective homomorphism $\mathcal{F}_x \hookrightarrow I_x$.

Denote the inclusion map $\{x\} \hookrightarrow X$ by f_x. The module I_x can be considered as a sheaf on $\{x\}$. Define $\mathcal{I} = \prod_{x \in X} (f_x)_* I_x$. We will show that $\mathcal{I}$ is injective.

For any $\mathcal{O}_X$-module $\mathcal{G}$, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) \cong \prod \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},(f_x)_* I_x).$$

On the other hand,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},(f_x)_* I_x) = \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, I_x)$$

(this is a special case of problem 10 on example sheet 2). In particular, using the homomorphisms $\mathcal{F}_x \hookrightarrow I_x$, we get a morphism $\mathcal{F} \to \mathcal{I}$; this morphism is injective, as can be checked on the stalks.

Now we show that $\mathcal{I}$ is an injective object in $\mathbf{Mod}_{\mathcal{O}_X}(X)$. Suppose we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r, "\varphi"] \arrow[d] & \mathcal{H} \\ & \mathcal{I} & \end{tikzcd} \tag{16.1}$$

where the top sequence is exact (i.e. φ is injective). We get a diagram on stalks

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_x \arrow[r, "\varphi"] \arrow[d] & \mathcal{H}_x \arrow[dl, dotted] \\ & \mathcal{I}_x = I_x & \end{tikzcd}$$

Since I_x is an injective object, the dotted arrow exists. By the remarks above, all the homomorphisms $\mathcal{H}_x \to I_x$ determine a morphism $\mathcal{H} \to \mathcal{I}$ that sits in the commutative diagram (16.1).

To get an injective resolution of $\mathcal{F}$, argue as follows:

Pick an injective $\mathcal{I}^0$ and an injective homomorphism $\mathcal{F} \hookrightarrow \mathcal{I}^0$. (So we have an exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0$.) Put $\mathcal{F}^0 = \mathcal{I}^0/\mathcal{F}$ (thinking of $\mathcal{F}$ as a subsheaf of $\mathcal{I}^0$), and we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{F}^0 \to 0.$$

Now pick an injective homomorphism $\mathcal{F}^0 \to \mathcal{I}^1$ where $\mathcal{I}^1$ is injective. Then we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1$$

(where the last map factors through $\mathcal{F}^0$). Now put $\mathcal{F}^1 = \mathcal{I}^1/\mathcal{F}^0$ and continue the process to get an injective resolution. ○

COROLLARY 16.3. *Let X be a topological space. Then $\mathbf{Sh}(X)$ has enough injectives.*

$\mathbf{Sh}(X)$ are $\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}$-modules, in the sense that rings are $\mathbb{Z}$-modules.

Proof. Let $\mathcal{O}_X$ be the constant sheaf associated to $\mathbb{Z}$. Then, $(X,\mathcal{O}_X)$ is a ringed space and $\mathbf{Sh}(X) = \mathbf{Mod}_{\mathcal{O}_X}(X)$ (this is analogous to saying that every abelian group is a module over $\mathbb{Z}$). Now apply the previous theorem. ○

DEFINITION 16.4. Let $(X,\mathcal{O}_X)$ be a ringed space. Let $F : \mathbf{Mod}_{\mathcal{O}_X}(X) \to \mathbf{Ab}$ be the functor $F : \mathcal{F} \to \mathcal{F}(X)$ (the global sections functor). This is left exact. We will denote

$H^i(X,\mathcal{F}) = R^iF(\mathcal{F})$. We take an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ in $\mathbf{Mod}_{\mathcal{O}_X}(X)$ and apply F to get a complex

$$\begin{tikzcd}
0 \arrow[r] & F(\mathcal{I}^0) \arrow[r] \arrow[d, equal] & F(\mathcal{I}^1) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & \mathcal{I}^0(X) \arrow[r, "d^0"] & \mathcal{I}^1(X) \arrow[r, "d^1"] & \cdots
\end{tikzcd}$$

and then $H^i(X,\mathcal{F}) = \ker d^i / \operatorname{im} d^{i-1}$ is the i^{th} cohomology group of $\mathcal{F}$.

We will see later that $H^i(X,\mathcal{F})$ depends only on X and $\mathcal{F}$ (not $\mathcal{O}_X$).

If we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

in $\mathbf{Mod}_{\mathcal{O}_X}(X)$ then we get a long exact sequence

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}) \to \cdots$$

For topological spaces we care about, we will see that this sequence reaches 0 after finitely many terms.

Remark 16.5. Note that $H^0(X,\mathcal{F}) = \mathcal{F}(X)$.

This is the cohomology theory we will mostly use, but there are other ways to define related cohomology constructions.

Example 16.6. Let $(X,\mathcal{O}_X)$ be a ringed space, and $\mathcal{L}$ an $\mathcal{O}_X$-module. Define $G : \mathbf{Mod}_{\mathcal{O}_X}(X) \to \mathbf{Ab}$ by $G(\mathcal{G}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{G})$. This is a left exact functor (but we won't prove that here). So we can define the right derived functors of G (denoted $Ext^i_{\mathcal{O}_X}(\mathcal{L},\mathcal{G}) := R^iG(\mathcal{G})$). In particular, if $\mathcal{L} = \mathcal{O}_X$ then $G = F$ and $R^iF = R^iG$.

Example 16.7. Let $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Define $E : \mathbf{Mod}_{\mathcal{O}_X}(X) \to \mathbf{Mod}_{\mathcal{O}_X}(Y)$ by $E(\mathcal{E}) \to f_*\mathcal{E}$. Check that E is a left exact functor. Define the functor R^iE, denoted $R^if_*\mathcal{E} = R^iE(\mathcal{E})$. (Note that $R^iE(\mathcal{E})$ is an $\mathcal{O}_Y$-module rather then just an abelian group.)

Definition 16.8. Suppose X is a topological space, and $\mathcal{F}$ is a sheaf on X. We say that $\mathcal{F}$ is *flasque* if for all $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

We will show that if $(X,\mathcal{O}_X)$ is a ringed space and $\mathcal{I} \in \mathbf{Mod}_{\mathcal{O}_X}(X)$ is injective, then $\mathcal{I}$ is flasque.

These are not necessarily injective, but they have trivial cohomology ($H^i(X,\mathcal{F}) = 0$ for $i > 0$). So we can use resolutions by flasque sheaves to calculate $H^i(X,-)$. In particular, this depends only on X, not $\mathcal{O}_X$.

Lecture 17: November 12

Theorem 17.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{I}$ an injective $\mathcal{O}_X$-module. Then, $\mathcal{I}$ is flasque.*

Proof. Pick an open $U \subset X$ and $t \in \mathcal{I}(U)$. We should find $s \in \mathcal{I}(X)$ restricting to t. Define $\mathcal{L}_U$ by

$$\mathcal{L}_U(W) = \begin{cases} 0 & \text{if } W \not\subset U \\ \mathcal{O}_X(W) & \text{if } W \subset U \end{cases}$$

Define a morphism $\mathcal{L}_U \to \mathcal{I}$ by

$$\mathcal{L}_U(W) \to \mathcal{I}(W) = \begin{cases} 0 & \text{if } W \not\subset U \\ a \mapsto at|_W \text{ if } W \subset U \end{cases}$$

Now that we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_U \arrow[r, hook] \arrow[d] & \mathcal{O}_X \arrow[dl, dotted] \\ & \mathcal{I} & \end{tikzcd}$$

of solid arrows, we have a morphism $\mathcal{O}_X \to \mathcal{I}$ making the diagram commutative, because $\mathcal{I}$ is injective. We get a commutative diagram

$$\begin{tikzcd} \mathcal{L}_U(X) \arrow[r] \arrow[d] \arrow[dr, dotted] & \mathcal{O}_X(X) \arrow[dl] \arrow[dr, dotted] & \\ \mathcal{I}(X) \arrow[dr, dotted] & \mathcal{L}_U(U) \arrow[r] \arrow[d] & \mathcal{O}_X(U) \arrow[dl] \\ & \mathcal{I}(U) & \end{tikzcd}$$

Since t is in the image of the red arrows, it is also in the image of the blue arrows:

$$\begin{tikzcd} \mathcal{L}_U(X) \arrow[r] \arrow[d] \arrow[dr, dotted] & 1 \arrow[dl, blue] \arrow[dr, dotted, red] & \\ s \arrow[dr, dotted, blue] & 1 \arrow[r] \arrow[d] & 1 \arrow[dl, red] \\ & t & \end{tikzcd}$$

and in particular there is some $s \in \mathcal{I}(X)$ making everything commute.

Equivalently, there is a surjection

$$\mathcal{I}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}_U, \mathcal{I}) = \mathcal{I}(U).$$

○

Theorem 17.2. *$(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is a flasque $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = 0$ if $i > 0$.*

Proof. We have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \xrightarrow{\varphi} \mathcal{G} \to 0$$

when $\mathcal{I}$ is injective and $\mathcal{G} = \mathcal{I}/im(\mathcal{F})$. We will show that $\mathcal{G}$ is flasque, and that α is surjective in the sequence

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \xrightarrow{\alpha} \mathcal{G}(X)$$

(the exactness of the above sequence comes for free). In general, you can't put 0 at the right end of this sequence; the term on the right is $H^1(X, \mathcal{F})$. So we are proving that $H^1(X, \mathcal{F}) = 0$.

Pick $t \in \mathcal{G}(X)$; we want to find a preimage in $\mathcal{I}(X)$. Since $\varphi : \mathcal{I} \to \mathcal{G}$ is surjective, for all $x \in X$ there is some open $U \subset X$ containing x and some $s \in \mathcal{I}(U)$ such that $s \mapsto t|_U$ (because $\varphi_x : \mathcal{I}_x \to \mathcal{G}_x$ is surjective). Consider pairs (U_1, s_1) and (U_2, s_2) where $s_i \in \mathcal{I}(U_i)$ and $s_i \mapsto t|_{U_i}$. First consider the black labels and arrows in the diagram:

$$\begin{tikzcd}
 & & \mathcal{I}(U_1) \ni s_1 \arrow[r] \arrow[dd] & \mathcal{G}(U_1) \arrow[dd] \\
 & \gamma \arrow[r, "\exists \text{ b/c of exactness}"'] & s_1 - s_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F}(U_1 \cap U_2) \arrow[r] & \mathcal{I}(U_1 \cap U_2) \ni s_1 \arrow[r] & \mathcal{G}(U_1 \cap U_2) \\
 & \mathcal{F}(U_2) \arrow[u] \arrow[r] & \mathcal{I}(U_2) \ni s_2 + q \arrow[u] \arrow[r] & \mathcal{G}(U_2) \arrow[u] \\
 & p \arrow[uuu, dashed] \arrow[r, dashed, "\exists \text{ b/c } \mathcal{F} \text{ flasque}"'] & q \arrow[uuu, dashed] &
\end{tikzcd}$$

By construction s_1 and s_2 map to the same element $(t|_{U_1 \cap U_2})$ in $\mathcal{G}(U_1 \cap U_2)$. So $s_1 - s_2 = 0$ in $\mathcal{G}(U_1 \cap U_2)$ and by exactness there is an element $\gamma \in \mathcal{F}(U_1 \cap U_2)$ mapping to $s_1 - s_2$.

Because $\mathcal{F}$ is flasque, $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ is a surjection. *Show that flasque is equivalent to: for all $V \subset U$, $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ is a surjection.* So γ has a preimage $p \in \mathcal{F}(U_2)$.

Recall the aim was to get a global section in $\mathcal{I}(X)$ mapping to t. You can't glue s_1 and s_2, because $s_1 - s_2$ might not be 0 in $\mathcal{I}(U_1 \cap U_2)$. Let q be the image of p in $\mathcal{F}(U_2)$. Then

$$s_2 + q|_{U_1 \cap U_2} = s_2 + s_1 - s_2 = s_1|_{U_1 \cap U_2}.$$

So $s_2 + q$ and s_1 glue together to give a section $s \in \mathcal{I}(U_1 \cup U_2)$. Define an ordering: $(U, s) \leq (U', s')$ if $U \subset U'$ and $s = s'|_U$, where $s \in \mathcal{I}(U)$ maps to $t|_U$, and $s' \in \mathcal{I}(U')$ maps to $t|_{U'}$. Now by Zorn's lemma there is a maximal element (U, s) among such pairs.

We will show that $U = X$. Assume not. Then pick $x \in X - U$. Since φ is surjective, there is some $U' \subset X$ and $s' \in \mathcal{I}(U')$ such that $s' \mapsto t|_{U'}$. By the arguments above we get a section $\ell \in \mathcal{I}(U \cup U')$ such that $\ell \mapsto t|_{U \cup U'}$. In other words, $(U \cup U', \ell)$ is a pair as defined above. We have $(U, s) \subsetneq (U \cup U', \ell)$ and this contradicts the fact that U is maximal. Therefore, $U = X$ and $s \in \mathcal{I}(X)$ maps to $t \in \mathcal{G}(X)$. So α is surjective.

The same arguments show that $\mathcal{I}(V) \to \mathcal{G}(V)$ is surjective for all open $V \subset X$. (Flasqueness is preserved when you restrict from X to V.)

Now we apply induction on i. *Recall, cohomology in this context is $H^i(X, \mathcal{A}) = R^iF(\mathcal{A})$ where F is the global sections functor $\boldsymbol{Sh} \to \boldsymbol{Ab}$.* The short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ of sheaves gives rise (by 15.7(2)) to a long exact sequence:

$$0 \longrightarrow \underbrace{H^0(X,\mathcal{F})}_{\mathcal{F}(X)} \longrightarrow \underbrace{H^0(X,\mathcal{I})}_{\mathcal{I}(X)} \xrightarrow{\alpha} \underbrace{H^0(X,\mathcal{G})}_{\mathcal{G}(X)}$$

$$\longrightarrow H^1(X,\mathcal{F}) \xrightarrow{\cong} \underbrace{H^1(X,\mathcal{I})}_{0} \longrightarrow H^1(X,\mathcal{G})$$

$$\longrightarrow H^2(X,\mathcal{F}) \xrightarrow{\lambda} \underbrace{H^2(X,\mathcal{I})}_{0} \longrightarrow \cdots$$

(where on the first line $H^0(X, \mathcal{A}) = R^0F(\mathcal{A}) = F(\mathcal{A}) = \mathcal{A}(X)$ (where F is the global sections functor) by 15.7(4)). We know that $\alpha : \mathcal{I}(X) \to \mathcal{G}(X)$ is surjective and $H^1(X, \mathcal{I}) = 0$, since $\mathcal{I}$ is injective. This immediately gives $H^1(X, \mathcal{F}) = 0$.

To show $\mathcal{G}$ is flasque, look at the second square in the commutative diagram: we want to show that the last vertical arrow is a surjection.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d, two heads] & \mathcal{I}(X) \arrow[r] \arrow[d, two heads] & \mathcal{G}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{I}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & 0
\end{tikzcd}$$

$\mathcal{I}(U) \to \mathcal{G}(U)$ is surjective, so we can pull an element of $\mathcal{G}(U)$ all the way back to $x \in \mathcal{I}(X)$ this way; so the image of x in $\mathcal{G}(X)$ is the thing we're looking for.

We showed above that $H^1(X, \mathcal{F}) = 0$. $\mathcal{F}$ was an arbitrary flasque sheaf, so $H^1(X, \mathcal{G}) = 0$ as well. Since $H^2(X, \mathcal{I}) = 0$, the exact sequence shows that $H^2(X, \mathcal{F}) = 0$.

Now keep going with this argument, showing that $H^2(X, \mathcal{G}) = 0$, etc. Step by step one shows that $H^i(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{G}) = 0$ for all $i > 0$. ○

Corollary 17.3. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{O}'_X$ the constant sheaf associated to $\mathbb{Z}$. (Note $(X, \mathcal{O}'_X)$ is a ringed space). Then $H^i(X, \mathcal{F})$ is the same whether calculated using $\boldsymbol{Mod}_{\mathcal{O}_X}(X)$ or $\boldsymbol{Mod}_{\mathcal{O}'_X}(X)$, a.k.a. $\boldsymbol{Sh}(X)$.*

Proof. Remember that $H^i(X,\mathcal{F})$ is calculated as follows (using $\mathbf{Mod}_{\mathcal{O}_X}(X)$). Take an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

We don't know if the $\mathcal{I}^i$'s are injective in $\mathbf{Mod}_{\mathcal{O}'_X}$. But we proved that every injective is flasque, so the sequence above is a resolution by flasque sheaves. We just proved that $H^i(X,\mathcal{I}^j) = 0$ (calculated using $\mathbf{Mod}_{\mathcal{O}'_X}(X) = \mathbf{Sh}(X)$). By the first lecture on cohomology, $H^i(X,\mathcal{F})$ (calculated using $\mathbf{Mod}_{\mathcal{O}'_X}(X)$) is given as follows:

$$H^i(X,\mathcal{F}) = \ker d^i / \operatorname{im} d^{i-1}$$

where the morphisms come from

$$0 \to \mathcal{I}^0(X) \xrightarrow{d^0} \mathcal{I}^1(X) \xrightarrow{d^1} \mathcal{I}^2(X) \to \cdots$$

Since $\mathcal{I}^j$ are injective in $\mathbf{Mod}_{\mathcal{O}_X}(X)$, $H^i(X,\mathcal{F})$ (calculated using $\mathbf{Mod}_{\mathcal{O}_X}(X)$) is given by $\ker d^i / \operatorname{im} d^{i-1}$. ○

I think what's meant here is: $H^i(X,\mathcal{F})$ w.r.t. $\boldsymbol{Mod}_{\mathcal{O}_X}(X)$ is calculated using the injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots.$$

This might not be an injective resolution w.r.t. $\boldsymbol{Mod}_{\mathcal{O}'_X}(X)$, but it is a flasque resolution, and flasque-ness doesn't depend on the ambient category $\boldsymbol{Mod}_{\mathcal{O}_X}(X)$ vs. $\boldsymbol{Mod}_{\mathcal{O}'_X}(X)$. Since it suffices to use a flasque resolution instead of an injective one, this is equally valid when the category is $\boldsymbol{Mod}_{\mathcal{O}'_X}(X)$.

Lecture 18: November 14

Cohomology of affine schemes.

Theorem 18.1. *Suppose that X is a Noetherian scheme. Then, TFAE:*

1. *X is affine;*
2. *$H^i(X,\mathcal{F}) = 0$ for all $i > 0$, for every quasicoherent sheaf $\mathcal{F}$;*
3. *$H^i(X,\mathcal{I}) = 0$ for all $i > 0$, for all coherent ideal sheaves $\mathcal{I}$.*

Definition 18.2. Suppose X is a scheme, $b \in \mathcal{O}_X(X)$. Define

$$D(b) = \{x \in X \,:\, b \text{ is invertible in } \mathcal{O}_x\}$$

(a.k.a. b is not in the maximal ideal of $\mathcal{O}_x$, a.k.a. b does not vanish at x). Locally on affine open sets $D(b)$ is the same as before, so $D(b)$ is an open subset of X.

Fact 18.3. *X is a Noetherian scheme. Then X is affine $\iff$ there are $b_1, \cdots, b_n \in \mathcal{O}_X(X)$ such that the $D(b_i)$ are affine and $\langle b_1, .., b_n\rangle = \mathcal{O}_X(X)$.*

Proof. Problem 11 on example sheet 3. ○

Definition 18.4. Suppose X is a scheme. A point $x \in X$ is called closed if $\{x\}$ is a closed subset of X.

For example, $\langle t-a\rangle \in \operatorname{Spec}\mathbb{C}[t]$ is a closed point but 0 is not a closed point.

Remark 18.5. Suppose X is Noetherian and $Z \subset X$ is a closed subset. Then, there is some closed point $x \in X$ such that $x \in Z$. Choose an open affine set $U \subset X$ such that $U \cap Z \neq \emptyset$. If $Z \not\subset U$, replace Z with $Z \cap (X \backslash U)$. Do "induction" (this is possible because X is Noetherian – you can't have an infinite sequence of closed sets). If $Z \subset U$, then Z is affine. Say $U = \operatorname{Spec} A$. Then there is some ideal $I \leq A$ such that $Z = V(I)$. If x is any maximal ideal such that $I \subset x$, then x is a closed point of X ($\{x\}$ is a subset of Z, which implies that it is also a closed subset of X).

Proof of Theorem. (1) $\implies$ (2): Suppose $X = \operatorname{Spec} A$, and $\mathcal{F}$ is quasicoherent (i.e. $\mathcal{F} = \widetilde{M}$). In the category of A-modules we an find an injective resolution

$$0 \to M \to I^0 \to I^1 \to \cdots$$

from which we get an exact sequence

$$0 \to \mathcal{F} = \widetilde{\mathcal{M}} \to \widetilde{\mathcal{I}}^0 \to \widetilde{\mathcal{I}}^1 \to \cdots \tag{18.1}$$

There is no reason why $\widetilde{\mathcal{I}}$ should be injective. But:

Fact 18.6. *If A is a Noetherian ring and I is an injective A-module then $\widetilde{I}$ is a flasque sheaf.*

(See Hartshorne, Proposition III.3.4. *Note this is not necessarily true when the Noetherian hypothesis is removed: see e.g.* `http://stacks.math.columbia.edu/tag/0273`.)

Thus, we can use (18.1) to calculate $H^i(X, \mathcal{F})$. In this case, (18.1) being exact implies that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \widetilde{I}^0(X) \arrow[r] \arrow[d, equal] & \widetilde{I}^1(X) \arrow[r] \arrow[d, equal] & \cdots \\ 0 \arrow[r] & M \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

So $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

(2) $\implies$ (3): Obvious.

(3) $\implies$ (1): Pick a closed point $x \in X$. We want to find $b \in \mathcal{O}_X(X)$ such that $D(b)$ is affine and $x \in D(b)$. Choose an open affine $U \subset X$ such that $x \in U$ and let $Y = X \backslash U$. Y and $Y \cup \{x\}$ are both closed subsets of X.

Claim 18.7. *If $Z \subset X$ is any closed subset we can put a (not necessarily unique) closed subscheme structure on Z.*

Proof. Define the ideal sheaf $\mathcal{I}_Z$ by

$$\mathcal{I}_Z(W) = \{a \in \mathcal{O}_X(W) : a \text{ is not invertible in } \mathcal{O}_z \ \forall z \in Z \cap W\}.$$

If $W = \operatorname{Spec} B$ then $\mathcal{I}_Z|_W = \widetilde{I}$ where $I \leq B$ is the largest ideal such that $Z \cap W = V(I) \subset \operatorname{Spec} B$. $\mathcal{I}_Z$ is quasicoherent, so $\mathcal{I}_Z$ gives the subscheme structure on Z that we wanted. ○

We apply this construction to define $\mathcal{I}_Y$ and $\mathcal{I}_{Y\cup\{x\}}$ to get closed subscheme structures on Y and $Y \cup \{x\}$. Since $Y \subset Y \cup \{x\}$, $\mathcal{I}_{Y\cup\{x\}} \subset \mathcal{I}_Y$. We get an exact sequence

$$0 \to \mathcal{I}_{Y\cup\{x\}} \to \mathcal{I}_Y \to \mathcal{L} := \mathcal{I}_Y/\mathcal{I}_{Y\cup\{x\}} \to 0 \tag{18.2}$$

By construction $\mathcal{I}_{Y\cup\{x\}}|_{X\setminus\{x\}} = \mathcal{I}_Y|_{X\setminus\{x\}}$. So $\mathcal{L}|_{X\setminus\{x\}} = 0$. So $\mathcal{L}$ is the skyscraper sheaf on X at x defined by the residue field $k(x)$ at x.

By assumptions, $H^1(X, \mathcal{I}_{Y\cup\{x\}}) = 0$. From (18.2), we get a long exact sequence

$$0 \to H^0(X, \mathcal{I}_{Y\cup\{x\}}) \to H^0(X, \mathcal{I}_Y) \xrightarrow{\alpha} \underbrace{H^0(X, \mathcal{L})}_{k(x)} \to \underbrace{H^1(X, \mathcal{I}_{Y\cup\{x\}})}_{0} \to \cdots$$

Since $H^0(X, \mathcal{L}) = k(x)$ and α is surjective, there is some $b \in H^0(X, \mathcal{I}_Y)$ with $b \overset{\alpha}{\mapsto} 1$. Since $\mathcal{I}_Y \subset \mathcal{O}_X$, $H^0(X, \mathcal{I}_Y) \subset H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ and we can consider $b \in \mathcal{O}_X(X)$. Since $b \mapsto 1 \in k(x)$, b gives an invertible element in $\mathcal{O}_x$. In particular, $x \in D(b)$. By construction, $D(b) \subset U$, which implies that $D(b)$ is affine.

We proved, for every closed point $x \in X$, there is some global section $b \in \mathcal{O}_X(X)$ such that $x \in D(b)$ and $D(b)$ is affine. This implies that there is a family $\{b_i\} \subset \mathcal{O}_X(X)$ such that $\bigcup_i D(b_i)$ contains all closed points (and $D(b_i)$ is affine). So $X = \bigcup D(b_i)$, otherwise there is a closed point in $X \setminus \bigcup D(b_i)$ which is not possible. Since X is Noetherian we can assume that there are finitely many b_i's, say $b_1, \cdots, b_n$.

It remains to show that the b_i's generate $\mathcal{O}_X(X)$. Define a morphism

$$\underbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}_{n \text{ copies}} \xrightarrow{\varphi} \mathcal{O}_X \text{ where } \underbrace{(s_1, \cdots, s_n)}_{\text{on } W} \mapsto \sum s_i b_i|_W$$

φ is surjective as a morphism of sheaves; check this locally. (φ_x is surjective for all x because, for all $x \in X$ there is some b_i such that b_i is invertible in $\mathcal{O}_x$.) It suffices to show that this is a surjection on global sections. Let $\mathcal{F} = \ker \varphi$. We get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X \to 0$$

Define a filtration

$$\underbrace{0}_{\mathcal{G}_0} \subset \underbrace{\mathcal{O}_X \oplus 0 \oplus \cdots \oplus 0}_{\mathcal{G}_1} \subset \underbrace{\mathcal{O}_X \oplus \mathcal{O}_X \oplus 0 \oplus \cdots}_{\mathcal{G}_2} \subset \cdots \subset \underbrace{\bigoplus_1^n \mathcal{O}_X}_{\mathcal{G}_n}.$$

Put $\mathcal{F}_n = \mathcal{F}$. Let $\mathcal{F}_{n-1} = \ker\left(\mathcal{F}_n \to \mathcal{G}_n/\mathcal{G}_{n-1}\right)$ and inductively define

$$\mathcal{F}_{i-1} = \ker\left(\mathcal{F}_i \to \mathcal{G}_i/\mathcal{G}_{i-1}\right)$$

The $\mathcal{F}_i$ are quasicoherent. Also, $\mathcal{F}_i/\mathcal{F}_{i-1}$ injects into $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \mathcal{O}_X$ so $\mathcal{F}_i/\mathcal{F}_{i-1}$ can be considered as a quasicoherent ideal sheaf.

So $H^1(X,\mathcal{F}_0)=H^1(X,0)=0$. The short exact sequence
$$0\to\mathcal{F}_{i-1}\to\mathcal{F}_i\to\mathcal{F}_i/\mathcal{F}_{i-1}\to 0$$
yields an exact sequence
$$0=H^1(X,\mathcal{F}_{i-1})\to H^1(X,\mathcal{F}_i)\to H^1(X,\mathcal{F}_i/\mathcal{F}_{i-1})=0$$
and so, by induction, $H^1(X,\mathcal{F}_i)=0$ for all i. In particular,
$$H^1(X,\mathcal{F}_n)=H^1(X,\mathcal{F})=0.$$
Now the sequence
$$0\to H^0(X,\mathcal{F})\to H^0(X,\mathcal{G}_n)\xrightarrow{\beta}\underbrace{H^0(X,\mathcal{O}_X)}_{\mathcal{O}_X(X)}\to H^1(X,\mathcal{F})=0$$
is exact and $\beta:\mathcal{G}_n(X)=\bigoplus_1^n\mathcal{O}_X(X)\to\mathcal{O}_X(X)$ is surjective. Hence $\langle b_1,\cdots,b_n\rangle=\mathcal{O}_X(X)$. ○

Lecture 19: November 16

Čech cohomology.

Definition 19.1. Let X be a topological space, and $\mathcal{F}$ a sheaf on X. Let $\mathscr{U}=(U_i)_{i\in I}$ be a collection of open subsets covering X that are indexed by a well-ordered set I.

For $i_0,\cdots,i_p$, let $U_{i_0,\cdots,i_p}=U_{i_0}\cap\cdots\cap U_{i_p}$. Let $C^p(\mathscr{U},\mathcal{F})=\prod_{i_0<\cdots<i_p}\mathcal{F}(U_{i_0,\cdots,i_p})$. An element of $C^p(\mathscr{U},\mathcal{F})$ is denoted by $(s_{i_0,\cdots,i_p})$ (this is a collection of elements $s_{i_0,\cdots,i_p}$, one for every intersection of $p+1$ sets). Define a complex
$$C^\bullet(\mathscr{U},\mathcal{F}):0\to C^0(\mathscr{U},\mathcal{F})\xrightarrow{d^0}C^1(\mathscr{U},\mathcal{F})\xrightarrow{d^1}\cdots$$
where $d^p(s_{i_0,\cdots,i_p})=(t_{i_0,\cdots,i_{p+1}})$ such that
$$t_{i_0,\cdots,i_{p+1}}=\sum_{\ell=0}^{p+1}(-1)^\ell s_{i_0,\cdots,\widehat{i_\ell},\cdots,i_{p+1}}|_{U_{i_0,\cdots,i_{p+1}}}$$

It is easy to see that $d^{p+1}d^p=0$. (So $C^\bullet(\mathscr{U},\mathcal{F})$ is indeed a complex.)

We define the p^{th} Čech cohomology
$$\check{H}^p(\mathscr{U},\mathcal{F})=\ker d^p/\operatorname{im}d^{p-1}$$

Theorem 19.2. *With the above notation, we have $\check{H}^0(\mathscr{U},\mathcal{F})=\mathcal{F}(X)=H^0(X,\mathcal{F})$.*

Proof. By definition,
$$\check{H}^0(\mathscr{U},\mathcal{F})=\ker d^0/\operatorname{im}d^{-1}\cong\ker d^0=\{(s_i)\,:\,s_i-s_j|_{U_i\cap U_j}=0\}$$

By the sheaf condition, the s_i glue together to give a global section $s \in \mathcal{F}(X)$ such that $s_i = s|_{U_i}$. Conversely, any $s \in \mathcal{F}(X)$ gives the element $(s_i) \in \ker d^0$ where we put $s_i = s|_{U_i}$. The equality, $\mathcal{F}(X) = H^0(X, \mathcal{F})$ was proved in earlier lectures. ○

EXAMPLE 19.3. Let k be a field, $X = \mathbb{P}^1_k = \operatorname{Proj} k[t_0, t_1]$, $U_0 = D_+(t_0)$, $U_1 = D_+(t_1)$, $I = \{0,1\}$ (with the obvious ordering), and $\mathcal{F} = \mathcal{O}_X$. Then we have

$$C^0(\mathscr{U}, \mathcal{O}_X) = \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_1) = k[t_0,t_1]_{(t_0)} \oplus k[t_0,t_1]_{(t_1)}$$
$$C^1(\mathscr{U}, \mathcal{O}_X) = \mathcal{O}_X(U_{0,1} = U_0 \cap U_1) = k[t_0,t_1]_{(t_0t_1)}$$

(since $U_0 \cap U_1 = D_+(t_0) \cap D_+(t_1) = D_+(t_0t_1)$), and

$$C^p(\mathscr{U}, \mathcal{O}_X) = 0 \ \forall p > 1$$

Introduce $u = \frac{t_1}{t_0}$ and $v = \frac{t_0}{t_1}$. Then the Čech complex can be written as

$$0 \xrightarrow{d^{-1}} k[u] \oplus k[v] \xrightarrow{d^0} k[u]_u \xrightarrow{d^1} 0$$

where the differential sends $(f,g) \mapsto f - g = f(u) - g(\frac{1}{u})$, because $uv = 1$ on $U_0 \cap U_1$. When is $f - g = 0$? f has nonnegative degree in u, and g has nonpositive degree in u, so $f = g$ only when f and g were originally (the same) constant. So

$$\ker d^0 = \{(f,g) \,:\, f - g = 0 \text{ in } k[u]_u\} = \{(f,g) \,:\, f = g \text{ in } k\} \cong k$$

So

$$\check{H}^0(\mathscr{U}, \mathcal{O}_X) = \ker d^0 / \operatorname{im} d^{-1} \cong \ker d^0 \cong k$$

Now calculate $\check{H}^1$: since $\mathscr{U}$ only has two open sets, $C^2 = 0$, so

$$\ker d^1 = C^1(\mathscr{U}, \mathcal{O}_X) = k[u]_u = k[u, \tfrac{1}{u}] = \operatorname{im} d^0$$

and

$$\check{H}^1(\mathscr{U}, \mathcal{O}_X) = \ker d^1 / \operatorname{im} d^0 \cong C^1(\mathscr{U}, \mathcal{O}_X)/C^1(\mathscr{U}, \mathcal{O}_X) \cong 0.$$

EXAMPLE 19.4. The data is the same as previously, except $\mathcal{F} = \Omega_{X/Y}$ where $Y = \operatorname{Spec} k$. There is a natural map $X = \mathbb{P}^1_k \to Y = \operatorname{Spec} k$.

The open sets are $U_0 \cong \mathbb{A}^1_k$ and $U_1 \cong \mathbb{A}^1_k$ as above. We have

$$\begin{aligned} C^0(\mathscr{U}, \Omega_{X/Y}) &= \Omega_{X/Y}(U_0) \oplus \Omega_{X/Y}(U_1) \\ &= k[u]du \oplus k[v]dv \\ C^1(\mathscr{U}, \Omega_{X/Y}) &= \Omega_{X/Y}(U_0 \cap U_1) = k[u]_u du \end{aligned}$$

The Čech complex is

$$0 \to k[u]du \oplus k[v]dv \to k[u]_u du$$

where $d^0 : (f\ du, g\ dv) \mapsto f\ du - g\ dv$. To write this in terms of only one variable, notice that $uv = 1$ on $U_0 \cap U_1$, so $dv = -\frac{1}{u^2}du$. Thus our map is $d^0 : (f\ du, g\ dv) \mapsto f(u)du + g(\frac{1}{u}) \cdot \frac{1}{u^2}du$. Now

$$\ker d^0 = \{(f\ du, g\ dv) \,:\, f(u)du + g\tfrac{1}{u} \cdot \frac{1}{u^2}du = 0\}$$

$$= \{\cdots : (f(u) + g(\tfrac{1}{u}) \cdot \frac{1}{u^2})du = 0\}$$
$$= \{(0,0)\} \cong 0$$

(again by issues with degree) So

$$\check{H}^0(\mathscr{U}, \Omega_{X/Y}) = \ker d^0 / \operatorname{im} d^{-1} \cong 0$$

(You can divide 0 by 0. If you don't believe me you need to take another course in algebra.)

Remember that we proved that $H^0(X, \Omega_{\mathbb{P}^n_k/Y}) = 0$, so this calculation is not surprising in view of Theorem 19.2.

Now calculate H^1. As before, we can write the second nontrivial term as $k[u, \frac{1}{u}]du$. Exercise: any monomial except $\frac{1}{u}$ can be given by the image of something in $k[u]du \oplus k[v]dv$. So the quotient $\ker / \operatorname{im}$ is a vector space that is generated by one element $\frac{1}{u}$, and hence is isomorphic to k:

$$\check{H}^1(\mathscr{U}, \Omega_{X/Y}) = \ker d^1 / \operatorname{im} d^0 = C^1(\mathscr{U}, \Omega_{X/Y}) \big/ \operatorname{im} d^0 \cong k$$

Example 19.5. Take the same data as before, except $\mathcal{F}$ is the constant sheaf defined by $\mathbb{Z}$. We have

$$0 \to \underbrace{C^0(\mathscr{U}, \mathcal{F})}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{d^0} \underbrace{C^1(\mathscr{U}, \mathcal{F})}_{\mathbb{Z}} \xrightarrow{d^1} 0$$

(every open set is irreducible, so the section is just $\mathbb{Z}$). Then $d^0 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is given by $(m, n) \mapsto m - n$.

So

$$\check{H}^0(\mathscr{U}, \mathcal{F}) \cong \mathcal{F}(X) \cong \mathbb{Z}$$
$$\check{H}^1(\mathscr{U}, \mathcal{F}) = \ker d^1 / \operatorname{im} d^0 = \mathbb{Z}/\mathbb{Z} \cong 0$$

Example 19.6. Let $X = S^1$ be the circle. Let α be the north pole, and β to be the south pole. Define $U_0 = X \backslash \{\alpha\}$ and $U_1 = X \backslash \{\beta\}$, so X is covered by U_0 and U_1. Let $\mathcal{F}$ be the constant sheaf defined by $\mathbb{Z}$.

The complex is

$$0 \to \underbrace{C^0(X, \mathcal{F})}_{\mathbb{Z}\oplus\mathbb{Z}} \to \underbrace{C^1(\mathscr{U}, \mathcal{F})}_{\mathbb{Z}\oplus\mathbb{Z}} \to 0$$

because the intersection is *not* irreducible this time. The differential $d^0 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $(m, n) \mapsto (m - n, m - n)$, and

$$\check{H}^0(\mathscr{U}, \mathcal{F}) \cong \mathbb{Z}$$
$$\check{H}^1(\mathscr{U}, \mathcal{F}) \cong \mathbb{Z}.$$

Lecture 20: November 19

Čech cohomology on schemes.

Definition 20.1. Let X be a topological space, $\mathscr{U} = (U_i)_{i \in I}$ finite, and $\mathcal{F} \in \mathbf{Sh}(X)$. Denote the inclusion $U \hookrightarrow X$ by f (for any U). Define

$$C^p(\mathscr{U}, \mathcal{F}) = \bigoplus_{i_0 < \cdots < i_p} f_* \mathcal{F}|_{U_{i_0, \cdots, i_p}}$$

Define morphisms

$$C^p(\mathscr{U}, \mathcal{F}) \to C^{p+1}(\mathscr{U}, \mathcal{F})$$

as in the previous lecture. We then form the *Čech sheaf complex*:

$$C^\bullet(\mathscr{U}, \mathcal{F}) : 0 \to C^0(\mathscr{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathscr{U}, \mathcal{F}) \xrightarrow{d^1} \cdots$$

As we said before, $d^{p+1} d^p = 0$. This is the same as taking the Čech complexes on all the open sets of X. So this makes a complex of sheaves, instead a complex of groups, as before.

Define a morphism $\mathcal{F} \to C^0(\mathscr{U}, \mathcal{F})$ by defining $\mathcal{F}(W) \to f_* \mathcal{F}|_{U_i}(W)$ to be $s \mapsto s|_{U_i \cap W}$. Then we have a sequence

$$0 \to \mathcal{F} \to C^0(\mathscr{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathscr{U}, \mathcal{F}) \to \cdots \tag{20.1}$$

Theorem 20.2. *The sequence* (20.1) *is exact.*

Proof. The exactness of

$$0 \to \mathcal{F} \to C^0(\mathscr{U}, \mathcal{F}) \to C^1(\mathscr{U}, \mathcal{F})$$

is just the sheaf condition of $\mathcal{F}$. (Exactness on the first map is saying that a collection of zero sections lifts to the zero section; the second map is the sheaf condition about intersections.) To prove exactness of the rest of (20.1) use stalks.

Fix $x \in X$. We will show that $C^{p-1}(\mathscr{U}, \mathcal{F})_x \to C^p(\mathscr{U}, \mathcal{F})_x \to C^{p+1}(\mathscr{U}, \mathcal{F})_x$ is exact for all $p \geq 1$. Exactness is local around x, and we can throw away any U_i such that $x \notin U_i$. By replacing X, and all the U_i by $U_0 \cap U_1 \cap \cdots \cap U_n$ we can assume $X = U_0 = U_1 = \cdots = U_n$.

The magician's hat: for each $p \geq 1$, we define a map $e^p : C^p(\mathscr{U}, \mathcal{F})_x \to C^{p-1}(\mathscr{U}, \mathcal{F})_x$ by sending $(W, s = (s_{i_0, \cdots, i_p}))$ to $(W, t = (t_{i_0, \cdots, i_{p-1}}))$ by:

$$t_{i_0, \cdots, i_{p-1}} = \begin{cases} s_{0, i_0, \cdots, i_{p-1}} & \text{if } i_0 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

A routine calculation shows that we have a formula

$$d^{p-1} e^p + e^{p+1} d^p = \text{ identity map of } C^p(\mathscr{U}, \mathcal{F})_x. \tag{20.2}$$

(This trick comes from algebraic topology. Claim that the zero chain map is homotopic to the identity chain map.)

Pick $(W,s) \in C^p(\mathscr{U},\mathcal{F})_x$ which belongs to the kernel of $C^p(\mathscr{U},\mathcal{F})_x \to C^{p+1}(\mathscr{U},\mathcal{F})$ (we have to show this comes from something in C^{p-1}). Applying (20.2), we get

$$d_x^{p-1}\Big(e^p((W,s))\Big) + \underbrace{e^{p+1}\Big(d_x^p((W,s))\Big)}_{0} = (W,s)$$

So $(W,s) \in \operatorname{im} d_x^{p-1}$. Therefore we have exactness of the desired sequence. ○

Theorem 20.3. *Let the setting be as above. Assume that $\mathcal{F}$ is flasque. Then, $\check{H}^p(\mathscr{U},\mathcal{F}) = 0$ for all $p \geq 1$.*

Proof. We have the exact sequence

$$0 \to \mathcal{F} \to C^0(\mathscr{U},\mathcal{F}) \to C^1(\mathscr{U},F) \to \cdots . \tag{20.3}$$

Since $\mathcal{F}$ is flasque, it is easy to see that each $C^p(\mathscr{U},\mathcal{F})$ is flasque. Therefore, (20.3) is a resolution of $\mathcal{F}$ by flasque sheaves. Apply the global sections functor to get a sequence

$$0 \to C^0(\mathscr{U},\mathcal{F})(X) \to C^1(\mathscr{U},\mathcal{F})(X) \to \cdots \tag{20.4}$$

that calculates $H^p(X,\mathcal{F})$. On the other hand, (20.4) is, by definition, the Čech complex, so its cohomology is also $\check{H}^p(\mathscr{U},\mathcal{F})$. Thus $\check{H}^p(\mathscr{U},\mathcal{F}) = H^p(X,\mathcal{F})$, and this is equal to zero by Theorem 17.2. ○

Theorem 20.4. *Let X be a Noetherian scheme, and $\mathscr{U} = (U_i)_{i\in I}$ be a finite cover by affine subschemes (even if they aren't affine to begin with, you can get a subcover by affine subschemes, and since X is Noetherian, you can ensure that this is still finite). Let $\mathcal{F}$ be a quasicoherent sheaf. We assume that if $V,W \subset X$ are open affines, then $V \cap W$ is also affine.*

Then, $\check{H}^p(\mathscr{U},\mathcal{F}) \cong H^p(X,\mathcal{F})$.

Remark 20.5. All quasi-projective schemes over a Noetherian ring satisfy the last condition about intersections of open affines.

Proof. Consider the exact sequence

$$0 \to \mathcal{F} \to C^0(\mathscr{U},\mathcal{F}) \to C^1(\mathscr{U},\mathcal{F}) \to \cdots \tag{20.5}$$

We will show that this is an acyclic resolution: that $H^\ell(X, C^p(\mathscr{U},\mathcal{F})) = 0$ for all $\ell > 0$, $p \geq 0$. Equivalently, we need to show that $H^\ell(X, f_*\mathcal{F}|_{U_{i_0,\cdots,i_p}}) = 0$ for all $\ell > 0$, and for all $U_{i_0,\cdots,i_p}$. (Just apply the global sections functor to get the cohomology of X.)

The main assumption of the theorem says that $U_{i_0,\cdots,i_p} =: U$ is affine. Since $\mathcal{F}$ is quasicoherent, $\mathcal{F}|_U$ is also quasicoherent. In particular, $H^\ell(U,\mathcal{F}|_U) = 0$ for all $\ell > 0$.

Take a resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

where the $\mathcal{I}^j$ are quasicoherent and flasque. (We know that this exists: if $\mathcal{F}|_U = \widetilde{M}$, then take an injective resolution of M, and apply tildes to the resulting I_i's). The complex

$$0 \to \mathcal{I}^0(U) \to \mathcal{I}^1(U) \to \cdots \tag{20.6}$$

calculates $H^\ell(U, \mathcal{F}|_U) = 0$. Now the complex

$$0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots \tag{20.7}$$

is exact (this is problem 2 on the fourth example sheet). Trivially, $f_*\mathcal{I}^j$ are flasque; they are quasicoherent by an earlier theorem. (A morphism from a Noetherian scheme to something else, push-downs of quasicoherent sheaves are quasicoherent). So (20.7) is a flasque resolution of $f_*\mathcal{F}$. The complex

$$0 \to (f_*\mathcal{I}^0)(X) \to (f_*\mathcal{I}^1)(X) \to \cdots$$

calculates $H^\ell(X, f_*\mathcal{F})$; this is the same as complex (20.6). Since the cohomology of (20.6) is zero, the cohomology here is also zero.

This implies that (20.5) is an acyclic resolution of $\mathcal{F}$. The complex

$$0 \to C^0(\mathscr{U}, \mathcal{F})(X) \to C^1(\mathscr{U}, \mathcal{F})(X) \to \cdots$$

calculates $H^\ell(X, \mathcal{F})$. But this is just

$$0 \to C^0(\mathscr{U}, \mathcal{F}) \to C^1(\mathscr{U}, \mathcal{F}) \to \cdots$$

and this calculates the Čech cohomology groups $\check{H}^\ell(\mathscr{U}, \mathcal{F})$.

Therefore, $\check{H}^\ell(\mathscr{U}, \mathcal{F}) \cong H^\ell(X, \mathcal{F})$. ○

Lecture 21: November 21

Cohomology of $\mathbb{P}^n$.

Theorem 21.1. *Let k be a field, and $X = \mathbb{P}^n_k := \operatorname{Proj} k[t_0, \cdots, t_n]$. Then*

(1) *$H^0(X, \mathcal{O}_X(d))$ is the k-vector space generated by monomials in $t_0, \cdots, t_n$ of degree d.*
(2) *$H^p(X, \mathcal{O}_X(d)) = 0$ if $p > n$. (More general phenomenon: once you pass the dimension of your scheme (whatever that means), the cohomology is trivial.)*
(3) $\dim_k H^n(X, \mathcal{O}_X(d)) = \dim_k H^0(X, \mathcal{O}_X(-d-n-1)) < \infty$
(4) *$H^p(X, \mathcal{O}_X(d)) = 0$ if $0 < p < n$.*

Fact 21.2. *The intersection of any two open affine subsets of $\mathbb{P}^n_k$ is again affine.*

Proof. See last example sheet. ○

Proof of Theorem 21.1. (1) By Fact 21.2, we may invoke Theorem 20.4 (equivalence of H^p and $\check{H}^p$ under certain conditions) and calculate Čech cohomology instead. Let $U_i = D_+(t_i) \subset \mathbb{P}^n_k$, and let $U = (U_i)_{i \in \{0, \cdots, n\}}$ be the covering by the U_i. An element $s \in H^0(X, \mathcal{O}_X(d))$ can be identified with (s_i) where $s_i \in \mathcal{O}_X(d)(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Remember, $\mathcal{O}_X(d)(U_i) = \widetilde{S(d)}(U_i) = S(d)_{(t_i)}$. Put $s_i = \frac{f_i}{t_i^\ell}$ where f_i is homogeneous of degree $d+\ell$ as an element of S.

$$\left.\frac{f_i}{t_i^{\ell_i}}\right|_{U_i\cap U_j} = \left.\frac{f_j}{t_j^{\ell_j}}\right|_{U_i\cap U_j} \iff \frac{f_i}{t_i^{\ell_i}} = \frac{f_j}{t_j^{\ell_j}} \text{ in } S(d)_{(t_it_j)} = \mathcal{O}_X(d)(U_i\cap U_j)$$
$$\iff \frac{f_i}{t_i^{\ell_i}} = \frac{f_j}{t_j^{\ell_j}} \text{ in } S_{(t_it_j)} \subset k(t_0,\cdots,t_n)$$
$$\iff f_i t_j^{\ell_j} = f_j t_i^{\ell_i} \text{ in } k[t_0,\cdots,t_n]$$

Since $k[t_0,\cdots,t_n]$ is a UFD (it's not a UFO!), $t_j^{\ell_j} \mid f_j$ and $t_i^{\ell_i} \mid f_i$, so there is some homogeneous degree-d polynomial g such that $g = \frac{f_i}{t_i^{\ell_i}} = s_i$ for all i.

Conversely, any homogeneous g of degree d gives a section $s=(s_i)$ in $\mathcal{O}_X(d)(X)$ where we can put $s_i = \frac{g}{1}$. In particular, if $d<0$, $H^0(X,\mathcal{O}_X(d)) = 0$. And if $d=0$, then $H^0(X,\mathcal{O}_X(d)) = H^0(X,\mathcal{O}_X) \cong k$.

(2) Since $H^p(X,\mathcal{O}_X(d)) \cong \check{H}^p(U,\mathcal{O}_X(d))$, and since the number of the U_i is $n+1$, $C^p(U,\mathcal{O}_X(d)) = 0$ if $p>n$ (there aren't enough open sets to intersect). Hence $\check{H}^p(U,\mathcal{O}_X(d)) = 0$ and so $H^p(X,\mathcal{O}_X(d)) = 0$ when $p>n$.

(3) We only prove the case when $d \geq -n-1$ (the other cases are similar, but require more combinatorics). We use Čech cohomology. The end of the Čech complex looks like

$$\cdots \to \underbrace{C^{n-1}(U,\mathcal{O}_X(d))}_{\prod_{i_0<\cdots<i_{n-1}} S(d)_{(t_{i_0}\cdots t_{i_{n-1}})}} \xrightarrow{\delta^{n-1}} \underbrace{C^n(U,\mathcal{O}_X(d))}_{S(d)_{(t_0\cdots t_n)}} \to 0$$

To calculate $H^n(X,\mathcal{O}_X(d)) = \check{H}^n(U,\mathcal{O}_X(d))$ we need to calculate $\operatorname{im}\delta^{n-1}$. To do this, we will try to find elements outside the image.

It's enough to consider elements $\alpha = \frac{t_0^{m_0}\cdots t_n^{m_n}}{(t_0\cdots t_n)^\ell} \in S(d)_{(t_0\cdots t_n)}$ where $\sum m_i = d+(n+1)\ell$. If $m_i \geq \ell$, then we could cancel t_i from the denominator. But then this element would definitely be in the image (it comes from the factor of C^{n-1} corresponding to a missing U_i). So we can assume $m_i < \ell$ for all i. Also, $0<\ell$ and assume there is some i such that $m_i = 0$: otherwise, we can make ℓ smaller.

So

$$d+(n+1)\ell = \sum m_i \leq n(\ell-1) = n\ell - n$$

so $d+\ell \leq -n$. By the assumption that $d \geq -n-1$, we have that $\ell = 1$. Then there is only one possibility: $m_i = 0$ for all i, and $\alpha = \frac{1}{t_0\cdots t_n}$. (This happens when $d=-n-1$.)

I claim that α is not in the image (we showed that everything else is). Finish this yourself.

To summarize:

$$H^0(X, \mathcal{O}_X(d)) = \check{H}^n(U, \mathcal{O}_X(d)) = \begin{cases} 0 & \text{if } d > -n-1 \\ k & \text{if } d = -n-1 \end{cases}$$

(4) Let $I = \langle t_n \rangle \triangleleft S$ and let Y be the closed subscheme defined by $\mathcal{I} = \widetilde{I}$; this is a coherent ideal sheaf. (Recall that closed subschemes are in 1-1 correspondence with ideal sheaves.) Let $f : Y \to X$ be the corresponding closed immersion. We then have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0 \tag{21.1}$$

(surjectivity by definition of being a closed immersion). This can be written as

$$0 \to \widetilde{\langle t_n \rangle} \to \widetilde{S} \to \widetilde{T} \to 0$$

where $T = S/\langle t_n \rangle$.

We have an isomorphism $S(-1) \overset{\cong}{\to} \langle t_n \rangle$ where $s \mapsto t_n \cdot s$ (the rings are all integral domains). The -1 is there to make the degrees work out: when you multiply by t_n, the degree increases by 1.

(21.1) can be written as

$$0 \to \widetilde{S(-1)} \xrightarrow{-\cdot t_n} \widetilde{S} \to \widetilde{T} \to 0$$

We get

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

Now tensor with $\mathcal{O}_X(d)$ (this is an invertible sheaf; it is locally free of rank 1 and when you tensor, you're just tensoring by the ring itself):

$$0 \to \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-1) \to \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to 0$$

which can be rewritten as

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to f_*\mathcal{O}_Y(d) \to 0$$

Recall $Y \cong \mathbb{P}^{n-1}_k$, because $T \cong k[u_0, \cdots, u_{n-1}]$. (TBC)

Lecture 22: November 23

(Proof of theorem, continued): $X = \mathbb{P}^n_k = \operatorname{Proj} S$, $S = k[t_0, \cdots, t_n]$, $U_i = D_+(t_i)$. We're proving $H^p(X, \mathcal{O}_X(d)) = 0$, for all $0 < p < n$ and all d.

Let Y be the closed subscheme defned by $\langle t_n \rangle$, and $f : Y \hookrightarrow X$ be the inclusion. We have an exact sequence

$$0 \to \mathcal{I}_Y = \widetilde{\langle t_n \rangle} = \mathcal{O}_X(-1) \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

After tensoring with $\mathcal{O}_X(d)$ we get

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to f_*\mathcal{O}_Y(d) \to 0$$

Recall that $Y = \operatorname{Proj} T$ where $T = S/\langle t_n\rangle$. We get a long exact sequence

$$0 \to H^0(X, \mathcal{O}_X(d-1)) \to H^0(X, \mathcal{O}_X(d)) \to \underbrace{= H^0(X, f_*\mathcal{O}_Y(d))}_{H^0(Y,\mathcal{O}_Y(d))}$$

$$\xrightarrow{\beta} H^1(X, \mathcal{O}_X(-1)) \xrightarrow{\alpha} H^1(X, \mathcal{O}_X(d)) \to \cdots$$

$$\cdots \to H^{n-1}(X, \mathcal{O}_X(d)) \to \underbrace{= H^{n-1}(X, f_*\mathcal{O}_Y(d))}_{H^{n-1}(Y,\mathcal{O}_Y(d))} \xrightarrow{\alpha'} H^n(X, \mathcal{O}_X(d)) \to \underbrace{= H^n(X, f_*\mathcal{O}_Y(d))}_{0}$$

See example sheet 3, problem 8 for the equalities. (The last term is zero because you can write it as cohomology of Y, by the problem set, and Y has dimension $< n$.) We know the dimensions of the first three and last four terms (by last lecture). In particular, α and α' are injective. Also β is zero, and $H^1(X, \mathcal{O}_X(-1)) = 0$. By induction on n, $H^p(Y, \mathcal{O}_Y(d)) = H^p(X, f_*\mathcal{O}_Y(d)) = 0$ for all $p, 0 < p < n-1$ and all d.

Thus all the maps $H^p(X, \mathcal{O}_X(d-1)) \xrightarrow{\beta^p} H^p(X, \mathcal{O}_X(d))$ are isomorphisms, for every $p, 0 < p < n$. The goal is to prove that one of these spaces is zero. Using Čech cohomology, we can see that β^p is induced by the maps

$$\mathcal{O}_X(d-1)(U_{i_0,\cdots,i_p}) \to \mathcal{O}_X(d)(U_{i_0,\cdots,i_p})$$

which can be written as a map

$$S(d-1)_{(t_{i_0}\cdots t_{i_p})} \to S(d)_{(t_{i_0}\cdots t_{i_p})}$$

But this map is simply multiplication by t_n (because we said earlier that the isomorphism $S(-1) \to \langle t_n\rangle$ was multiplication by t_n). From the Čech complexes we deduce that β^p is also multiplication by t_n.

Let $\mathcal{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_X(d) = \widetilde{\bigoplus_{d\in\mathbb{Z}} S(d)}$. We will calculate the Čech complex of $\mathcal{F}$.

$$\mathcal{F}(U_{i_0}, \cdots, i_p) = \bigoplus_{d\in\mathbb{Z}} S(d)_{(t_{i_0}\cdots t_{i_p})} \xrightarrow{\cong} S_{t_{i_0}\cdots t_{i_p}}$$

where the isomorphism is $(\lambda_d) \mapsto \sum \lambda_d$ (the second localization includes only fractions of total degree zero, while the third one does not). The Čech complex is then

$$0 \to \underbrace{C^0(\mathscr{U}, \mathcal{F})}_{\prod S_{t_{i_0}}} \to \underbrace{C^1(\mathscr{U}, \mathcal{F})}_{\prod S_{t_{i_0}t_{i_1}}} \to \cdots$$

Now localize this complex at t_n (check that $(\prod S_{t_{i_0}})_{(t_n)} = \prod S_{t_{i_0}t_n}$):

$$0 \to \prod S_{t_{i_0}t_n} \to \prod S_{t_{i_0}t_{i_1}t_n}$$

This is simply the Čech complex of $\mathcal{F}|_{U_n}$ using the cover $\mathscr{U}' = (U_0 \cap U_n, \cdots, U_n \cap U_n)$.

But U_n is affine, $\mathcal{F}|_{U_n}$ is quasicoherent, so

$$\check{H}^p(\mathscr{U}', \mathcal{F}|_{U_n}) = H^p(U_n, \mathcal{F}|_{U_n}) = 0 \;\forall p > 0$$

so

$$H^p(X, \mathcal{F})_{t_n} = 0 \;\forall p, 0 < p < n.$$

Equivalently, for all $w \in H^p(X,\mathcal{F})$, then there is some r such that $t_n^r w = 0$. Since $H^p(X,\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}} H^p(X,\mathcal{O}_X(d))$, for all $w \in H^p(X,\mathcal{O}_X(d))$.

Recall that the β^p were all multiplication by t_n. By what we just showed, β^p eventually kills every element. In order for these to be an isomorphism,

$$H^p(X,\mathcal{O}_X(d-1)) \cong H^p(X,\mathcal{O}_X(d)) = 0$$

for all p, $0 < p < n$, for all d.

Theorem 22.1. *Let k be a field, $X \overset{f}{\hookrightarrow} \mathbb{P}_k^n$ a closed subscheme of $\mathbb{P}_k^n$ (i.e. X is projective), and $\mathcal{F}$ is a coherent sheaf on X. Then*

$$H^p(X,\mathcal{F}(d)) = 0$$

for all $p > 0, d \gg 0$, where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(d)$ and $\mathcal{O}_X(d) = f^*\mathcal{O}_{\mathbb{P}^n}(d)$.

Proof. We have (exercise) $f_*(\mathcal{F}(d)) = (f_*\mathcal{F})(d) := f_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(d)$. Since cohomology does not change when you push down – $H^p(X,\mathcal{F}(d)) \cong H^p(\mathcal{O}_k^n, (f_*\mathcal{F})(d))$ – we can replace X by $\mathbb{P}_k^n$ and $\mathcal{F}$ by $f_*\mathcal{F}$, and thereby assume that $X = \mathbb{P}_k^n$. Remember the fact:

Fact 22.2. *There are some m, ℓ (depending on the sheaf and the space) and an exact sequence*

$$\bigoplus_1^\ell \mathcal{O}_X \to \mathcal{F}(m) \to 0$$

(This should remind you of the following fact: if A is a ring, M is a finitely-generated A-module, then there is some ℓ such that $\bigoplus_1^\ell A \to M \to 0$.)

Let $\mathcal{G} = \ker\varphi$. We have the exact sequence

$$0 \to \mathcal{G} \to \bigoplus_1^\ell \mathcal{O}_X \xrightarrow{\varphi} \mathcal{F}(m) \to 0$$

Tensor the sequence with $\mathcal{O}_X(d-m)$ to get

$$0 \to \mathcal{G}(d-m) \to \bigoplus_1^\ell \mathcal{O}_X(d-m) \to \mathcal{F}(d) \to 0$$

Then, the long exact sequence of the above gives

$$H^p(X,\mathcal{G}(d-m)) \to H^p(X,\bigoplus \mathcal{O}_X(d-m)) \to H^p(X,\mathcal{F}(d)) \to H^{p+1}(X,\mathcal{G}(d-m))$$

If $p > 0$ and d is sufficiently large, by the theorem we proved in the previous lecture $H^p(X,\bigoplus \mathcal{O}_X(d-m)) = 0$.

Apply decreasing induction on p; we get $H^{p+1}(X,\mathcal{G}(d-m)) = 0$, if $p+1 > n$. $H^{p+1}(X,\mathcal{L}) = 0$ for every quasicoherent $\mathcal{L}$ because of Čech cohomology (because you don't have any more open sets to intersect). This forces $H^p(X,\mathcal{F}(d)) = 0$, for $p > 0$ and d sufficiently large. ○

Theorem 22.3. *Take the same setting as the previous theorem. Then $H^p(X,\mathcal{F})$ is a finite-dimensional k-vector space for every p.*

Proof. Almost identical to the previous proof: take $d = 0$, and try to trap your ring in between two zero rings. ○

Lecture 23: November 26

Euler characteristic and Hilbert polynomial.

Definition 23.1. Let X be a subscheme of $\mathbb{P}^n_k$ (k is a field), $\mathcal{O}_X(d) = f^*\mathcal{O}_{\mathbb{P}^n_k}(d)$, and $f : X \to \mathbb{P}^n_k$ a closed immersion. For a coherent sheaf on X we define the Euler characteristic as

$$\chi(X, \mathcal{F}) = \sum_{p \geq 0} (-1)^p \dim_k H^p(X, \mathcal{F})$$

If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of coherent sheaves on X, then we get a long exact sequence

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \cdots \to H^n(X, \mathcal{F}_3) \to 0.$$

From simple linear algebra, we get

$$\dim_k H^0(X, \mathcal{F}_1) - \dim_k H^0(X, \mathcal{F}_2) + \cdots = 0$$

So

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

We can generalize this to a longer exact sequence.

Proposition 23.2. *If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_m \to 0$$

is an exact sequence of coherent sheaves on X, then we have a similar formula:

$$\chi(X, \mathcal{F}_1) - \chi(X, \mathcal{F}_2) + \cdots = 0.$$

Proof. Proof by induction on m: if $\mathcal{G} = \mathrm{im}(\mathcal{F}_2 \to \mathcal{F}_3)$, then we have the two exact sequences

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{G} \to 0$$

$$0 \to \mathcal{G} \to \mathcal{F}_3 \to \mathcal{F}_4 \to \cdots \to \mathcal{F}_m \to 0$$

so

$$\chi(X, \mathcal{F}_1) - \chi(X, \mathcal{F}_2) + \chi(X, \mathcal{G}) = 0$$

and by induction,

$$\chi(X, \mathcal{G}) - \chi(X, \mathcal{F}_3) + \cdots = 0$$

Subtracting the second sequence from the first gives the result. ○

If $\mathcal{F}$ is coherent on X, then for d sufficiently large we have

$$\chi(X, \mathcal{F}(d)) = \dim H^0(X, \mathcal{F}(d))$$

THEOREM/ DEFINITION 23.3. Let $\mathcal{F}$ be coherent on X, a subscheme of $\mathbb{P}^n_k$. Then, the function

$$\chi(X,\mathcal{F}(-)) : \mathbb{Z} \to \mathbb{Z} \text{ where } d \mapsto \chi(X,\mathcal{F}(d))$$

is a polynomial in $\mathbb{Q}[d]$. This polynomial is called the *Hilbert polynomial of* $\mathcal{F}$.

(Note that this polynomial depends on $\mathcal{F}$, X, and the embedding $f : X \hookrightarrow \mathbb{P}^n_k$.)

Proof. By replacing X by $\mathbb{P}^n_k$, and replacing $\mathcal{F}$ by $f_*\mathcal{F}$, we can assume that $X = \mathbb{P}^n_k$ (show on the example sheet that pushing down doesn't change cohomology). Now do induction on n. Let Y be the closed subscheme of X defined by the ideal $\langle t_n \rangle$, where $\mathbb{P}^n_k = \operatorname{Proj} k[t_0, \cdots, t_n]$. Let $g : Y \hookrightarrow \mathbb{P}^n_k$ be the closed immersion. Since Y is defined by one equation, $Y \cong \mathbb{P}^{n-1}_k$.

If $n = 0$, $\chi(X,\mathcal{F}(d)) = \chi(X,\mathcal{F})$ is constant. So we can assume $n > 0$. We have the exact sequence

$$0 \to \widetilde{\langle t_n \rangle} \cong \mathcal{O}_X(-1) \to \mathcal{O}_X \to g_*\mathcal{O}_Y \to 0$$

(where $\langle t_n \rangle$ means the ideal $t_n \cdot k[t_0, \cdots, t_n]$). Tensor the sequence with $\mathcal{F}$ to get

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{F} \to g_*\mathcal{O}_Y \otimes \mathcal{F} \to 0$$

where $\mathcal{G}$ is just the kernel of $\mathcal{F}(-1) \to \mathcal{F}$. Now tensor with $\mathcal{O}_X(d)$:

$$0 \to \mathcal{G}(d) \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \underbrace{g_*\mathcal{O}_Y \otimes \mathcal{F}(d)}_{=g_*((g^*\mathcal{F})(d))} \to 0$$

Take Euler characteristic:

$$\chi(X,\mathcal{F}(d)) - \chi(X,\mathcal{F}(d-1)) = \chi(X, g_*(g^*\mathcal{F})(d)) - \chi(X,\mathcal{G}(d))$$

If we can show that each of the terms on the right are polynomials, then the LHS is a polynomial. This is sufficient, by Hartshorne I, 7.3.

$$\chi(X, g_*(g^*\mathcal{F})(d)) = \chi(Y, (g^*\mathcal{F})(d))$$

so this is a polynomial in $\mathbb{Q}[d]$ by induction.

So now it is enough to show that $\chi(X,\mathcal{G}(d))$ is in $\mathbb{Q}[d]$. For any $\mathcal{O}_X$-module $\mathcal{M}$, we have a natural homomorphism $\mathcal{M} \to g_*g^*\mathcal{M}$ (special case of example sheet 2, problem 10). We have the following exact sequences

$$0 \to \mathcal{G}_1 = \ker\varphi \to \mathcal{G} \xrightarrow{\varphi} g_*g^*\mathcal{G} \to 0$$

$$0 \to \mathcal{G}_2 = \ker\varphi_1 \to \mathcal{G}_1 \xrightarrow{\varphi_1} g_*g^*\mathcal{G}_1 \to 0$$

(Surjectivity of φ can be seen by looking at it locally.) If we show that $\mathcal{G}_\ell = 0$ for some ℓ, then we are done by tensoring each of the sequences with $\mathcal{O}_X(d)$: look at

$$0 \to \mathcal{G}_\ell(d) \to \mathcal{G}(d) \to g_*(g^*\mathcal{G})(d) \to 0$$

If the first term is zero, then $\mathcal{G}(d) \cong g_*(g^*\mathcal{G})(d)$.

We argue locally: for example, $D_+(t_0) \cong \operatorname{Spec} k[u_1, \cdots, u_n]$ where $u_i = \frac{t_i}{t_0}$. Suppose $\mathcal{G}|_{D_+(t_0)} \cong \widetilde{G}$ for some module $\mathcal{G}$. Then the above sequences are given on $D_+(t_0)$ as

$$0 \to \widetilde{\langle u_n \rangle G} \to \widetilde{G} \to \widetilde{G/\langle u_n \rangle G} \to 0$$
$$0 \to \widetilde{\langle u_n^2 \rangle G} \to \widetilde{\langle u_n \rangle G} \to \widetilde{\langle u_n \rangle G/\langle u_n^2 \rangle G}$$

etc. On the other hand, by construction, $\mathcal{G}|_{D_+(t_n)} = 0$ which implies that $\mathcal{G}|_{D_+(t_n)\cap D_+(t_0)} = 0$. So $G_{u_n} = 0$ which implies that there is some ℓ such that $\langle u_n^\ell \rangle G = 0$. Furthermore, $\langle u_n^m \rangle G = 0$ for any $m \geq \ell$.

So $\mathcal{G}_\ell|_{D_+(t_0)} = 0$ for all ℓ sufficiently large. The same arguments show that $\mathcal{G}_\ell|_{D_+(t_i)} = 0$ for all $\ell \gg 0$, and for all i. So $\mathcal{G}_\ell = 0$ for all $\ell \gg 0$. ○

Example 23.4. Let $X = \mathbb{P}^n_k$, $\mathcal{F} = \mathcal{O}_X$. If $d \gg 0$,

$$\begin{aligned}\chi(X, \mathcal{O}_X(d)) &= \dim_k H^0(X, \mathcal{O}_X(d)) \\ &= \#\text{ monomials of degree } d \text{ in } t_0, \cdots, t_n \\ &= \binom{d+n}{d} = \frac{(d+n)!}{d!n!} = \frac{1}{n!}(d+n)\cdots(d+1)\end{aligned}$$

(If you have two polynomials that are equal for an infinite number of integers, then they are equal.) In particular, deg $\chi(X, \mathcal{O}_X(d)) = n$.

Example 23.5. Let F be a homogeneous polynomial of degree r, and X the closed subscheme of $\mathbb{P}^n_k$ defined by $\langle F \rangle$. Let $\mathcal{F} = \mathcal{O}_X$. We have the following exact sequence

$$0 \to \widetilde{\langle F \rangle} \cong \mathcal{O}_{\mathbb{P}^n}(-r) \to \mathcal{O}_{\mathbb{P}^n} \to f_*\mathcal{O}_X \to 0$$

Tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-r) \to \mathcal{O}_{\mathbb{P}^n}(d) \to f_*\mathcal{O}_X(d) \to 0$$

We can calculate the Euler characteristic of X:

$$\begin{aligned}\chi(X, \mathcal{O}_X(d)) = \ \chi(\mathbb{P}^n, f_*\mathcal{O}_X(d)) &= \ \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) - \ \chi\ \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-r)) \\ &= \frac{1}{n!}(d+n)\cdots(d+1) - \frac{1}{n!}(d-r+n)\cdots(d-r+1)\end{aligned}$$

We can see $\chi(X, \mathcal{O}_X(d))$ is of degree $n-1$. If you define dimension precisely, you will see that X has dimension $n-1$.

Now assume $n = 2$: then

$$\chi(X, \mathcal{O}_X(d)) = \frac{1}{2}rd + 1 - \frac{1}{2}(r-1)(r-2)$$

Observations:

- $r = \deg F$ appears in the leading coefficient.
- $\frac{1}{2}(r-1)(r-2)$ arising in the constant term is called the *genus* of X. For example, if $k = \mathbb{C}$, and if X is smooth (i.e. $\Omega_{X/\operatorname{Spec} k}$ is locally finite), then X is a Riemann surface with the complex topology. Then the genus is the number of holes in the Riemann surface.

Lecture 24: November 28

Duality and Riemann-Roch Theorem.

Definition 24.1. Throughout this lecture, k will be an algebraically closed field.

- A *quasi-projective variety* over k is an integral scheme X with a quasi-projective morphism $X \to \operatorname{Spec} k$.
- A *projective variety* over k is an integral scheme X with a projective morphism $X \to \operatorname{Spec} k$.
- We say that a quasi-projective variety is *smooth* if the following equivalent conditions hold:
  - $\mathcal{O}_x$ is a regular ring for every $x \in X$
  - $\Omega_{X/\operatorname{Spec} k}$ is a locally free sheaf on X
- The *dimension* of a quasi-projective variety X is
$$\dim X = \max\{\dim \mathcal{O}_x \,:\, x \in X\}$$
(where dimension is Krull dimension).

If $X = \operatorname{Spec} A$, then $\dim X = \dim A$. In particular, $\dim \mathbb{P}^n_k = \dim \mathbb{A}^n_k = 1$. If $X = V(\langle f \rangle) \subset \mathbb{A}^n$ (where $f \neq 0$ and is not a unit), then $\dim V(I) = n - 1$.

Theorem 24.2 (Duality). *Assume that X is a smooth projective variety over k of dimension n. Then, there is a Cartier divisor K_X (called a canonical divisor) such that, for every locally free sheaf $\mathcal{F}$ on X we have*
$$\dim_k H^p(X, \mathcal{F}) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{F}^\vee)$$
where we define $\mathcal{F}^\vee = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

In particular, if $\mathcal{F} = \mathcal{O}_X(D)$ then the theorem says that
$$\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X - D)).$$

Definition 24.3 (Weil divisors). Suppose X is a quasi-projective variety. Then a *Weil divisor* on X is of the form $M = \sum_{finite} m_i M_i$ where $m_i \in \mathbb{Z}$ and M_i is a closed subscheme of X which is a quasi-projective variety of dimension $\dim X - 1$.

Now assume that X is smooth, and that $D = (U_\alpha, f_\alpha)$ is a Cartier divisor on X. Then we can define a Weil divisor M associated to D informally as the collection of zeroes and poles of f_α, counted with multiplicity. More precisely, for each U_α, we define a Weil divisor for f_α and the M will be the divisor obtained by putting together all the divisors on U_α. Define the divisor of f_α to be
$$M_\alpha = \sum_{\substack{M_{i,\alpha} \subset U_\alpha \\ \text{codim. } 1}} m_{i,\alpha} M_{i,\alpha}$$
and $m_{i,\alpha} = v_{i,\alpha}(f_\alpha)$ where $v_{i,\alpha}$ is the function $K(\eta_{i,\alpha}) \backslash \{0\} \to \mathbb{Z}$ where $\eta_{i,\alpha}$ is the generic point of $M_{i,\alpha}$ as follows:

(1) if $0 \neq g \in \mathcal{O}_{\eta_{i,\alpha}}$, then put $v_{i,\alpha}(g)$ to be the largest ℓ such that $g \in (\mathfrak{m}_{\mathcal{O}_{\eta_{i,\alpha}}})^\ell$
(2) for general $0 \neq h \in K(\eta_{i,\alpha})$, write $h = g/\ell$ for some $g, \ell \in \mathcal{O}_{\eta_{i,\alpha}}$ and put $v_{i,\alpha}(h) = v_{i,\alpha}(g) - v_{\ell,\alpha}(\ell)$

This is well-defined because $\mathcal{O}_{\eta_{i,\alpha}}$ is a regular Noetherian ring of dimension 1 ($\mathcal{O}_{\eta_{i,\alpha}}$ is a DVR).

Now further assume $\dim X = 1$ and X is projective. We define $\deg D = \sum m_i$ where $M = \sum m_i M_i$ is the Weil divisor associated to D. One can show that

$$D \sim D' \implies \deg D = \deg D'.$$

In particular, if $D \sim 0$ then $\deg D = 0$. (Think of a compact Riemann surface and a meromorphic function.)

Theorem 24.4 (Riemann-Roch). *Suppose X is a smooth projective variety of dimension 1. For every Cartier divisor D we have*

$$\chi(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D))$$

by the duality theorem. Then

$$\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - \dim_k H^0(X, \mathcal{O}_X(K_X)) = \textit{ genus}.$$

Sketch of proof. This is not examinable.

$$\begin{aligned}\chi(X, \mathcal{O}_X(D)) &= \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(1)) \\ &\overset{\textit{duality}}{=} \dim_k H^0(X, \mathcal{O}_X(1)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D))\end{aligned}$$

Note that $\mathcal{O}_X(K_X) = \Omega_{X/\operatorname{Spec} k}$ in dimension 1. Also note that $\mathcal{O}_{\mathbb{P}^n_k}(K_{\mathbb{P}^n_k}) = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$.

For the second equality, here is a sketch. Pick any closed point $x \in X$. We will construct a Cartier divisor D' such that $M' =$ the Weil divisor of D, which is just x. Since $\mathcal{O}_x$ is a DVR, the maximal ideal is $\langle t \rangle$ for some $t \in \mathcal{O}_x$. Then find some open affine $U \subset X$ such that $x \in U$, $t \in \mathcal{O}_X(U)$, $V_U(t) = \{x\}$. Put $W = X \backslash \{x\}$, and define D' to be given by (U, t) and $(W, 1)$. Then M' is the Weil divisor of D', which is just x (the only point you need to worry about vanishing is x).

We next define a morphism $\mathcal{O}_X(D') \to \mathcal{F}$, where $\mathcal{F}$ is the skyscraper sheaf at x define by $K(x) = k$. Define the morphism on open sets:

$$\mathcal{O}_X(D')(C) \to \mathcal{F}(C) \text{ is } \begin{cases} 0 & \text{if } x \notin C \\ e \mapsto \substack{\text{image of } et \text{ in } K(x) \text{ in } K(x) \\ \text{under the map } \mathcal{O}_X(C \cap U) \to K(x)} & \text{if } x \in C \end{cases}$$

It is easy to see that $\ker \varphi = \mathcal{O}_X$ and then φ is surjective.

Then we get an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D') \xrightarrow{\varphi} \mathcal{F} \to 0$$

By tensoring we get

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + D') \to \mathcal{F} \to 0$$

(because $\mathcal{F}$ is concentrated in only one point). Tensor again:

$$0 \to \mathcal{O}_X(D - D') \to \mathcal{O}_X(D) \to \mathcal{F} \to 0$$

So the theorem holds for D iff it holds for $D + D'$ iff it holds for $D - D'$.

Now, $\deg(D + D') = \deg D + 1$, and $\deg(D - D') = \deg D - 1$. Applying the above sequences finitely many times, we arrive at the case where $M =$ the Weil divisor of D is zero.

So we only have to do the case where $D \sim 0$. Then $\mathcal{O}_X(D) = \mathcal{O}_X$. But in this case,

$$\begin{aligned}\chi(X, \mathcal{O}_X) &= \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) \\ &\overset{duality}{=} \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^0(X, \mathcal{O}_X(K_X))\end{aligned}$$

It also turns out that $\dim_k H^0(X, \mathcal{O}_X)$, because X is projective and integral, so we have

$$\cdots = 1 - \dim_k H^0(X, \mathcal{O}_X(K_X)).$$

○

Appendix A: Examples classes

Here are my notes from this course's two examples classes, in which Prof. Birkar presented his solutions to many of the exercises on the examples sheets.

Example sheet 1

Question 7: Take Y to be one point, and $\mathcal{O}_Y$ a DVR R. There is no ring A such that $(Y, \mathcal{O}_Y) = (X = \operatorname{Spec} A, \mathcal{O}_X)$. If there were, then A has only one prime ideal. But $\mathcal{O}_X(X) = A = \mathcal{O}_Y(Y) = R$. This is a contradiction because R has two prime ideals.

Question 11: Prime ideals of $\mathcal{F}_p[t] = \{0, \langle f \rangle\}$ for f irreducible. $\dim k[t] = 1$ (no nontrivial prime is contained in another); so there can be no more primes.

If $x = 0$ then $\mathcal{O}_x$ is the fraction field of $\mathcal{F}_p[t]$, i.e. $\mathcal{F}_p(t)$. The maximal ideal is (t), so the residue field is just $\mathcal{F}_p$.

If $x \neq 0$ then $K(x) = \mathcal{O}_x/\mathfrak{m}_x = \mathcal{F}_p[t]_x/x_x = \mathcal{F}_p[t]/x$ is a field. If $x = \langle f \rangle$, then $K(x) = \mathcal{F}_p[t]/\langle f \rangle$ the natural homomorphism $\mathcal{F}_p \to K(x) = \mathcal{F}_p[t]/\langle f \rangle$ is a finite extension. If $\overline{\mathcal{F}_p}$ is the algebraic closure of $\mathcal{F}_p$, $K(x)$ can be injected into $\overline{\mathcal{F}_p}$ (the image of t gives an element of $\overline{\mathcal{F}_p}$). Conversely, if we pick $w \in \overline{\mathcal{F}_p}$, then w has a minimal polynomial, say h irreducible, and $x = \langle h \rangle$ is a prime ideal of $\mathcal{F}_p[t]$ with $K(x) \cong \mathcal{F}_p(w)$.

Question 12: If $P \in \operatorname{Spec} \mathbb{Z}[t]$, then $P = 0$, $P = \langle p \rangle$ (where $p \in \mathbb{Z}$ is prime), $P = \langle f \rangle$ where f is irreducible, or $P = \langle p, f \rangle$ for $p \in \mathbb{Z}$ prime and f monic, irreducible $\pmod p$.

This is sufficient, because $\dim \mathbb{Z}[t] = 2$ (note $\dim R[t] = \dim R + 1$ where R is Noetherian and finite-dimensional). Why monic? $P/\langle p\rangle$ is a prime of $\mathcal{F}_p[t]$, say $\langle g\rangle$, and $g \in P$.

Consider the morphism $\operatorname{Spec}\mathbb{Z}[t] \to \operatorname{Spec}\mathbb{Z}$ induced from $\mathbb{Z} \to \mathbb{Z}[t]$: $0 \mapsto 0$, $\langle p\rangle \mapsto \langle p\rangle$, $\langle f\rangle \mapsto 0$ (no nonzero multiple of f is an integer), and $\langle p, f\rangle \mapsto \langle p\rangle$ (otherwise the next bigger "prime" is $\mathbb{Z}$, but 1 is not in this ideal).

Question 13:

$$\operatorname{Spec}\mathbb{R}[t_1,t_2] = \{0, \langle f\rangle, \langle t_1 - a_1, t_2 - a_2\rangle, \langle f, g\rangle : \deg f = 1, \deg g = 2\}$$

where f, g are irreducible. Suppose $p \in \operatorname{Spec}\mathbb{R}[t_1,t_2]$. If $\dim \mathbb{R}[t_1,t_2]/p = 2$ then $p = 0$. If $\dim \mathbb{R}[t_1,t_2]/p = 1$ then $p = \langle f\rangle$ (for an irreducible f). If $\dim \mathbb{R}[t_1,t_2]/p = 0$ then $\mathbb{R} \to \mathbb{R}[t_1,t_2]/p$ is a finite extension of degree 1 or 2 (this is where the last two terms in the set come from).

$$\operatorname{Spec}\mathbb{C}[t_1,t_2] = \{0, \langle h\rangle, \langle t_1 - b_1, t_2 - b_2\rangle\}$$

where h is irreducible. This is simpler than the above, because $\mathbb{C}$ is algebraically closed, so any $\langle f, g\rangle$ as above can be factored into some $\langle h\rangle$.

Suppose $q \mapsto p$ under the morphism $\operatorname{Spec}\mathbb{C}[t_1,t_2] \to \operatorname{Spec}\mathbb{R}[t_1,t_2]$. If $q = 0$, then $p = 0$. If $q = \langle h\rangle$ is nonzero, then I claim $p = \langle f\rangle$ for some f. If $h = h' + ih''$ (for real polynomials h', h''), then ${h'}^2 + {h''}^2 \in P$. If these are nonzero, then there must be at least one nonzero element in P. So $P \neq 0$. We have the map $\mathbb{R}[t_1,t_2]/p \hookrightarrow \mathbb{C}[t_1,t_2]/q$ where the latter is one-dimensional. Now what to do?

Question 15: Define a map $f : \operatorname{Spec} A_b \to D(b)$ where $P_b \mapsto P$. This s a 1-1 map. f is also continuous: $f^{-1}(D(c)) = D(\frac{c}{1}) \subset \operatorname{Spec} A_b$ (where $D(c) \subset D(b)$). (Remember that $D(c) \subset D(b) \iff \sqrt{\langle c\rangle} \subset \sqrt{\langle b\rangle}$.) f^{-1} is also continuous for similar reasons. Define

$$\varphi : \mathcal{O}_{D(b)} \to f_*\mathcal{O}_{\operatorname{Spec} A_b}$$

by defining it on sets of the form $D(c) \subset D(b)$. Do this by considering $A_c = \mathcal{O}_{D(b)}(D(c)) \to f_*\mathcal{O}_{\operatorname{Spec} A_b}(D(c)) = \mathcal{O}_{\operatorname{Spec} A_b}(D(c)) = (A_b)_c = A_c$ (last equality because $\sqrt{\langle c\rangle} \subset \sqrt{\langle b\rangle}$ means $c = db^r$). So there is an isomorphism $\mathcal{O}_{D(b)}D(c) \to f_*\mathcal{O}_{\operatorname{Spec} A_b}D(c)$ induced by the identity $A_c \cong A_c$. So φ can be extended to every open set and it is an isomorphism.

Y is a scheme, and $U \subset Y$ is an open subset. Let $p \in U$. By the definition of schemes, there is an open affine scheme $V \subset Y$ such that $p \in V = \operatorname{Spec} B$. There is some $b \in B$ such that $p \in D(b) \subset U \cap V$. Take $W = D(b)$ which is affine because $D(b) \cong \operatorname{Spec} B_b$.

Question 16: (i) Assume that α is injective. For any $a \in A$, $A_a = \mathcal{O}_X(D(a)) \to f_*\mathcal{O}_Y(D(a)) = \mathcal{O}_Y D(\alpha(a)) = B_{\alpha_a}$. We know that $A_a \hookrightarrow B_{\alpha(a)}$ is injective. So $|o_X \to f_*\mathcal{O}_Y$ is injective (because the sets $D(a)$ form a base for the topology on X).

Assume $\varphi : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Then it is injective on global sections $A = \mathcal{O}_X(X) \hookrightarrow f_*\mathcal{O}_Y(X) = \mathcal{O}_Y(Y) = B$.

(ii) Assume α is surjective. Then there is some $I \subset A$ such that $B = A/I$. f is just the closed immersion $\operatorname{Spec} A/I \to \operatorname{Spec} A$. So f is a homeomorphism onto $V(I)$, and φ is surjective.

Now assume f is a homeomorphism onto $Z \subset X$, and φ is surjective. Let $I = \ker \alpha$. Then we have morphisms $\operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} A/I \xrightarrow{h} \operatorname{Spec} A$ where the composition is f. ($A/I \to B$ is injective.) By (i), $\mathcal{O}_{\operatorname{Spec} A/I} \to g_*\mathcal{O}_{\operatorname{Spec} B}$ is injective. Since $\mathcal{O}_{\operatorname{Spec} A} \to h_*\mathcal{O}_{\operatorname{Spec} A/I}$ is surjective and $\mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} B} = h_*(g_* \operatorname{Spec} B)$ is surjective, we have that $\mathcal{O}_{\operatorname{Spec} A/I} \to g_*\mathcal{O}_{\operatorname{Spec} B}$ is surjective. Together with (), $\mathcal{O}_{\operatorname{Spec} A/I} \to g_*\mathcal{O}_{\operatorname{Spec} B}$ s an isomorphism. $B = A/I$ and therefore $\alpha : A \to B$ is surjective.

Question 17: Define $f : U \to \operatorname{Proj} S$ where $p \mapsto \alpha^{-1}P$. Define $\varphi : \mathcal{O}_{\operatorname{Proj} S} \to f_*\mathcal{O}_U$ by considering open sets of the form $D_+(b)$ and diagrams

$$\begin{tikzcd}
S_{(b)} = \mathcal{O}_{\operatorname{Proj} S} D_+(b) \arrow[rr] \arrow[dr] & & T_{(\alpha(b))} = \mathcal{O}_{\operatorname{Proj} T}(D_+(\alpha(b))) \arrow[dl, "restr."] \\
& f_*\mathcal{O}_U(D_+(b)) = \mathcal{O}_{\operatorname{Proj} T}(U \cap D_+(\alpha(b))) &
\end{tikzcd}$$

Example: Let $T = \mathbb{C}[t_1, t_2]$, $S = T$ except $T_1 = 0$. If $S_d \to T_d$ is an isomorphism for $d \geq n$ (for fixed n), then $U = \operatorname{Proj} T$ and f is an isomorphism.because for any b homogeneous of degree ≥ 2, $D(b)$ for such b cover $\operatorname{Proj}(S)$ and coker $\operatorname{Proj} T$ and $S_{(b)} \cong T_{(b)}$. (If $\frac{t}{b^r} \in T_{(b)}$ then $\frac{t}{b^r} \in S_{(b)}$.)

More generally, you can drop as many degrees as you want (finite), and this will still work as an example.

Question 18: Let $X = \operatorname{Spec}(K \oplus K)$ for a field K; this consists of two points $\{a, b\}$. But $\mathcal{O}_x = K$ for all $x \in X$. So $\mathcal{O}_x$ is an integral domain, but X is not integral because $\mathcal{O}_X(X)$ is not integral.

Question 20: Let K be a field. Take $X = \operatorname{Spec}(K \oplus K[t]/\langle t^2 \rangle)$. This consists of two points $\{a, b\}$ (where a is the prime ideal corresponding to K, and b comes from $\operatorname{Spec} K[t]/\langle t^2 \rangle$). $\mathcal{O}_a = K$ (an integral domain), but $\mathcal{O}_b = \left(K[t]/\langle t^2 \rangle\right)_{\langle t \rangle}$ has a nilpotent element (the class of t).

Another example: $X = \operatorname{Spec} \mathbb{C}[t_1, t_2]/\langle t_1^2, t_1 t_2 \rangle$, where the anomalous point is the origin.

Example sheet 2

Question 3: We have a diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} K(y) \arrow[r] & Y \end{tikzcd}$$

We get a map $h : X_y \to f^{-1}\{y\}$. Since these are all continuous maps, we get that h is continuous. We need that h is a homeomorphism. This is a local problem, so we can assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$. Then we have the diagram

$$\begin{tikzcd} B_y/\mathfrak{m}_y B_y & B_y \arrow[l] & B \arrow[l] \\ K(y) \arrow[u] & A_y \arrow[l] \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

where $\mathfrak{m}_y \subset A_y$ is the maximal ideal. The fibre is given by the prime ideals of $B_y/\mathfrak{m}_y B_y$. This is in 1-1 correspondence with

$$\{primes\ P_y \subset B_y \,:\, \mathfrak{m}_y B_y \subset P_y\}$$

and this is in 1-1 correspondence with

$$\{primes\ P \subset B \,:\, f(p) = y\} = f^{-1}\{y\}.$$

This shows that h is a 1-1 map.

To show h is a homeomorphism, it is enough to show that h^{-1} is continuous (the image of any closed [open] set is closed [open]). Take a closed subset of X_y which is of the form $V(I_y/\mathfrak{m}_y B_y)$ for some ideal $I \subset B$.

$$h(V(I_y/\mathfrak{m}_y B_y)) = \{P \subset B \,:\, I \subset P, f(P) = y\} = f^{-1}\{y\} \cap V(I)$$

and that's closed.

Question 4: Assume $X \to Y$ is an open immersion. We have a diagram

$$\begin{tikzcd} g^{-1}X \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

For any scheme W and a diagram

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

we get a larger diagram (since $e(W) \subset g^{-1}X$)

$$\begin{tikzcd}
W \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& g^{-1}X \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

so $g^{-1}X$ satisfies the universal property of products, so $g^{-1}X = X \times_Y Z$.

Now assume $X \to Y$ is a closed immersion: to show $X \times_Y Z \to Z$ is a closed immersion is a local problem. So we can assume $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$. Moreover, we can write $C = A/I$. So $X \times_Y Z \to Z$ is just

$$\operatorname{Spec} B/IB = \operatorname{Spec}(A/I \otimes_A B) \to \operatorname{Spec} B$$

So $X \times_Y Z \to Z$ is a closed immersion.

Question 5: The diagram

$$\begin{tikzcd}
A[t_0, \cdots, t_n] \arrow[r] & \mathbb{Z}[t_0, \cdots, t_n] \\
A \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

gives (using problem 8 on the first sheet)

$$\begin{tikzcd}
\mathbb{P}^n_A = \operatorname{Proj} A[t_0, \cdots, t_n] \arrow[r] \arrow[d] & \operatorname{Proj} \mathbb{Z}[t_0, \cdots, t_n] = \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

By the universal property of products, I get a diagram

$$\begin{tikzcd}
\mathbb{P}^n_A \arrow[dr, "h"] \arrow[drr] \arrow[ddr] & & \\
& Y \times \mathbb{P}^n_{\mathbb{Z}} = \mathbb{P}^n_Y \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
& Y = \operatorname{Spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

It is enough to show that h is an isomorphism. This can be done locally on the open sets $D_+(t_i)$. This boils down to

$$\operatorname{Spec} \mathbb{Z}[u_1, \cdots, u_n] \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A \cong \operatorname{Spec}(\mathbb{Z}[u_1 \cdots, u_n] \otimes_{\mathbb{Z}} A) \cong \operatorname{Spec} A[u_1, \cdots, u_n]$$

Question 6: The fibre Y_y over $y = \langle s - a \rangle$ is, by definition,

$$\operatorname{Spec} K(y) \times_{\mathbb{A}^1_k} Y = \operatorname{Spec}(K(y) \otimes_{k[s]} k[s, t_1, t_2] / \langle h \rangle)$$

$K(y) = k[s]/\langle s-a\rangle \cong k$ via $s \mapsto a$. So Y_y can be rewritten

$$\operatorname{Spec} k[t_1,t_2]/\left\langle t_2^2 - t_1(t_1-1)(t_1-a)\right\rangle$$

So this is naturally a closed subscheme.

Question 7: We show that the map $D(t) \overset{f}{\hookrightarrow} \mathbb{A}^1_k = \operatorname{Spec} k[t]$ is not projective. Assume that f is projective: there is a diagram

$$\begin{tikzcd} D(t) \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{P}^n_{\mathbb{A}^1_k} \arrow[d] \\ & \mathbb{A}^1_k \end{tikzcd}$$

for some map where g is a closed immersion. f corresponds to the homomorphism

$$k[t]_t \leftarrow k[t] = A$$

Let $y = \langle t\rangle$. The homomorphism $A \to A_y$ gives a diagram

$$\begin{tikzcd} & \mathbb{P}^n_Z = Z \times \mathbb{P}^n_{\mathbb{A}^1} \arrow[rr] \arrow[dd] & & \mathbb{P}^n_{\mathbb{A}^1_k} \arrow[dd] \\ Z \times_{\mathbb{A}^1} D(t) \arrow[ur, "cl.imm."] \arrow[rr] \arrow[dr, "open\ imm."'] & & D(t) \arrow[ur, "g"] \arrow[dr, "f"] & \\ & Z = \operatorname{Spec} A_y = \operatorname{Spec} k[t]_{\langle t\rangle} \arrow[rr] & & \mathbb{A}^1_k \end{tikzcd}$$

$k[t]_{\langle t\rangle}$ is a DVR so $Z \times_{\mathbb{A}^1} D(t) = \operatorname{Spec} L$ where L is the fraction field of A_y. So we have

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r, "e"] \arrow[dr, "op.imm\ d"'] & \mathbb{P}^n_Z \arrow[d, "\pi"] \\ & Z \end{tikzcd}$$

Since e is a closed immersion, $x = h(\operatorname{Spec} L)$ is a closed point n $\mathbb{P}^n_Z$. But the image in Z is not a closed point.

It is enough to show that $\pi(x)$ is a closed point, because then we get a contradiction, since $d(\operatorname{Spec} L)$ is the generic point.

Take an affine neighborhood U of x ($U = \operatorname{Spec} B$) and consider the diagram

$$\begin{tikzcd} & B \arrow[dl] \\ L & A_y \arrow[l, "\alpha"] \arrow[u, "\beta"'] \end{tikzcd}$$

B contains x, the inverse image of the zero ideal in L. $\ker\alpha$ is a maximal ideal $\beta^{-1}x$ which implies that $\pi(x)$ is a closed point.

Question 8: Proposition 3.2 in Hartshorne chapter II.

Question 10: The idea is that there are naturally-defined morphisms as follows:

$$f^* f_* \mathcal{F} \xrightarrow{\varphi} \mathcal{F}$$

$$\mathcal{G} \xrightarrow{\psi} f_* f^* \mathcal{G}$$

Then you can define maps

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

where

$$(f^*\mathcal{G} \to \mathcal{F}) \to (f_* f^* \mathcal{G} \to f_*\mathcal{F}) \to \text{ composition with } \mathcal{G} \to f_* f^* \mathcal{G}$$

In the other direction,

$$(\mathcal{G} \to f_*\mathcal{F}) \to f^*\mathcal{G} \to f^* f_* \mathcal{F} \to \text{ composition with } f^* f_* \mathcal{F} \to \mathcal{F}$$

Question 13: The general case is treated in Hartshorne (II, Prop. 5.7).

Assume that X is Noetherian. We can assume that $X = \operatorname{Spec} A$, where A is Noetherian. By assumption, $\mathcal{F} = \widetilde{M}$, $\mathcal{E} = \widetilde{N}$, for some A-modules M and N. We have the exact sequence

$$0 \to M = \mathcal{F}(X) \to \mathcal{G}(X) \to N = \mathcal{E}(X) \to H^1(X, \mathcal{F})$$

But $H^1(X, \mathcal{F}) = 0$ when $X = \operatorname{Spec} A$ and A is Noetherian. Then we get

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M} \arrow[r] \arrow[d, "\cong"] & \widetilde{\mathcal{G}(X)} \arrow[r] \arrow[d, "\varphi"] & \widetilde{N} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E} \arrow[r] & 0
\end{tikzcd}$$

using question 12. So φ is an isomorphism.

Question 17: It's a local issue, so assume $X = \operatorname{Spec} A, \mathcal{F} = \widetilde{M}$; assume we're in some $D(b)$ so elements of $\widetilde{M}(D(b))$ correspond to elements of M

- $\bigoplus_1^n \mathcal{O}_X = \bigoplus_1^n m_i' \cdot \mathcal{O}_X \cong M_x$
- $\bigoplus_1^n A \to M = \mathcal{F}(X)$ (send $(a_1, \cdots, a_n) \mapsto \sum a_i m_i$)
- $\varphi : \bigoplus_1^n \mathcal{O}_X \to \mathcal{F}$ by Question 12
- Kernel and cokernel are $\widetilde{K}$ and $\widetilde{C}$ become zero after localizing, so are zero on some neighborhood.

Question 18: This is a local problem, so we can assume that X and Y are both affine: $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $\mathcal{G} = \bigoplus_1^n \mathcal{O}_Y = \widetilde{\bigoplus_1^n A}$. Then $f^*\mathcal{G} = \widetilde{(\bigoplus A)} \otimes_A B \cong \widetilde{\bigoplus_1^n B} = \bigoplus_1^n \widetilde{B} = \bigoplus_1^n \mathcal{O}_X$.

Question 19: Take $S = \mathbb{C}[t_0, t_1]$, $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj} S$. $N = S$, $M = S$ with degree 1 stuff removed (i.e. declare $S_1 = 0$). So $\widetilde{M} = \widetilde{N} = \mathcal{O}_{\mathbb{P}^1_k}$, because for any homogeneous $b \in S$ of degree ≥ 2, $M_{(b)} \cong N_{(b)}$; so the morphism $\widetilde{M} \to \widetilde{N}$ is an isomorphism.

Question 20: Straightforward – take a covering by open affines.

Example sheet 3

Question 2: It is enough to find D such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(1)$ (then take tensor powers). $X = \operatorname{Proj} k[x_0, \cdots, s_n]$. Let $U_i = D_+(s_i)$ and $f_i = \frac{s_0}{s_i} \in K$. Let $D = (U_i, f_i)$; this is the right Cartier divisor. Let K be the function field. By definition

$$\mathcal{O}_X(D)(U_i) = \{h \in K \ : \ h\frac{s_0}{s_i} \in \mathcal{O}_X(U_i)\}$$

and $\mathcal{O}_X(1)(U_i) = S(1)_{s_i}$. Define a morphism

$$\varphi_i : \mathcal{O}_X(D)(U_i) \to \mathcal{O}_X(1)(U_i)$$

by sending $h \mapsto h \cdot s_0$. Write $h = \frac{F}{G}$ where F, G are homogeneous of the same degree. The φ_i give isomorphisms

$$\widetilde{\varphi}_i : \mathcal{O}_X(D)|_{U_i} \to \mathcal{O}_X(1)|_{U_i}$$

which are compatible on $U_i \cap U_j$ and this gives an isomorphism $\mathcal{O}_X(D) \to \mathcal{O}_X(1)$.

Question 3: Let $X = Y = \operatorname{Proj} k[s_0, \cdots, s_n]/S$. Define a homomorphism $\alpha : S \to S$ where $s_i \mapsto s_i^m$. If α preserved degrees then $\mathcal{O}(1)$ would pull back to $\mathcal{O}(1)$; but that's not what we want. This induces a morphism $f : X \to Y$. Recall from Q2, if $D = (U_i, \frac{s_0}{s_i})$ then $\mathcal{O}_X(D) \cong \mathcal{O}_X(1)$. Now $f^{-1}U_i = U_i$ because $U_i = D_+(s_i)$, so

$$f^{-1}D_+(s_i) = D_+(\alpha(s_i)) = D_+(s_i^m) = D_+(s_i) = U_i$$

So, $f^*D = (U_i, \left(\frac{s_0}{s_i}\right)^m)$ and so $\mathcal{O}_X(f^*D) \cong \mathcal{O}_X(m)$. Then $f^*\mathcal{O}_X(D) \cong \mathcal{O}_X(m)$.

Question 4: First we show $\operatorname{Pic}(\mathbb{A}^n) = 0$. By problem 1, it suffices to show $\operatorname{Div}(\mathbb{A}^n) = 0$. Let $D = (U_i, f_i)$ be a Cartier divisor on $\mathbb{A}^n$. Let K be the function field $k(t_1, \cdots, t_n)$; every f_i is just a quotient of polynomials. By Noetherian-ness, there are finitely many U_i's. We can find $h \in k[t_1, \cdots, t_n]$ such that $hf_i \in \mathcal{O}_{\mathbb{A}^n}(U_i)$ (clearing denominators). In particular, $\mathcal{O}_{\mathbb{A}^n}(-D) \subset \mathcal{O}_{\mathbb{A}^n}$. This is a coherent ideal sheaf so there is some ideal $I \leq k[t_1, \cdots, t_n]$ such that $\widetilde{I} = \mathcal{O}_{\mathbb{A}^n}(-D)$. Since $\widetilde{I}$ is an invertible sheaf, I is locally principal.

Let Z be an irreducible component of $\operatorname{Spec} k[t_1, \cdots, t_n]/I$, where Z has the unique structure of an integral scheme. Z corresponds to some prime ideal $P \leq k[t_1, \cdots, t_n]$. Since I is locally principal, $\dim Z = \dim \mathbb{A}^n - 1 = n - 1$ (by commutative algebra). P is principal: pick any irreducible $e \in P$; $\langle e \rangle$ is prime, so we have $0 \subset \langle e \rangle \subset P$ and that can only happen if $\langle e \rangle = P$. By construction, $V(P) \subset V(I)$, so

$$\langle e \rangle = P = \sqrt{P} \supset \sqrt{I} \supset I.$$

Now
$$J = \{a \in k[t_1, \cdots, t_n] \,:\, ae \in I\}$$
is an ideal. So $\widetilde{J}$ is an ideal sheaf which is $\mathcal{O}_{\mathbb{A}^n}(-D')$ where $D' = (U_i, \frac{f_i}{e})$. Repeat the argument until the ideal J is prinicpal. This finishes the proof because $D \sim 0 \iff D' \sim 0$.

Now we show that $\mathrm{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$. Again by problem 1 and theorems in lectures, $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathrm{Div}(\mathbb{P}^n_k)$. So it's enough to show $\mathrm{Div}(\mathbb{P}^n) = \mathbb{Z}$. Let D be a Cartier divisor on $\mathbb{P}^n$. Since $\mathrm{Div}(\mathbb{A}^n) = 0$ and $U_i = D_+(s_i) \cong \mathbb{A}^n$ we can write $D = (U_i, f_i)$. We could also assume $f_0 = 1$ (this is true up to linear equivalence). By definition, $\frac{f_i}{f_0}, \frac{f_0}{f_i} \in \mathcal{O}_{\mathbb{P}^n}(U_0 \cap U_i)$. Note $U_i \backslash U_0 = V(s_0)$. By arguments similar to the case $\mathbb{A}^n$, we could assume $f_i = \left(\frac{s_0}{s_i}\right)^{m_i}$ for some $m_i \in \mathbb{Z}$ since $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i} \in \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$. (So this has an inverse.) This implies that $m_i = m_j$ for all i, j. So there is some $m \in \mathbb{Z}$ such that $D = (U_i, f_i) = (U_i, \left(\frac{s_0}{s_i}\right)^m)$.

From problem 2, we see that $\mathcal{O}_{\mathbb{P}^n}(D) = \mathcal{O}_{\mathbb{P}^n}(m)$. So, $\mathrm{Pic}(\mathbb{P}^n)$ is generated by the single element $\mathcal{O}_{\mathbb{P}^n}(1)$. To show that $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, it suffices to show that $\mathcal{O}_{\mathbb{P}^n}(1)$ is not torsion; that is, that there is no $\ell \in \mathbb{Z}$ such that $\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes \ell} \cong \mathcal{O}_{\mathbb{P}^n}$: otherwise we could choose $\ell < 0$ and so
$$0 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) = H^0(\mathcal{O}^n, \mathcal{O}_{\mathbb{P}^n}) = k$$
which is a contradiction.

Question 5: Let $S = k[s_0, s_1]$. Remember that there is an exact sequence
$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$$
which is constructed by the homomorphism
$$\begin{aligned} S(-1) \oplus S(-1) &\xrightarrow{\alpha} S \\ (F_0, G_1) &\mapsto F_0 s_0 + F_1 s_1 \end{aligned}$$
And $\Omega_{X/Y} \cong \widetilde{\ker \alpha}$. If $(F_0, F_1) \in \ker \alpha$ then there is some G such that $(F_0, F_1) = (Gs_1, -Gs_0)$. So we can get an isomorphism
$$\begin{aligned} S(-2) &\to \ker \alpha \\ H &\mapsto (Hs_1, -Hs_0) \end{aligned}$$
So $\Omega_{X/Y} \cong \widetilde{S(-2)} \cong \mathcal{O}_X(-2)$.

Question 6: See solutions on Prof. Birkar's webpage.

Question 9: Let $\mathcal{K}$ be the constant sheaf defined by K. Then $\mathcal{K}$ is a flasque sheaf because X is integral *hence irreducible*. In fact $\mathcal{K}/\mathcal{O}_X$ is also a flasque sheaf. Let $U \subset X$ be an open set. If $U = X$ then $(\mathcal{K}/\mathcal{O}_X)(X) \to (\mathcal{K}/\mathcal{O}_X)(U)$ is surjective. If $U \neq X$ then, since K is algebraically closed, by a change of variables we can assume $U \subset D_+(s_0) \cong \mathbb{A}^1$ where U is affine. If $A = \mathcal{O}_X(U)$ then $\mathcal{K}/\mathcal{O}_X|_U = \widetilde{K/A}$, and also $\mathcal{K}(U) \to \mathcal{K}/\mathcal{O}_X(U)$ is surjective.

So we have

$$\begin{tikzcd} \mathcal{K}(X) \arrow[r] \arrow[d, two heads] & \mathcal{K}/\mathcal{O}_X(X) \arrow[d, two heads] \\ \mathcal{K}(U) \arrow[r, two heads] & \mathcal{K}/\mathcal{O}_X(U) \end{tikzcd}$$

So

$$0 \to \mathcal{O}_X \xrightarrow{d^0} \mathcal{K} \xrightarrow{d^1} \mathcal{K}/\mathcal{O}_X \to 0$$

is a flasque resolution. We can use this to calculate cohomology: In general we know $H^0(X, \mathcal{O}_X) = \ker d^0$. It is not hard to show that d^0 is surjective. This shows that $H^1(X, \mathcal{O}_X) = 0$.

Question 10: Trivial, since $\mathcal{F}$ is flasque.

Question 12: Let $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$. Let $x = \langle t \rangle$, $y = \langle t-1 \rangle$, and $f : \{x\} \to X$ and $g : \{y\} \to X$ be the closed immersions. Let $\mathcal{G}$ be the constant sheaf on X defined by $\mathbb{Z}$. Now we have natural morphisms

$$\mathcal{G} \to f_* f^{-1} \mathcal{G} = f_* \mathcal{G}_x$$
$$\mathcal{G} \to g_* g^{-1} \mathcal{G} = g_* \mathcal{G}_y$$

So we get a surjective morphism

$$\mathcal{G} \xrightarrow{\varphi} f_* \mathcal{G}_x \oplus g_* \mathcal{G}_y$$

giving an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \xrightarrow{\varphi} f_* \mathcal{G}_x \oplus g_* \mathcal{G}_y \to 0$$

where $\mathcal{F}$ is defined to be the kernel. I claim $\mathcal{F}$ has the required property.

By the long exact sequence of cohomology,

$$0 \to H^0(\mathcal{F}) \to \underbrace{H^0(\mathcal{G})}_{\mathbb{Z}} \to \underbrace{H^0(f_* \mathcal{G}_x \oplus g_* \mathcal{G}_y)}_{\mathbb{Z} \oplus \mathbb{Z}} \to H^1(\mathcal{F}) \to \underbrace{H^1(\mathcal{G})}_{0} \to \cdots$$

Since $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ can't be surjective, we have $H^1(\mathcal{F}) \neq 0$.

Question 15: Take any X and $\mathcal{F}$ such that $H^1(X, \mathcal{F}) \neq 0$. For example, take $X = \mathbb{P}^1$, $\mathcal{F} = \mathcal{O}_X(-3)$; by calculations we've done on projective space, this is $H^0(\mathcal{O}_X(-2+3)) \neq 0$. Now let $\mathscr{U} = (X)$ be the covering given by X alone. Then $\check{H}^p(\mathscr{U}, \mathcal{F}) = 0$ for all $p > 0$.

Question 16: Take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X (by finitely many open sets). Let $\mathcal{F}$ be a quasicoherent sheaf over X. Then $F|_{U_i} = \widetilde{M_i}$. By commutative algebra, there is an injective A_i-module I_i such that $M_i \subset I_i$. Now $\mathcal{I}_i = \widetilde{I_i}$ is injective in $\mathbf{Qcoh}(U_i)$. Moreover, $f_{i*}\mathcal{I}_i$ is injective in $\mathbf{Qcoh}(X)$: suppose we are given

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \\ & f_{i*}\mathcal{I}_i & \end{tikzcd}$$

Restrict to U_i to get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}|_{U_i} \arrow[r] \arrow[d] & \mathcal{N}|_{U_i} \arrow[dl, dotted] \\ & \mathcal{I}_i & \end{tikzcd}$$

where the dotted arrow exists by injectivity of $\mathcal{I}$. We have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\ 0 \arrow[r] & f_{i*}\mathcal{M}|_{U_i} \arrow[r] \arrow[d] & f_{i*}\mathcal{N}|_{U_i} \arrow[dl, dotted] \\ & f_{i*}\mathcal{I}_i & \end{tikzcd}$$

which gives a morphism $\psi : \mathcal{N} \to f_{i*}\mathcal{N}|_{U_i} \to f_{i*}\mathcal{I}_i$. Finally we have

$$0 \to \mathcal{F} \to \bigoplus f_{i*}\mathcal{F}_i|_{U_i} \subset \bigoplus f_{i*}\mathcal{I}_i$$

and the last term is injective in $\mathbf{Qcoh}(X)$.

Question 17: First we show that if

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{L} \to 0 \tag{A.1}$$

is exact in $\mathbf{Qcoh}(X)$ then

$$0 \to f_*\mathcal{M} \to f_*\mathcal{N} \to f_*\mathcal{L} \to 0$$

is exact in $\mathbf{Qcoh}(Y)$. This is a local problem: pick an open affine set $U \subset Y$. By assumption, $V = f^{-1}U$ is affine. Now

$$0 \to \mathcal{M}|_V \to \mathcal{N}|_V \to \mathcal{L}|_V \to 0$$

is exact corresponding to an exact sequence

$$0 \to M \to N \to L \to 0$$

of $\mathcal{O}_X(V)$-modules. Now take tildes:

$$0 \to f_*\mathcal{M}|_U \to f_*\mathcal{N}|_U \to f_*\mathcal{L}|_U \to 0$$

is an exact sequence considered as a sequence of $\mathcal{O}_Y(U)$-modules.

This is exact for all open affine sets, and therefore it's exact globally:

$$0 \to f_*\mathcal{M} \to f_*\mathcal{N} \to f_*\mathcal{L} \to 0$$

We can take an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ in $\mathbf{Qcoh}(X)$. By the proof of Question 16, we can assume the $\mathcal{I}^i$'s are also flasque. So,

$$0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots$$

is a flasque resolution of $f_*\mathcal{F}$. The following complexes are identical:

$$0 \to \mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \cdots \qquad \text{(calculates } H^p(X,\mathcal{F})\text{)}$$

$$0 \to f_*\mathcal{I}^0(Y) \to f_*\mathcal{I}^1(Y) \to \cdots \qquad \text{(calculates } H^p(Y, f_*\mathcal{F})\text{)}$$

so $H^p(X, \mathcal{F}) \cong H^p(Y, f_*\mathcal{F})$.

Question 19: Let $S = k[s_0, s_1, s_2]$. The ideal sheaf of X is $\widetilde{\langle F \rangle}$. We have the exact sequence

$$0 \to \langle F \rangle \to S \to S/\langle F \rangle \to 0$$

which is isomorphic to

$$0 \to S(-d) \overset{-\times F}{\to} S \to S/\langle F \rangle \to 0$$

We get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to f_*\mathcal{O}_X \to 0$$

where $f : X \to \mathbb{P}^2$ is the closed immersion. By the long exact sequence of cohomology,

$$\begin{aligned}
0 &\to \underbrace{H^0(\mathcal{O}_{\mathbb{P}^2}(-d))}_{0} \to \underbrace{H^0(\mathcal{O}_{\mathbb{P}^2})}_{k} \overset{\alpha}{\to} H^0(f_*\mathcal{O}_X) \overset{Q8}{=} H^0(\mathcal{O}_X) \\
&\to \underbrace{H^1(\mathcal{O}_{\mathbb{P}^2}(-d))}_{0} \to \underbrace{H^1(\mathcal{O}_{\mathbb{P}^2})}_{0} \to H^1(f_*\mathcal{O}_X) = H^1(\mathcal{O}_X) \\
&\overset{\beta}{\to} H^2(\mathcal{O}_{\mathbb{P}^2}(-d)) \to H^2(\mathcal{O}_{\mathbb{P}^2}) = 0
\end{aligned}$$

so α and β are isomorphisms.

So $\dim_k H^0(\mathcal{O}_X) = 1$ and

$$\begin{aligned}
\dim_k H^1(\mathcal{O}_X) &= \dim_k H^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = \dim_k H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) \\
&= \#\text{ monomials of degree } d-3 \\
&= \frac{1}{2}(d-1)(d-2)
\end{aligned}$$

Question 20: You can omit this question.

Example sheet 4

Question 1: In the sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 0 \tag{A.2}$$

note that the group structure is multiplicative. A Cartier divisor D is written as (U_i, f_i) for $f_i \in K$. We also determine $H^*(\mathcal{K}^*/\mathcal{O}^*)$. Pick $s \in H^*(\mathcal{K}^*/\mathcal{O}^*)$. There is an open cover $X = \bigcup V_\alpha$ and $t_\alpha \in \mathcal{K}^*(V_\alpha)$ such that $t_\alpha \mapsto s|_{V_\alpha}$. Of course s is determined by the V_α and t_α. Since $t_\alpha|_{V_\alpha \cap V_\beta}$ and $t_\beta|_{V_\alpha \cap V_\beta}$ both go to $s|_{V_\alpha \cap V_\beta}$. So $\frac{t_\alpha}{t_\beta}$ and $\frac{t_\beta}{t_\alpha}$ both map to $1 \in \mathcal{K}^*/\mathcal{O}^*(V_\alpha \cap V_\beta)$, and hence they are in $\mathcal{O}_X^*(V_\alpha \cap V_\beta)$. Then s determines the Cartier divisor (V_α, t_α).

This gives the 1-1 correspondence between Cartier divisors and elements in $H^0(\mathcal{K}^*/\mathcal{O}^*)$. The long exact sequence of cohomology of (A.2) gives

$$0 \to H^0(\mathcal{O}_X^*) \to \underbrace{H^0(\mathcal{K}^*)}_{K^*} \to H^0(\mathcal{K}^*/\mathcal{O}_X^*) \to H^1(\mathcal{O}_X^*) \to H^1(\mathcal{K}^*) = 0$$

where $\mathcal{K}^*$ is flasque. Then

$$\begin{aligned}\operatorname{Div}(X) &= \text{Cart. div.}/\sim \\ &\cong H^0(\mathcal{K}^*/\mathcal{O}_X^*)/\operatorname{im}\varphi \cong H^1(\mathcal{O}^*X).\end{aligned}$$

Question 2: The problem is local, so we could replace X by some open affine set $V \subset X$; here we need to replace U by $U \cap V$, which is (by assumption) affine. So we could assume from the start that X is affine, say $X = \operatorname{Spec} A$. Let $U = \operatorname{Spec} B$. The exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

corresponds to an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{A.3}$$

of B-modules, where $\mathcal{F}' = \widetilde{M'}$, etc. Now if we consider this as an exact sequence of A-modules, we deduce that the sequence

$$0 \to \underbrace{f_*\mathcal{F}'}_{{}_A\widetilde{M'}} \to \underbrace{f_*\mathcal{F}}_{{}_A\widetilde{M}} \to \underbrace{f_*\mathcal{F}''}_{{}_A\widetilde{M''}} \to 0$$

is exact because it corresponds to (A.3) as A-modules.

Question 3: By assumption V, W are open subschemes of X, and we have inclusions

$$X \overset{open}{\hookrightarrow} \overline{X} \overset{closed}{\hookrightarrow} \mathbb{P}^n_A.$$

So V, W are open subschemes of $\overline{X}$. $V \cap W$ is the same inside X or $\overline{X}$, so we can replace X by $\overline{X}$. That is, we could assume X is a closed subscheme of $\mathbb{P}^n_k$.

We have

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_A \arrow[d] \\ & Y = \operatorname{Spec} A \end{tikzcd}$$

By the universal property of products we get $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$, where, given any diagram

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ \mathbb{P}^n \arrow[dr] & & \mathbb{P}^n \arrow[dl] \\ & Y & \end{tikzcd}$$

we get a map $? \to \mathbb{P}^n \times \mathbb{P}^n$. Applying this to $? = \mathbb{P}^n$ (with identity maps to $\mathbb{P}^n$ on each side), we get the diagonal map $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$.

It turns out that Δ is a closed immersion.

Similarly, we get a morphism $\Delta_X : X \to X \times_Y X$. One can check (using the universal property of products) that we have a diagram

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[dr, "\Delta_X"'] & pullback \arrow[r] \arrow[d, "\Delta_X"] & \mathbb{P}^n \arrow[d, "\Delta"] \\ & X \times X \arrow[r] & \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

If you have a closed immersion, and take the product with any other scheme, you get a closed immersion. So Δ_X is also a closed immersion.

Then $V \cap W \cong V \times W \cap \Delta(X)$; since $\Delta(X)$ is closed, $V \cap W$ can be identified with a closed subscheme of $V \times W$.

Question 4: The main point is that $\mathbb{P}^n$ is covered by $U_i = D_+(s_i)$. We have $n+1$ open sets $U_0, \cdots, U_n$. If you have any closed subscheme, take intersections with these open sets to get a cover with at most $n+1$ open sets. Now apply Čech cohomology (the Čech complex will stop at C^n).

Question 5: Take one point.

Question 6: Remember some facts: $H^p(\mathcal{O}_X(m)) = 0$ if $p \neq 0, n$, and by duality, $\dim_k H^n(\mathcal{O}_X(m)) = \dim_k H^0(\mathcal{O}_X(-n-1-m))$ for all $m \in \mathbb{Z}$.

Remember that the exact sequence

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

The long exact sequence gives

$$\begin{aligned} 0 \to H^0(\Omega_{X/Y}) &\to \underbrace{H^0(\mathcal{O}_X(-1)^{n+1})}_{0} \to \underbrace{H^0(\mathcal{O}_X)}_{K} \\ &\to H^1(\Omega_{X/Y}) \to \underbrace{H^1(\mathcal{O}_X(-1)^{n+1})}_{0} \to \underbrace{H^1(\mathcal{O}_X)}_{0} \to \cdots \\ &\cdots \to H^{n-1}(\mathcal{O}_X(-1)^{n+1}) \to H^{n-1}(\mathcal{O}_X) \\ &\to H^n(\Omega_{X/Y}) \to H^n(\mathcal{O}_X(-1)^{n+1}) \to \underbrace{H^n(\mathcal{O}_X)}_{0} \to 0 \end{aligned}$$

(The rest is zero.) Since the second term is zero, so is the first. $H^{n-1}(\mathcal{O}_X) = 0$ unless $n = 1$. $H^n(\mathcal{O}_X(-1)) = 0$ by duality.

If $1 < p < n$ then we have

$$\underbrace{H^{p-1}(\mathcal{O}_X)}_{0} \to H^p(\Omega_{X/Y}) \to \underbrace{H^p(\mathcal{O}_X(-1))}_{0} \to \underbrace{H^p(\mathcal{O}_X)}_{0}$$

so $H^p(\Omega_{X/Y}) = 0$. The only thing that remains is H^1; but $H^1(\Omega_{X/Y}) \cong H^0(\mathcal{O}_X) \cong K$.

Now calculate the Euler characteristic:

$$\chi(\Omega_{X/Y}) = \sum_{p=0}(-1)^p \cdot \dim_k H^p(\Omega_{X/Y}) = -1$$

To calculate the Hilbert polynomial, consider the exact sequence

$$0 \to \Omega_{X/Y}(d) \to \mathcal{O}_X(d-1)^{n+1} \to \mathcal{O}_X(d)$$

We understand the middle and right sheaves:

Hilbert polynomial of $\Omega_{X/Y}$ = Hilbert polynomial of $\mathcal{O}_X(-1)^{n+1}$ − Hilbert polynomial of $\mathcal{O}_X$

In other words, the Hilbert polynomial of $\Omega_{X/Y}$ is

$$\chi(\Omega_{X/Y}(d)) = \chi(\mathcal{O}_X(d-1)^{n+1}) - \chi(\mathcal{O}_X(d))$$

Do the calculations to get a polynomial in terms of d.

Question 7: This is the case most of the time; just don't pick something trivial.

Question 8: Let $\mathbb{P}^1 = \operatorname{Proj} k[s_0, s_1]$, $\mathbb{P}^1 = \operatorname{Proj} k[u,u_1]$, $\mathbb{P}^3 = \operatorname{Proj} k[t_0, t_1, t_2, t_3]$. Let S_d be the degree-d part of $k[s_0, s_1]$, and U_d be the degree-d part of $k[u_0, u_1]$. Let $R = \bigoplus_{d \geq 0} S_d \otimes U_d$ (this is *not* the tensor product of rings!). We have $D_+(s_0u_0)$, $D_+(s_0u_1)$, $D_+(s_1, u_0)$, and $D_+(s_1u_1)$ in $\operatorname{Proj} R$, and these give a cover of $\operatorname{Proj} R$. Now

$$\begin{aligned} D_+(s_0u_0) &\cong \operatorname{Spec} R_{(s_0u_0)} \cong \operatorname{Spec} k[\frac{s_1}{s_0}, \frac{u_1}{u_0}] \\ &\cong \mathbb{A}^2 \cong D_+(s_0) \times D_+(u_0) \subset \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

We have similar isomorphisms $D_+(s_iu_j) \to$ some open set in $\mathbb{P}^1 \times \mathbb{P}^1$. It is not hard to glue all these isomorphisms to get an isomorphism

$$\operatorname{Proj} R \xrightarrow{\cong} \mathbb{P}^1 \times \mathbb{P}^1$$

We have a diagram of homomorphisms

$$\begin{tikzcd} k[t_0, \cdots, t_3] \arrow[rr] \arrow[dr] & & R \\ & T = k[t_0, \cdots, t_3]/\langle t_0t_1 - t_2t_3 \rangle \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$$\begin{aligned} t_0 &\mapsto s_0u_0 \\ t_1 &\mapsto s_0u_1 \\ t_2 &\mapsto s_1u_0 \\ t_3 &\mapsto s_1u_1 \end{aligned}$$

This is the Segre embedding.

After a bit of elementary calculation, you can see that α is an isomorphism. We then get a corresponding morphism of schemes

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \operatorname{Proj} R \overset{\cong}{\to} X = \operatorname{Proj} T$$

that is an isomorphism.

Let $V_i = D_+(s_i)$, $W_i = D_+(u_i)$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by the $V_i \times W_j$, all of which are isomorphic to $\mathbb{A}^2$. So $\operatorname{Div}(V_i \times W_j) = 0$ by some problem on Example sheet 3.

Any Cartier divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$ can be written as a system $(V_i \times W_j, f_{ij})$. We can assume $f_{00} = 1$ in $\mathcal{O}_{V_0 \times W_0}(V_0 \times W_0)$ (after multiplying by a rational function). Now $\mathbb{P}^1 \times \mathbb{P}^1 \backslash V_0 \times W_0 = F \cup G$ where $F = \mathbb{P}^1 \times *$, $G = * \times \mathbb{P}^1$. Define

$$\begin{aligned} \varphi : \operatorname{Div}(\mathbb{P}^1 \times \mathbb{P}^1) &\to \mathbb{Z} \oplus \mathbb{Z} \\ D &\mapsto (\nu_F(f_{11}), \nu_G(f_{11})) \end{aligned}$$

where $\nu_F(f_{11})$ is the vanishing order of f_{11} along F. (This is calculated using the fact that $(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1})_{\substack{\text{gen. pt.} \\ \text{of } F}}$ is a DVR.)

φ is injective: If $\varphi(D) = 0$, then f_{11} is a unit in the local rings of F and G, so there are finitely many closed points $x_1, \cdots, x_\ell \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $D|_{\mathbb{P}^1 \times \mathbb{P}^1 \backslash \{x_1, \cdots, x_\ell\}} \sim 0$. So the f_{ij} are units in $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(V_i \times W_j)$. (Don't worry about this if you're not familiar with valuations.)

φ is surjective: We have a diagram

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[drr, dotted, "\cong" near start] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dr] & F = fibre \arrow[l, hook'] \arrow[dll, dotted, "\cong"' near start] \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{tikzcd}$$

which can be used to get a D such that $\varphi(D) = (m, n)$ for any given $m, n \in \mathbb{Z}$.

Now we have to calculate $H^1(X, \mathcal{O}_X(D))$ where D corresponds to D' on $\mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi(D') = (1, 1)$. $\mathcal{O}_{\mathbb{P}^3}(1)$ pulls back to $\mathcal{O}_X(1)$ on X, and $\mathcal{O}_X(1) \cong \mathcal{O}_X(1)$. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to f_* \mathcal{O}_X \to 0$$

where $f : X \to \mathbb{P}^3$. Tensor with $\mathcal{O}_X(1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to f_* \mathcal{O}_X(1) \to 0$$

The long exact sequence contains:

$$\underbrace{H^1(\mathcal{O}_{\mathbb{P}^3 \mathcal{O}(1)})}_{0} \to H^1(\mathcal{O}_X(1)) \to \underbrace{H^2(\mathcal{O}_{\mathbb{P}^3}(-1))}_{0}$$

so $H^1(\mathcal{O}_X(1)) = 0$.

Question 9: Let $X = \operatorname{Proj} k[s_0, \cdots, s_n]$. Pick a closed point $x = \langle s_1, \cdots, s_n \rangle$, and let $f : \langle x \rangle \to X$ be the closed immersion. Let $\mathcal{F} = f_* \mathcal{O}_{\{x\}}$. Now $\dim_k H^0(\mathcal{F}) = \dim_k H^0(\mathcal{O}_{\{x\}}) =$

1. Let $U = X\backslash\{x\}$. Then $\mathcal{F}|_U = 0$ which implies $\mathcal{F}^\vee|_U = 0$. Tensoring doesn't change this property: $\mathcal{F}^\vee(-n-1)|_U = 0$. Since this sheaf is trivial outside the point x, it comes from a sheaf on x: that is, there is some sheaf $\mathcal{G}$ on $\{x\}$ such that $f_*\mathcal{G} = \mathcal{F}^\vee(-n-1)$. So

$$\dim_k H^n(\mathcal{F}^\vee(-n-1)) = \dim_k H^n(\mathcal{G}) = 0$$

Question 10: Algebraic topology.

Question 11: Straightforward.