Geometry of Manifolds
(lecture notes)

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Fall 2013

Last updated: August 23, 2013

Disclaimer

These are my notes from Prof. Seidel's course on differential topology and differential geometry, given at MIT in Fall 2013. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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Lecture 1: September 4

Let $U, V \subset \mathbb{R}^n$ be open subsets. A smooth map $\varphi : U \to V$ is called a diffeomorphism if it has a smooth inverse $\varphi^{-1} : V \to U$.

Lemma 1.1. φ is a diffeomorphism iff it is bijective and $D\varphi_X : \mathbb{R}^n \to \mathbb{R}^n$ is invertible for all x.

This is a consequence of the inverse function theorem.

Example 1.2. $\mathbb{R}^n \backslash \{0\}$ is diffeomorphic to $\{x \in \mathbb{R}^n \,:\, a < |x| < b\}$ for $0 < a < b$.

Suppose $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ and I want to understand a smooth map $f : U \to V$. There is a linear map $Df_x : \mathbb{R}^m \to \mathbb{R}^n$ associated to every point.

Theorem 1.3 (Theorem of constant rank)**.** Let $U \subset \mathbb{R}^m$ be an open subset and $f : U \to \mathbb{R}^m$ be a smooth map such that the rank of Df_x is constant. Choose $x_0 \in U, y_0 = f(x_0)$. Then there are open neighborhoods $x_0 \in V \subset U$, $y_0 \in W$ and diffeomorphisms $\varphi : (V, x_0) \to (\widetilde{V}, 0)$, $\psi : (W, y_0) \overset{\cong}{\to} (\widetilde{W}, 0)$ such that $\psi \circ \varphi^{-1}$ is a linear map.

See Bröcker-tom Dieck chapter 5.

Remark 1.4. The most frequent application is when $\mathrm{rank}(Df_x)$ is maximal, because that is an open condition (i.e. the space of matrices with that rank is open). If $D_x f$ is surjective [injective] for all $x \in U$, f is called a submersion [immersion].

Proof. By reduction to the inverse function theorem. To simplify write $f : \mathbb{R}^m \to \mathbb{R}^n$, and assume $x_0 = f(x_0) = 0$. All the argument will (implicitly) be local near the origin in both source and target spaces. We will also assume that $(Df)_0$ is a block matrix $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Take $\varphi : \mathbb{R}^m \to \mathbb{R}^m, \varphi(x_1, \cdots, x_m) = (f_1(x), \cdots, f_r(x), x_{r+1}, \cdots, x_m)$. (This makes sense because $r \leq m$ and $r \leq n$.) Then $\varphi(0) = 0$, $(D\varphi)_0 = I_m$ hence locally near 0 there is an inverse φ^{-1}.

$$\begin{tikzcd} (\mathbb{R}^m, 0) \arrow[rr, "f(x)"] \arrow[dr, "{(f_1(x),\cdots,f_r(x),x_{r+1},\cdots)}"'] & & (\mathbb{R}^n, 0) \\ & (\mathbb{R}^m, 0) \arrow[ur, "g = f\circ\varphi^{-1}"'] & \end{tikzcd}$$

By construction g is the identity on the first r coordinates.

$$(Dg)_0 = \begin{pmatrix} I & 0 \\ * & * \end{pmatrix}$$

We know $\mathrm{rank}(Dg)_x = r$. This allows us to say the bottom right of this matrix is zero (otherwise if something was nonzero you would get another vector that is linearly independent, making the rank $r + 1$). So g_{r+1} doesn't depend on x_{r+1}, etc. So we can write

$g(x) = (x_1, \cdots, x_r, g_{r+1}(x_1, \cdots, x_r), \cdots, g_n(x_1, \cdots, x_r))$. The image is a graph; we have to make another change of variables to flatten it out.

Define $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ where

$$\psi(y_1, \cdots, y_n) = (y_1, \cdots, y_r, y_{r+1} - g_{r+1}(y_1, \cdots, y_r), \cdots, y_n - g_n(y_1, \cdots, y_r)).$$

Then $\psi \circ g = \psi \circ f \circ \varphi^{-1} = (x_1, \cdots, x_r, 0, \cdots, 0)$. $\square$

Submanifolds. Take an open subset $U \subset \mathbb{R}^n$. A closed subset $M \subset U$ is called a k-dimensional submanifold of U if "it locally looks like a flat subspace" – if for each $x \in M$ there is an open neighborhood $x \in V \subset U$ and a diffeomorphism $\varphi : (V, x) \to (\widetilde{V}, 0)$ such that $\varphi(M \cap V) = (\mathbb{R}^k \times 0) \cap \widetilde{V}$.

Lemma 1.5. If $f : U \to \mathbb{R}^n$ is a submersion, all the fibres $f^{-1}(y)$ are submanifolds of U.

This follows from the constant rank theorem.

Example 1.6. Fix $\lambda_1, \cdots, \lambda_n$. The subspace of matrices with eigenvalues $\{\lambda_i\}$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - n$. To do this, take the map $A \mapsto$ coefficients of the characteristic polynomial $\det(I - A)$.

Lecture 2: September 6

Let $U \subset \mathbb{R}^m$ be an open set, $f : U \to \mathbb{R}^n$ be smooth. If f is a submersion then for all y, $f^{-1}(y) \subset U$ is a submanifold. Call $y \in \mathbb{R}^n$ a *regular value* if Df_x is onto for all $x \in f^{-1}(y)$ (otherwise it's a *critical value*).

Lemma 2.1. If y is a regular value, $f^{-1}(y)$ is a submanifold.

Definition 2.2. Let $U \subset \mathbb{R}^n$ be open. A closed subset $M \subset U$ is called a smooth neighborhood retract if there is an open set V with $M \subset V \subset U$ and a smooth map $r : V \to V$ with $r(V) = M$ and $r|_M = Id$.

Lemma 2.3. A smooth neighborhood retract is automatically a submanifold.

This follows from the theorem of constant rank.

Let $Gr(k, m)$ be the set of k-dimensional linear subspaces of $\mathbb{R}^n$. To give it a topology, we can identify it with the subset of $\mathbb{R}^{\binom{n}{2}}$ consisting of symmetric matrices that satisfy $A^2 = A$ and have rank k. (A is just projection onto a k-dimensional subspace.)

Take

$$M = \{A \in \mathbb{R}^{\binom{n}{2}} : \operatorname{rank} A = k, A^2 = A\},$$

which sits inside

$$U = \{A \in \mathbb{R}^{\binom{n}{2}} \ : \ k \text{ eigenvalues in } (\tfrac{1}{2}, \tfrac{3}{2}) \text{ and } n-k \text{ eigenvalues in } (-\tfrac{1}{2}, \tfrac{1}{2})\}$$

and there is a smooth retraction $U \to M$: take the eigenvalues in the first range and replace them by 1, and take the eigenvalues in the second interval and replace them by 0.

The retraction is constructed by spectral projection:

$$r(A) = \frac{1}{2\pi i} \oint \frac{dz}{z \cdot Id - A}$$

where the integral is taken over the counterclockwise circle of radius $\frac{1}{2}$ around 1. None of the eigenvalues lie on the circle, so you can invert the matrix in the denominator. Recall that

$$\frac{1}{2\pi i} \oint \frac{dz}{z-\lambda} = \begin{cases} 1 & \text{if } \lambda \text{ is inside the circle} \\ 0 & \text{otherwise} \end{cases}.$$

This moves the eigenvalues as desired and is smooth.

Lemma 2.4. Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open subsets, and $i : U \to V$ a *proper injective immersion*. Then $i(U) \subset V$ is a submanifold.

In the case of the hypotheses of this theorem we say that i is an embedding.

This follows from the theorem of constant rank.

Example 2.5. Take the injective immersion $\mathbb{R} \to \mathbb{R}^2$ where $\mathbb{R}$ loops around but doesn't self-intersect. This is not proper.

Let $U \subset \mathbb{R}^m$ be open, $M \subset U$ be a submanifold. At $x \in M$ we have a tangent space $T_xM \subset \mathbb{R}^m$ that we will define in three ways. First choose a neighborhood V so $x \in V \subset U$, and a diffeomorphism $\varphi : V \to \widetilde{V}$ so $\varphi(x) = 0$, $\varphi(M \cap V) = (\mathbb{R}^m \times 0) \cap \widetilde{V}$. Define $T_xM = D\varphi_x^{-1}(\mathbb{R}^m \times 0)$.

Alternatively, define

$$T_xM = \{\xi \in \mathbb{R}^m \ : \ \text{if } f : U \to \mathbb{R} \text{ is smooth and vanishes along } M \text{ then } Df_x(\xi) = 0\}.$$

If we just required the functions to be defined in small neighborhoods then this is the pullback of the previous. This is not a problem: you can always extend a function to U.

Alternatively, define T_xM to be the set $\{\xi = c'(0) \ : \ c : (-\varepsilon, \varepsilon) \to U, c(0) = x, c(t) \in M \ \forall t\}$. This is the same as the first definition: for every path you get a tangent direction and vice versa. However, this is not obviously a linear subspace (how do you add paths?).

Transversality. Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n, f : U \to V$ and a submanifold $N \subset V$. We say that f is transverse to N if for all x such that $f(x) = y \in N$, $(Df_x) + TN_y = \mathbb{R}^n$. This is a weakening of the notion of regular value.

Lemma 2.6. If f is transverse to N then $f^{-1}(y)$ is a submanifold of U.

(This generalizes the theorem about the regular value.)

Proof. Without loss of generality $N = \mathbb{R}^k \times 0 \subset \mathbb{R}^n$. Write $f = (f_1, f_2) : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then $(Df_2)_x$ is onto so 0 is a regular value of f_2. □

Let M be a submanifold in $U \subset \mathbb{R}^m$. Then $T_xM \subset \mathbb{R}^m$, and $\mathbb{R}^m/T_xM = \nu_xM$ is called the normal space. (Think about this as the orthogonal complement of T_xM... but this isn't always useful because diffeomorphism don't preserve orthogonality.)

Then the transversality condition is that $\mathbb{R}^m \overset{D_xf}{\to} \mathbb{R}^n \overset{proj}{\to} \nu_yN$ is onto.

Bump functions and cutoff functions. The classic bump function is

$$k(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t^2}} & t > 0. \end{cases}$$

From this, construct $\ell(t) = \frac{k(t+1)k(1-t)}{k(1)^2}$; this is nonzero only on $(-1, 1)$. Also consider

$$m(t) = \frac{\int_{-1}^{t} \ell(y)dy}{\int_{-\infty}^{\infty} \ell(y)dy}$$

which is nonzero on $(-1, \infty)$ and constant on $[1, \infty]$. This is called a *cutoff function*. Finally, we can get an improved bump function

$$n(t) = m(3 - 2|t|)$$

that is nonzero on $(-2, 2)$ and constant $= 1$ on $(-1, 1)$.

Since k was smooth, all of these functions are smooth.

Lecture 3: September 9

Proposition 3.1. Let $U \subset \mathbb{R}^n$ be open, $M \subset U$ a closed subset. Suppose there is an open neighborhood V with $M \subset V \subset U$ and a smooth retraction $r : V \to M$ (that is, $r(V) = M$ and $r|_M = Id$, which implies r is idempotent).

Then M is a submanifold.

Proof. Pick $x \in M$ and a small neighborhood $W \ni x$ in V. Since r is idempotent, $Dr_x \circ Dr_x = Dr_x$ so Dr_x is an idempotent matrix, say of rank r. For $y \in W$, $\text{rank}(Dr_y) \geq r$. If additionally $y \in M \cap W$, $\text{rank}(Dr_y) = r$, because Dr_y is idempotent (two idempotent matrices near each other have the same rank – *if A is an idempotent matrix then $tr(A) =$*

$rank(A)$). In fact, equality (of rank) holds for all $y \in W$, since $Dr_{r(y)} \circ Dr_y = Dr_y$ and $\mathrm{rank}(Dr_y) \le r$ (because $Dr_{r(y)}$ has rank r). $\square$

(Source: Bröcker-Jänich, chapter 4 and 7.)

More about bump functions. Recall we had a smooth map $\psi : \mathbb{R}^n \to [0,1]$ such that $\psi(0) > 0$ and $\psi(x) = 0$ outside $(-2,2)$. We also had an improved bump function satisfying $\psi(x) = 1$ for $\|x\| \le 1$ and $\psi(x) = 0$ for $\|x\| \ge 2$. These are useful for "smearing functions". Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, and let ψ be a bump function. Write $\psi_r(x) = \frac{\psi(\frac{x}{r})}{\int_{\mathbb{R}^n} \psi(\frac{z}{r})dz}$ (the denominator is just a normalization factor). This is supported in $B_{2r}(0)$. Then the convolution

$$(f * \psi_r) = \int_{\mathbb{R}^n} f(x-y)\psi_r(y)dy$$

is smooth and, as $r \to 0$, this converges to f uniformly on compact subsets (using uniform continuity).

Why is it smooth? Change variables from y to $y + x$

$$(f * \psi_r)(x) = \int f(-y)\psi_r(y+x)dy$$

Use the derivative of ψ to come up with the putative derivative of the convolution, and then show it's actually the right thing.

Definition 3.2. Let U be a topological space. A cover of U is a collection $(V_\alpha)_{\alpha \in A}$ of open subsets such that $\bigcup_{\alpha \in A} V_\alpha = U$. We say that $(W_\beta)_{\beta \in B}$ refines $(V_\alpha)_{\alpha \in A}$ if for all $\beta \in B$ there is some α such that $V_\beta \subset V_\alpha$.

A cover $(V_\alpha)_{\alpha \in A}$ is called *locally finite* if each point has a neighborhood that intersects only finitely many V_α.

A Hausdorff space U is called paracompact if every open cover has a locally finite refinement.

Theorem 3.3 (Stone)**.** Metric spaces are paracompact.

In particular, open subsets of $\mathbb{R}^n$ are paracompact. (Even better – they're locally compact.)

Lemma 3.4. Let $U \subset \mathbb{R}^n$ be an open subset. Then every cover $(V_\alpha)_{\alpha \in A}$ has a locally finite refinement of the form $(W_1 = B_{r_1}(x_1), W_2 = B_{r_2}(x_2), \cdots)$ where $(B_{\frac{r_i}{3}}(x_i))$ is still an open cover.

Sketch of proof. Step 1: write $U = \bigcup K_i$ where each K_i is compact and $K_i \subset int(K_{i+1})$. For example, $K_i = \{x \in U : \|x\| \le i,\ dist(x, \mathbb{R}^n \backslash U) \ge \frac{1}{i}\}$.

Step 2: Cover $K_{i+1}\backslash int(K_i)$ with a union of balls such that each ball lies in some V_α and is also contained in $int(K_{i+1})\backslash K_{i-1}$ (this eventually guarantees local finiteness).

Step 3: Do the same thing with finitely many balls for each i.

Step 4: Take the balls for all i together.

We haven't shown the last statement in the lemma; you have to be careful in step 2. $\square$

Take a topological space U with an open cover $(V_\alpha)_{\alpha\in A}$. A partition of unity subordinate to $(V_\alpha)_{\alpha\in A}$ is a collection of functions $f_\beta : U \to [0,1]$ such that:

- for all β there exists α such that $supp(f_\beta) \subset V_\alpha$ (the support is the closure of the set of points where f_β is nonzero)
- each point has neighborhood on which all but finitely many f_β vanish
- $\sum_{\beta\in B} f_\beta(x) = 1$ for all x.

Note that the second condition guarantees that the sum in the third condition is not infinite.

Theorem 3.5. If U is paracompact then every cover has a subordinate partition of unity.

The converse is also true. (Given a cover $(V_\alpha)_\alpha$, consider sets $A_\beta = \{x : f_\beta(x) \neq 0\}$; these form a locally finite refinement of $(V_\alpha)_\alpha$.)

Corollary 3.6. Let U be paracompact, $X, Y \subset U$ disjoint closed subsets. Then there exist continuous functions $f : U \to [0,1]$ if $f|_X = 0$ and $f|_Y = 1$.

Spaces satisfying the corollary are called T4.

Proof of corollary. Take the cover $\{U\backslash X, U\backslash Y\}$ and find a subordinate partition of unity $(f_\beta)_{\beta\in B}$. Write $B = B_X \sqcup B_Y$ such that $\beta \in B_X$, $supp(f_\beta) \subset U\backslash Y$, $\beta \in B_Y$, $supp(f_\beta) \subset U\backslash X$. $\square$

Write

$$1 = \sum_{\beta\in B} f_\beta = \sum_{\beta\in B_X} f_\beta + \sum_{\beta\in B_Y} f_\beta.$$

The first sum vanishes on Y and the second sum vanishes on X. So the first sum is 1 on X and the second sum is 1 on Y.

Lecture 4: September 11

Lemma 4.1. Let $Y \subset \mathbb{R}^m$ be a open subset, $(V_\alpha)_{\alpha \in A}$ a cover of U. Then there is a *countable* partition of unity $(f_1, f_2, \cdots)$ consisting of *smooth functions* subordinate to $(V_\alpha)_{\alpha \in A}$.

Proof. We know (from last time) that $(V_\alpha)_{\alpha \in A}$ has a refinement of the form $(B_{r_1}(x_1), B_{r_2}(x_2), \cdots)$ which is locally finite and such that the $B_{\frac{r_i}{3}}(x_i)$ still cover U. Take a smooth bump function $\psi : \mathbb{R}^m \to [0,1]$ where

$$\psi(x) > 0 \text{ if } \|x\| \leq 1$$
$$\psi(x) = 0 \text{ if } \|x\| \geq 2$$

Set $\psi_i(x) = \psi\left(3i\left(\frac{x-x_i}{r_i}\right)\right)$. Then $supp(\psi_i) \subset B_{r_i}(x_i)$ an $\psi_i > 0$ on $B_{\frac{r_i}{3}}(x_i)$. For each point of U there is a neighborhood where only finitely many ψ_i are nonzero (each point has a neighborhood that intersects finitely many balls, and there are correspondingly finitely many functions that are nonzero). On the other hand, $\psi = \sum_i \psi_i > 0$ everywhere. Set $f_i = \frac{\psi_i}{\psi}$. This is a smooth partition of unity subordinate to the cover of balls, and hence to the original open cover. □

Corollary 4.2. Let $U \subset \mathbb{R}^n$ be open, $X, Y \subset U$ closed and disjoint. Then there is a smooth $f : U \to [0,1]$ such that $f|_X = 0$ and $f|_Y = 1$.

(You can also prove the zero set is exactly X, but that is much harder to prove.)

Corollary 4.3. Let $U \subset \mathbb{R}^m$ be an open subset. Then there is a proper smooth function $f : U \to [0, \infty)$ (the preimage of compact sets is compact).

This is obvious when U is a ball, but open sets can be weird...

Proof. Let $(f_1, f_2, \cdots)$ be a smooth partition of unity such that $supp(f_i)$ is compact. (Take a cover of U by bounded open sets, or go back to the construction and notice that the functions are supported on balls which are clearly compact.) Set

$$\begin{aligned} f &= f_1 + 2f_2 + 3f_3 + 4f_4 + \cdots \\ &= \sum f_i + (\sum f_i - f_1) + (\sum f_i - f_1 - f_2) + \cdots \\ &= 1 + (1 - f_1) + (1 - f_1 - f_2) \end{aligned}$$

This is well-defined by local finiteness. If x is not in the support of the first N, then the first N terms are N, so $f(x) \geq N$. This shows that $\{x : f(x) \leq N\}$ is compact. □

Theorem 4.4 (Approximating continuous functions by smooth functions)**.** Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}$ be continuous. For each $\varepsilon > 0$ there is a smooth $g : U \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all x.

If we want this on a compact set, this is easy – convolution with a bump function. Here's a stronger version.

Theorem 4.5. Let $\varepsilon : U \to (0, \infty)$ be continuous. Then there is a smooth function $g : U \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon(x)$. for all ε.

Proof. Fix a locally finite cover of U by balls $B_{r_i}(x_i)$ so that the $B_{\frac{r_i}{3}}(x_i)$ still cover U. On each $B_{\frac{3}{4}r_i}(x_i)$ we find a smooth map $g_i : B_{\frac{3}{4}r_i}(x_i) \to \mathbb{R}$ such that $|g_i(x) - f(x)| < \varepsilon$ and $x \in \overline{B}_{\frac{2}{3}r_i}(x_i)$ (using convolution with a bump function, defined on a slightly smaller ball). (In the book, they make the balls small enough that you can approximate by a constant function.) Take a partition of unity by functions f_i so that $supp(f_i) \subset B_{\frac{2}{3}r_i}(x_i)$ and set $g = \sum f_i g_i$. Then $|f(x) - g(x)| \leq \sum_i f_i(x)|f(x) - g_i(x)| \leq \varepsilon \sum_i f_i(x) = \varepsilon$. $\square$

Measure zero and Sard's theorem.

Definition 4.6. A subset $C \subset \mathbb{R}^m$ has *measure zero* if for each $\varepsilon > 0$ there is a countable collection of cubes $(Q_0, Q_1, \cdots)$ such that $C \subset \bigcup_{i=0}^{\infty} Q_i$ and $\sum_i vol(Q_i) < \varepsilon$.

Lemma 4.7. If C has measure zero, its complement is everywhere dense.

This amounts to showing that an open ball does not have measure zero.

Lemma 4.8. A countable union of measure zero sets has measure zero.

Cantor diagonalization argument: cover the first measure zero set with balls of total volume $< \frac{\varepsilon}{2}$, cover the second measure zero set with balls of total volume $< \frac{\varepsilon}{4}$, etc. The total volume will be $< \varepsilon$.

Lemma 4.9. Let $U \subset \mathbb{R}^m$ be open. $f : U \to \mathbb{R}^m$ is C^1. If $C \subset U$ has measure zero then so does $f(C)$.

Proof. Without loss of generality there are compact sets K_1, K_2 such that $K_1 \subset int(K_2) \subset K_2$ and $C \subset K_1$. We know that $\mathbb{R}^m$ can be covered by countably many K_i; we will show this is true for $C \cap K_1$ and then use the previous lemma.

Write $C \subset \bigcup_{i=1}^{\infty} Q_i$, where $\sum vol(Q_i) < \varepsilon$ and $Q_i \subset K_2$ (decompose Q into smaller cubes to make this work). On K_2, $\|Df\| \leq A$ for some $A > 0$, hence by MVT, $f(Q_i)$ is contained in a cube Q_i' of size A times the size of Q_i. (Recall: differentiable almost everywhere implies Lipschitz on compact subsets.) Therefore $f(C) \subset \bigcup Q_i'$. Then $\sum vol(Q_i') \leq \sum A^m vol(Q_i) = A^m \cdot \varepsilon$. $\square$

(This rules out things like the plane-filling curve.)

Corollary 4.10 (Trivial Sard's theorem)**.** If $U \subset \mathbb{R}^m$ is open and $f : U \to \mathbb{R}^n$ is smooth for $m < n$, then $f(U)$ has measure zero.

(Consider the map $\mathbb{R}^{n-m} \to U \xrightarrow{f} \mathbb{R}^n$.)

Lecture 5: September 13

Lemma 5.1. Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$ be a smooth function. Assume $n \geq 2m$. Then f can be approximated arbitrarily closely (in the sense of C^r-convergence (first r derivatives converge) on compact subsets, for any r) by immersions.

It's clear this doesn't work for functions $\mathbb{R} \to \mathbb{R}$ – if there's a local maximum, then any perturbation still has a local maximum near there. But if it's a function $\mathbb{R} \to \mathbb{R}^2$ then you can ensure that the components of the derivative aren't zero at the same time.

Proof. We will construct $f = g_0, g_1, g_2, \cdots : U \to \mathbb{R}^n$ such that

$$\left(\frac{\partial g_i}{\partial x_1}, \cdots, \frac{\partial g_i}{\partial x_i}\right)$$

are everywhere linearly independent. Then g_m is an immersion.

Step 1. Consider $\frac{\partial f}{\partial x_1} = \frac{\partial g_0}{d x_1} : U \to \mathbb{R}^n$. Since $U \subset \mathbb{R}^m$ for $m < n$, the image of this map has measure zero. Find small $v_1 \in \mathbb{R}^n$, $v_1 \notin im(\frac{\partial g_0}{\partial x_1})$ and set $g_1(x) = g_0(x) - v_1 x_1$. Then $\frac{\partial g_1}{\partial x_1} = \frac{\partial g_0}{\partial x_1} - v_1$ is nowhere zero.

Step 2. Consider $\mathbb{R} \times U \to \mathbb{R}^n$ given by

$$(\lambda_1, x) \mapsto \lambda_1 \frac{\partial g_1}{\partial x_1} + \frac{\partial g_1}{\partial x_2}. \tag{5.1}$$

The image of this again has measure zero (by dimension counting). Take a small $v_2 \notin im(5.1)$. Set $g_2 = g_1 - v_2 x_2$. Then

$$\frac{\partial g_2}{\partial x_1} = \frac{\partial g_1}{\partial x_1}$$

is everywhere nonzero, and $\frac{\partial g_2}{\partial x_2} = \frac{\partial g_1}{\partial x_2} - v_2$. If $\frac{\partial g_2}{\partial x_2}$ is linearly dependent on $\frac{\partial g_2}{\partial x_1}$ at some point, then

$$\frac{\partial g_2}{\partial x_2} = -\lambda \frac{\partial g_2}{\partial x_1}$$

$$\lambda \frac{\partial g_1}{\partial x_1} + \frac{\partial g_1}{\partial x_2} = v_2$$

Step 3. Consider $\mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}^n$ given by

$$(\lambda_1, \lambda_2, x) \mapsto \lambda_1 \frac{\partial g_2}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_2} + \frac{\partial g_2}{\partial x_3}.$$

This works the same way.

Step $m-1$. We produce a map $\mathbb{R}^{m-1} \times U \to \mathbb{R}^n$; the dimensions work out because $\dim(\mathbb{R}^{m-1} \times U) = 2m-1$ and $\dim(\mathbb{R}^n) \geq 2m$. □

Example 5.2. Suppose $f = (x_1 + x_2, *, *)$. Then $\frac{\partial f}{\partial x_1} = (1, *, *)$ and $\frac{\partial f}{\partial x_2} = (1, *, *)$. This is not an immersion iff $\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} = 0$.

Theorem 5.3. Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$ be smooth, $n \geq 2m$. Then for every $\varepsilon > 0$ there is an immersion $g : U \to \mathbb{R}^n$ such that $\|f(x) - g(x)\| < \varepsilon$.

Sketch of proof. Choose a locally finite cover of U by balls $B_{r_i}(x_i)$ so that the $B_{\frac{r_i}{3}}(x_i)$ still form an open cover of U. Let (ψ_i) be a collection of functions supported in $B_{r_i}(x_i)$ so that $\psi_i|_{B_{\frac{r_i}{3}}(x_i)} = 1$. Start with $f = g_0$ and construct g_1, g_2 such that

- $g_i = g_{i-1}$ outside $B_{r_i}(x_i)$
- Dg_i has maximum rank on $\overline{B_{\frac{r_1}{3}}}(x_1) \cup \cdots \cup \overline{B_{\frac{r_i}{3}}}(x_i)$
- $\|g_i - g_{i-1}\| < \varepsilon \cdot 2^{-i}$

The construction is by $g_i = g_{i-1} + \psi_i(h_i - g_{i-1})$ where h_i is an immersion on $\overline{B_{\frac{r_i}{3}}}(x_i)$, and $h_i - g_{i-1}$ is small (this exists by the previous theorem). (Make an immersion on the first ball; we still have some leeway to perturb this to make sure it stays an immersion on the first ball. So perturb it a little bit so that it's also an immersion on the second ball. Then you have a very small leeway to perturb it so it stays an immersion on the first and second balls, etc.) □

Corollary 5.4. Let $U \subset \mathbb{R}^m$ be open. Then there is a *proper* immersion $U \to \mathbb{R}^{2m}$.

Proof. Take a proper function $U \xrightarrow{f_1} [0, \infty)$, and set $f = (f_1, 0, \cdots, 0) : U \to \mathbb{R}^{2m}$. Find an immersion g with $\|f - g\| < 1$ everywhere. (Since g is uniformly close to f, it is also a proper map.) □

Theorem 5.5. Let $U \subset \mathbb{R}^m$ be an open subset. Then there is an embedding (a proper injective immersion) $U \hookrightarrow \mathbb{R}^{2m+1}$.

Proof. You need to check that there are no self-intersections. This is an application of the Weak Sard theorem. If you have a map $f : U \to \mathbb{R}^n$ look at $f(x) - f(x') : \{(x, x') \in$

$U \times U : x \neq x'\} \to \mathbb{R}^n$. You need to make some change to f to make sure you never hit zero. $\square$

Lecture 6: September 16

Theorem 6.1 (Easy Sard)**.** Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^m$ smooth (at least C^1). Then the set of critical values of f,

$$f(\{x : Df_x \text{ is not onto }\})$$

has measure zero. (Equivalently it's $f(\{x : \det(Df_x) = 0\}$.)

The set of critical *points* is not always measure zero! (Consider a constant function.)

Corollary 6.2. For almost all $y \in \mathbb{R}^m$, $f^{-1}(y)$ is a discrete subset of U.

(This is what we'd expect for m equations in m variables.)

Corollary 6.3. If f is proper then for almost all y, $f^{-1}(y)$ is a finite set.

Why does the dimension matter in Sard's theorem? Think about mapping m-dimensional cubes to n-dimensional cubes. If $m < n$ then a cube of volume ε^m goes to a smaller cube of volume ε^n. If $n < m$ things are harder.

Theorem 6.4 (Sard)**.** Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$ be smooth. Then the set of critical values has measure zero.

Note that in this case you probably need more than 1 derivative. "This is the reason for ever caring about manifolds: the solution set to a random set of equations is a smooth manifold, and if it's not you can jiggle it a little bit so that is the case."

Corollary 6.5. For almost all y, $f^{-1}(y) \subset U$ is a submanifold of dimension $m - n$.

Proof of Easy Sard. Take a compact subset $K \subset Crit(f) = \{x : Df_x \text{ is not onto}\}$. It is enough to show that $f(K)$ has measure zero. (Closed subsets are the union of countably many compact ones.)

Claim 6.6. For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $Q \subset U$ is a cube of side length $< \delta$ with $Q \cap K$ nonempty then $f(Q)$ is contained in a finite union of cubes whose total volume is $< \varepsilon \cdot vol(Q)$.

By taking ε small enough, I can ensure that for all $x \in Q$, $\| \left(\frac{\partial f}{\partial x_i}\right)_x \| \le C$ (find a slightly larger compact subset on containing $f(Q)$; derivatives are bounded there). Then $f(Q)$ is contained in a cube of size $\le C' \cdot size(Q)$. There is a point $x \in Q$ where Df_x is not an isomorphism. So there is a $\xi \in \mathbb{R}^m$, $\|\xi\| = 1$, $Df_x(\xi) = 0$. Using uniform continuity of Df a compact subsets, one can achieve that $f(Q)$ lies in a ball neighborhood of a hyperplane whose size is $< \varepsilon \cdot size(Q)$. (If $\langle \xi, Df_x \rangle = 0$ and p is close to x, then $\langle \xi, Df_p(\xi) \rangle \le \varepsilon$, so (the integral) $| \langle f(p) - f(x), \xi \rangle$ is small. So you have an n-dimensional cube squashed into a skinny neighborhood of a hyperplane.) □

Sketch of proof of Sard's theorem. Let D be the critical point set of f,

$$D_1 = \{x \in U \,:\, Df_x = 0\}$$
$$D_2 = \{x \in U \,:\, Df_x = 0, D^2 f_x = 0\}$$
$$\cdots$$

Claims:

(1) $f(D \backslash D_1)$ has measure zero
(2) $f(D_i \backslash D_{i+1})$ has measure zero for all i
(3) $f(D_k)$ has measure zero for large k

(3) is an argument of the kind we've seen before. If Q is a small cube intersecting D_k of size ρ, then $f(Q)$ is contained in a cube of size $< \rho^k$ (Taylor's theorem). For large k, $(\rho^k)^n$ goes to zero faster than ρ^m (for $kn > m$).

(2) omitted (same as proof of part (1)).

(1) By induction on m ($m = 1$ is trivial, since then $D = D_1$). Take a point p in $D \backslash D_1$. By assumption, Df is nonzero at p. Locally near p, there is a diffeomorphism ψ with $\psi(p) = 0$ such that

$$(f \circ \psi^{-1})(x) = (x_1 + c, g_2(x), \cdots, g_n(x))$$

(see Bröcker-Jänich). Then locally,

$$\text{critical values of } f = \bigcup_t \{t\} \times \text{critical values of } g(t, (x_2, \cdots, x_m)).$$

The first column of the Jacobian is $(1, 0, \cdots)$ so the map is not onto iff the remaining minor is not onto.

Now use Fubini's theorem (if the set's intersection with a series of hyperplanes has measure zero then the set has measure zero). □

Lecture 7: September 18

Definition 7.1. A topological manifold of dimension n is a second-countable Hausdorff topological space M which is locally homeomorphic to $\mathbb{R}^n$ – that is, for all $x \in M$ there exists $U \subset M$ containing x such that U is homeomorphic to an open subset in $\mathbb{R}^n$.

Definition 7.2. A space is second-countable if there is a countable collection of open sets which generates the topology.

Why is $\mathbb{R}^n$ second-countable? Take balls of rational radius and rational center.

Hausdorff-ness ensures that a finite set of points in M looks like a finite set of points in $\mathbb{R}^n$. If you're using the quotient topology this is something to watch out for. It's really hard to come up with a case that's not second-countable and the other conditions work – maybe a subset of a non-separable Hilbert or Banach space.

We want functions out of neighborhoods to look like smooth functions out of $\mathbb{R}^n$. To make this reasonable we have to restrict the kind of local neighborhoods we use.

Let M be a topological n-manifold. A smooth atlas $\mathcal{A} = (U_\alpha, g_\alpha)_{\alpha \in A}$ consists of an open cover $M = \bigcup_{\alpha \in A} U_\alpha$ and for each α a homeomorphism $g_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ (for open V_α), such that, for every $\alpha, \beta \in A$, the transition map

$$h_{\beta\alpha} = g_\beta \circ (g_\alpha|_{U_\alpha \cap U_\beta})^{-1}$$

is smooth.

$$\begin{tikzcd}
V_\alpha & & U_\alpha \cap U_\beta \arrow[ll, "g_\alpha"'] \arrow[rr, "g_\beta"] & & V_\beta \\
g_a(U_\alpha \cap U_\beta) \arrow[u, hook] \arrow[urr, "(g_\alpha|U_\alpha \cap U_\beta)^{-1}"] \arrow[urrrr, "h_{\beta\alpha}"'] & & & &
\end{tikzcd}$$

Note:

$$h_{\beta\alpha} : g_\alpha(U_\alpha \cap U_\beta) \to g_\beta(U_\alpha \cap U_\beta)$$

is a homeomorphism. Also note that $h_{\alpha\alpha} = \mathbb{1}_{V_\alpha}$, and $h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha}$ (where both sides are defined). This is called the cocycle condition. It's important because otherwise the definition would be internally inconsistent.

In particular, $h_{\alpha\beta}$ and $h_{\beta\alpha}$ are inverse homeomorphisms. Hence, if they are smooth, they are diffeomorphisms.

Definition 7.3. Let $(M, \mathcal{A} = (U_\alpha)_{\alpha \in A}, g_\alpha)$ and $(\widetilde{M}, \widetilde{\mathcal{A}} = (\widetilde{U}_\beta, g_\beta)_{\beta \in B})$ be topological manifolds with atlases. A continuous map $f : M \to \widetilde{M}$ is called *smooth* if

$$g_\beta \circ f \circ g_\alpha^{-1} : g_\alpha(f^{-1}(\widetilde{U}_\beta)) \to \widetilde{V}_\beta$$

is smooth for all α, β. This makes sense because the transition maps are assumed to be smooth.

Definition 7.4. Let M be a topological n-manifold with two smooth atlases A and $\widetilde{A}$. These are equivalent if the identity map $(M, A) \to (M, \widetilde{A})$ and $(M, \widetilde{A} \to (M, A)$ are smooth.

Intuition: if you have a smooth map with respect to A, then the second identity gives a smooth map with respect to $\widetilde{A}$, and vice versa.

Definition 7.5. A smooth manifold is a topological manifold with an equivalence class of atlases.

Lemma 7.6. *A and $\widetilde{A}$ are equivalent iff the disjoint union $A \sqcup \widetilde{A}$ is again a smooth atlas.*

(This is because we need the transition functions in A to be smooth with respect to $\widetilde{A}$.) Some people say this in terms of a maximal atlas, but that's such a huge thing...

Example 7.7. Let $M = \mathbb{R}, A = \{(\mathbb{1} : \mathbb{R} \to \mathbb{R})\}, \widetilde{A} = \{x^3 : \mathbb{R} \to \mathbb{R}\}$. These are *in*equivalent atlases, because $x^{\frac{1}{3}}$ is not smooth. These manifolds are still diffeomorphic, just not via the identity.

This does happen occasionally and it is a source of grief.

Tangent space. Let M be a smooth n-dimensional manifold (we usually omit the notation for atlases). For $x \in M$ the tangent space T_xM (or written TM_x) is an n-dimensional real vector space, defined below. Elements $\xi \in T_xM$ are called tangent vectors. Take an atlas $(U_\alpha, g_\alpha)_{\alpha \in A}$. Let $A_x \subset A$ be the subset of those α where $x \in U_\alpha$. A tangent vector is a collection $\xi = (\eta_\alpha)_{\alpha \in A_x}$ for $\eta_\alpha \in \mathbb{R}^n$, such that $D_{g_\alpha(x)}(h_{\beta\alpha})\eta_\alpha = \eta_\beta$. (So it's a compatible collection of points in the Euclidean space corresponding to each neighborhood containing x.) ξ is completely specified by one η_α, which can be arbitrary. Therefore the set of such $\xi = (\eta_\alpha)_{\alpha \in A}$ is naturally an n-dimensional vector space.

Why didn't I just specify the tangent vector to be one η_α? It is nicer to add them this way.

A smooth map $f : M \to \widetilde{M}$ has derivatives $Df_x : TM_x \to T\widetilde{M}_{f(x)}$, defined by the action of

$$D(\widetilde{g}_\beta \circ f \circ g_\alpha^{-1}).$$

The chain rule ensures that the compatibility condition is satisfied on the target space.

In particular, a smooth path $(-\varepsilon, \varepsilon) \xrightarrow{\gamma} M$ yields $D\gamma_0 : \mathbb{R} \to TM_{\gamma(0)}$. Conversely, a smooth function $f : M \to \mathbb{R}$ yields $Df_x : TM_x \to \mathbb{R}$ hence an element of $(TM_x)^\vee$.

You can use both approaches to give alternative definitions of the tangent space. If you define it in terms of paths, it's clear that you get all the tangent vectors this way, but it's not clear why it's a vector space.

NO CLASS ON FRI.

Lecture 8: September 23

Let M be a smooth manifold with atlas (U_α, g_α). General principle: define objects that live on a manifold by their local counterparts and transition rules.

DEFINITION 8.1. Let $x \in M$. A tangent vector at x is given by $(\xi_\alpha)_\alpha$, where $\xi_\alpha \in \mathbb{R}^n$ for each $\alpha \in A$ such that $x \in U_\alpha$ satisfying

$$\xi_\beta = D(h_{\beta\alpha})_{g_\alpha(x)}\xi_\alpha$$

where $h_{\beta\alpha} = g_\beta g_\alpha^{-1}$.

Note that any ξ_α determines all the others.

Tangent vectors at x form a vector space T_xM of dimension n.

A cotangent vector (an element of T_xM^* (the dual vector space)) is given by a collection $(\eta_\alpha)_\alpha$ for $\eta_\alpha \in \mathbb{R}^n$ satisfying

$$\eta_\alpha = D(h_{\beta\alpha})^t_{g_\alpha(x)}\eta_\beta.$$

For $\xi \in T_xM$ and $\eta \in T_xM^*$, the pairing

$$\langle \eta, \xi \rangle = \eta_\alpha^t \cdot \xi_\alpha \in \mathbb{R}$$

is well-defined (independent of the choice of α):

$$\begin{aligned} \langle \eta_\beta, \xi_\beta \rangle \, \eta_\beta^t \xi_\beta &= \eta_\beta^t D(h_{\beta\alpha})\xi_\alpha \\ &= (D(h_{\beta\alpha}^t \eta_\beta)^t \xi_\alpha) \\ &= \eta_\alpha^t \xi_\alpha. \end{aligned}$$

Let M be a manifold, $x \in M$. Consider smooth paths (i.e. smooth maps of manifolds) $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = x$. Two such paths c, d are called tangent at x if for some (hence every) $g_\alpha : U_\alpha \to V_\alpha$, $x \in U_\alpha$,

$$\frac{d}{dt} g_\alpha(c(t))|_{t=0} = \frac{d}{dt} g_\alpha(d(t))|_{t=0}.$$

(Note that this requires the tangent vectors to be the same, not just parallel; maybe we should just call this equivalent.)

DEFINITION 8.2. T_xM is the set of tangency equivalence classes of paths at x.

This is equivalent to the previous definition: take a path c and map it to the collection $(\xi_\alpha = \frac{d}{dt} g_\alpha(c(t))|_{t=0})_\alpha$. You need to check this is onto, but you can just construct a path in some local chart.

Let $C^\infty(M, \mathbb{R})$ be the algebra of smooth functions on M. Given a point x, we have the ideal $\mathcal{I}_x \subset C^\infty(M, \mathbb{R})$ of smooth functions that vanish at x. Then $\mathcal{I}_x^2$ is the ideal generated by products fg where $f, g \in \mathcal{I}_x$.

DEFINITION 8.3. T_x^*M is the quotient vector space $\mathcal{I}_x/\mathcal{I}_x^2$.

Lemma 8.4. Take $f \in \mathcal{I}_x$. Suppose that for some chart $g_\alpha : U_\alpha \to V_\alpha$ the derivative of $f \circ g_\alpha^{-1}$ at $g_\alpha(x)$ is zero. Then $f \in \mathcal{I}_x^2$. (The converse is obviously true.)

Using this one shows that the natural pairing

$$\mathcal{I}_x/\mathcal{I}_x^2 \times T_xM \to \mathbb{R} \quad \text{where } (f,[c]) \mapsto \frac{d}{dt} f(c(t))|_{t=0}$$

is nondegenerate. This yields $\mathcal{I}_x/\mathcal{I}_x^2 \cong T_xM^*$.

Let M, N be smooth manifolds, $f : M \to N$ a smooth map.

Definition 8.5. f is an *immersion* if T_xf is injective for all x.

f is a submersion if T_xf is surjective for all x.

A closed subset $P \subset M$ is a k-dimensional submanifold if for each chart (g_α) in an atlas of M, $g_\alpha(P \cap U_\alpha) \subset V_\alpha$ is a k-dimensional submanifold of V_α (i.e. in the sense of Chapter 1).

Note: around each $x \in P$ there is another chart $\widetilde{g} : \widetilde{h} \to \widetilde{V}$, with $\widetilde{g}(x) = 0$, such that $\widetilde{g}(P \cap \widetilde{U}) = (\mathbb{R}^k \times 0^{n-k}) \cap \widetilde{V}$.

Lemma 8.6. If $P \subset M$ is a submanifold, P itself is a manifold in a way so that its inclusion into M is an injective immersion.

For example, if your charts were $\mathbb{R} \to \mathbb{R}, t \mapsto t^3$ then this is not an immersion.

Theorem 8.7. If $f : M \to N$ is a proper injective immersion, then $f(M) \subset N$ is a submanifold.

Theorem 8.8. If $f : M \to N$ is a submersion and $y \in N$ is arbitrary, then $f^{-1}(y) \subset M$ is a submanifold of dimension $\dim M - \dim N$.

Given $f : M \to N$, $y \in N$ is called a regular value of f if T_xf is onto for all $x \in f^{-1}(y)$.

Theorem 8.9. If y is a regular value, then $f^{-1}(y)$ is a submanifold.

If $P \subset N$ is a submanifold, then $T_xP \subset T_xN$. Define the normal space at $x \in P$ to be

$$\nu_xP = T_\alpha N/T_xP.$$

(You want to say it's the orthogonal directions, but that isn't preserved across different charts.)

DEFINITION 8.10. Let $f : M \to N$ be a smooth map, and $P \subset N$ a submanifold. Then f is transverse to P if for all $x \in f^{-1}(P)$,

$$im(T_x f) + T_{f(x)}P = T_{f(x)}N$$

or equivalently, $T_x M \overset{T_x f}{\to} T_{f(x)}N \overset{\text{proj}}{\to} \nu_{f(x)}P$ is onto.

Theorem 8.11. If $f : M \to N$ is transverse to P then $f^{-1}(P)$ is a submanifold of dimension $\dim M - \dim N + \dim P$.

This follows from the corresponding local result.

For example, two surfaces meeting transversely intersect in a smooth path.

LECTURE 9: SEPTEMBER 27

Constructions of submanifolds:

- If $f : M \to N$ is an embedding (proper injective immersion), then $f(M) \subset N$ is a submanifold
- If $f : M \to N$ is a submersion, then $f^{-1}(y)$ is a submanifold for all y (also, generalizations)
- Suppose $M \subset N$ is a closed connected subset. Suppose there is an open neighborhood $M \subset U \subset N$ and a smooth map $f : U \to U$, $r(U) = M$, $r|_M = \mathbb{1}_M$. Then M is a submanifold. (The converse is true but we can't prove it yet. Also notice that this doesn't work if the retraction is just *continuous*, not smooth – there is a continuous retraction of the figure 8.)

DEFINITION 9.1. Let M be a Hausdorff space. An action of a group G on M is called *properly discontinuous* if for all $x, y \in M$ there are neighborhoods $x \in U, y \in V$ such that

$$g(U) \cap V = \emptyset$$

for all but finitely many $g \in G$.

"It should actually be called discontinuously proper."

REMARK 9.2. This implies that the G-action has finite stabilizers $G_x = \{g \in G : gx = x\}$.

REMARK 9.3. If G is finite then every G-action is properly discontinuous.

Lemma 9.4. If G acts properly discontinuously then X/G (with the quotient topology) is Hausdorff.

Proof. Take x, y which do not lie in the same orbit. Choose $x \in U, y \in V$ such that $g(U) \cap V \neq \emptyset$ only for $g_1, \cdots, g_r \in G$. Note $g_i(x) \neq y$, hence I can find a $U_i \ni g_i(x)$ open,

$V_i \ni y$ open such that $U_i \cap V_i = \emptyset$. Then

$$\widetilde{U} = U \cap \bigcap_{i=1}^{r} g_i^{-1}(U_i), \quad \widetilde{V} = V \cap \bigcap_{i=1}^{r} V_i.$$

Then $g(\widetilde{U}) \cap \widetilde{V} = \emptyset$ for all g. Hence, $\bigcup_{g\in G} g(\widetilde{U})$ and $\bigcup_{g\in G} g(\widetilde{V})$ are G-invariant disjoint open sets, one containing x, the other containing y. □

Theorem 9.5. Let M be a smooth manifold, G a group acting freely and properly discontinuously by diffeomorphism of M. Then M/G is a smooth manifold.

That is, $\pi : M \to M/G$ will be a smooth map with $T_x\pi$ invertible for all x. (So by the inverse function theorem you can get a local inverse and get charts by composing with the charts of M.)

Proof. For all $x \in M$ there is an open neighborhood $U \ni x$ such that $U \cap g(U) = \emptyset$ for all $g \neq e$ (similar to earlier argument). Use charts whose closure is contained in such U. □

Example 9.6.

- $\mathbb{R}P^n = S^n \big/ (\mathbb{Z}/2)$ (quotient by the antipodal map)
- Space forms S^n/G (quotients by a finite subgroup $G \subset O(n+1)$ such that nontrivial elements of G do not have 1 as an eigenvalue – this guarantees there are no fixed points).
- A particular example of a space form: S^3/G where $G \subset SU(2)$ is a finite subgroup (think of S^3 as a subset of $\mathbb{C}^2$). The finite subgroups of $SU(2)$ correspond (with one exception) to the Platonic solids.
- $T^n = \mathbb{R}^n/\mathbb{Z}^n$ ($\mathbb{Z}^n$ acting by translation)
- Given a manifold M and a diffeomorphism $f : M \to M$, the mapping torus is $(\mathbb{R} \times M)/\mathbb{Z}$ with $k \in \mathbb{Z}$ acting by $(t, x) \mapsto (t - k, f^k(x))$. The point is to change a discrete action (think of f as a discrete dynamical system) into a continuous action, with the price that you add an extra dimension.

Partitions of Unity.

Theorem 9.7. Let M be a smooth manifold. Then every open cover of M has a subordinate smooth partition of unity.

Proof. Essentially the same as in the $\mathbb{R}^n$ case, except you have to use second-countability. □

Corollary 9.8. For every smooth manifold M there is a proper smooth function $f : M \to [0, \infty)$.

Corollary 9.9. Any smooth n-manifold has a proper immersion into $\mathbb{R}^{2n}$.

Corollary 9.10. Any smooth manifold M has an embedding into $\mathbb{R}^{2n+1}$.

Theorem 9.11 (Sard)**.** If $f : M \to N$ is a smooth map of manifolds, the set of critical values

$$\{y \in N \,:\, y = f(x) \text{ for some } x \text{ such that } T_x f \text{ is not onto}\}$$

has measure zero.

Note that manifolds don't have a notion of *volume* – even locally, this depends on the chart used. But, *measure zero* does make sense – it's a set that has measure zero in every chart.

There are infinitely many 4-manifolds which are homeomorphic to the 4-sphere.

Lecture 10: September 30

Theorem 10.1 (Ehresmann)**.** Let $\pi : M \to N$ be a proper smooth submersion. For every $x \in N$ there is an open $x \in U \subset N$ and a diffeomorphism $f : U \times \pi^{-1}(y) \to \pi^{-1}(U)$ such that

$$\begin{tikzcd} U \times \pi^{-1}(y) \arrow[rr, "f", "\cong"'] \arrow[dr, "proj"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

commutes.

Corollary 10.2. If $\widetilde{y} \in U$, $\pi^{-1}(\widetilde{y})$ is diffeomorphic to $\pi^{-1}(y)$.

Corollary 10.3. If N is connected, any two fibers of π are diffeomorphic. (So π is a fibre bundle with structure group $Diff(\pi^{-1}(y))$).

This is a nice property, and it fails in many other settings such as algebraic geometry.

If the dimensions are equal, then every point in N has finitely many preimages, and you can see pretty easily that it is a finite cover. This is easy because there is no ambiguity in choosing the trivialization: given a point in one fiber, it is easy to shift it into a nearby fiber. In the general case, you want to do this locally and patch them together using partitions of unity. This is kind of annoying – how do you add diffeomorphisms? Fix this by generating the diffeomorphisms by integrating vector fields. This has the advantage that you can add vector fields.

Definition 10.4. Let M be a smooth manifold, with atlas $(U_\alpha, g_\alpha)_{\alpha\in A}$. A smooth vector field X is a collection of tangent vectors $X_x \in T_xM$ such that the map $g_\alpha(U_\alpha) \to \mathbb{R}^n$ sending $x \mapsto (X_{g_\alpha^{-1}(x)})_\alpha$ is smooth (where $(\cdots)_\alpha$ means looking at the value in the α^{th} chart).

Lemma 10.5. Let $N \subset M$ be a smooth submanifold. Given a vector field Y on N, there is a vector field X on M such that $X|_N = Y$.

(What does $X|_N$ mean? If $y \in N$ then $T_yN \subset T_yM$.) This is called a vector field extension property.

Proof. Take an open cover $(U_\alpha)_{\alpha\in A^+\sqcup A^-}$ of M such that

(1) if $\alpha \in A^-$, then $U_\alpha \cap N = \emptyset$;
(2) if $\alpha \in A^+$ then there is a diffeomorphism $g_\alpha : U_\alpha \to \mathbb{R}^n$, and $g_\alpha(U_\alpha \cap N) = B \cap (\mathbb{R}^m \times 0)$.

Let (ψ_α) be a subordinate partition of unity. We define our desired extension as $X = \sum_\alpha \psi_\alpha X_\alpha$, where the X_α are vector fields on U_α. This is automatically smooth. If $\alpha \in A^-$, choose arbitrary X_α, and if $\alpha \in A^+$, choose X_α such that $X_\alpha|_{U_\alpha\cap N} = Y|_{U_\alpha\cap N}$ (it is easy to extend a vector field on $\mathbb{R}^m \times 0$ to the rest of $\mathbb{R}^n$). Finally, we need to check that $X|_N = Y$. But at $y \in N$, $\sum_\alpha \psi_\alpha X_{\alpha,y} = (\sum_{\alpha\in A^+\psi_\alpha})Y_y = Y_y$. □

Remark 10.6. If $X_1, \cdots, X_m$ satisfy $X_i|_N = Y$ and $f_1, \cdots, f_m$ are functions defined on them such that $\sum_{i=1}^m f_i = 1$, then $X = \sum f_i X_i$ also satisfies $X|_N = Y$.

Lemma 10.7. Let $\pi : M \to N$ be a submersion. If Y is a vector field on N, there is a vector field X on M such that $Y_{\pi(x)} = (T\pi)_x(X_x)$.

This is called a vector field lifting property.

Proof. Solve locally; glue together. (Locally, it looks like linear projection, and is easy. Then use partition of unity, and it's easier than before because there are not two types of neighborhoods.) □

If $\dim M = \dim N$ then this is unique.

Take $M \ni x$, X a smooth vector field. The flow line of X through x is a smooth map $c : \Omega_x \to M$, where

(1) $\Omega_x \subset \mathbb{R}$ is an open interval containing 0

(2) $c(0) = x$, $\frac{dc}{dt} = X_{c(t)}$
(3) Ω_x is maximal among open intervals for which such a c exists

Theorem 10.8. Flow lines exist and are unique.

What happens when you reach the endpoints of the interval? If $a = \sup \Omega_x \in \mathbb{R}$ then $c(x), x \to a^-$ has no convergent subsequence ("goes to ∞"). The same thing is true for inf.

Corollary 10.9. If M is compact then $\Omega_x = \mathbb{R}$.

Theorem 10.10. Let X be a smooth vector field on M. Take $\Omega = \bigcup \Omega_x \times \{x\} \subset \mathbb{R} \times M$. Then Ω is open, and the map $\Phi : \Omega \to M$ such that $\Phi_t(x) := \Phi(t, x) = c(t)$ if c is the flow line through x, is smooth.

Moreover, $\Phi_0(x) = x$ and $\Phi_s(\Phi_t(x)) = \Phi_{s+t}(x)$ wherever defined.

Φ is called the flow of X.

In particular, if $\Omega = \mathbb{R} \times M$, then each Φ_t is a diffeomorphism (because $\Phi_t \circ \Phi_{-t} = \mathbb{1}$).

Lecture 11: October 2

Theorem 11.1 (Ehresmann)**.** Let $\pi : M \to N$ be a proper submersion. For every $y \in N$ there is an open set $V \subset N$ around y and a diffeomorphism

$$\begin{tikzcd} V \times \pi^{-1}(y) \arrow[rr, "\cong"] \arrow[dr, "proj"'] & & \pi^{-1}(V) \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

If you drop properness, this is false: just take any immersion of an open subset.

Choose tangent vector fields $Y_1, \cdots, Y_n$ on N (where $n = \dim N$) so that $Y_1(y), \cdots, Y_n(y)$ is a basis. (Why can you do this? Take a basis at y, extend it locally and then use bump functions.) For small enough ε we can define a map $G : (-\varepsilon, \varepsilon)^n \to N$ which takes $(t_1, \cdots, t_n) \mapsto \varphi_{t_1}^{Y_1} \circ \cdots \circ \varphi_{t_n}^{Y_n}(y)$ where $\varphi_{t_i}^{Y_i}$ is the flow of the vector field Y_i at time t_i (so this is a single vector). (Note that this depends on some arbitrary choice of ordering; I can make the flows pairwise commute if I work harder, but it doesn't matter.)

This is defined for small times, i.e. you can choose ε so this is well-defined (we're using the fact that the domain is open). A diffeomorphism from $(-\varepsilon, \varepsilon)^n$ onto an open neighborhood of $y \in N$. The latter statement follows from the inverse mapping theorem. (If $(t_1, \cdots, t_n) = (0, \cdots, t_i, \cdots, 0)$ then you are just flowing in the direction of Y_i; so $\frac{\partial G}{\partial t_i}(0, \cdots, 0) = Y_i(y)$).

Choose vector fields $X_1, \cdots, X_n$ on M such that $(T\pi)_x)(X_i = (U_i)_{\pi(x)}$ (lifting theorem). Define

$$\begin{aligned} (-\varepsilon, \varepsilon)^n \times \pi^{-1}(y) &\xrightarrow{F} M \\ (t_1, \cdots, t_n, x) &\mapsto \varphi_{t_1}^{x_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(x) \end{aligned}$$

Note this is well-defined for small ε: the flow is defined if t_i are small, but *how small* depends on x. But we're in a compact set $\pi^{-1}(y)$, so there's a nonzero minimum that makes sense. Also, note that it fits into a commutative diagram

$$\begin{tikzcd} (-\varepsilon, \varepsilon)^n \times \pi^{-1}(y) \arrow[r, "F"] \arrow[d, "proj"] & M \arrow[d, "\pi"] \\ (-\varepsilon, \varepsilon)^n \arrow[r, "G"] & N \end{tikzcd}$$

Going on the top right solves the same ODE as going through the bottom left, so this commutes by uniqueness of ODE solutions. So $F|_{\{(0,\cdots,0)\} \times \pi^{-1}(y)}$ is the inclusion, and $T_{(0,\cdots,0,x)F}$ is an isomorphism for all $x \in \pi^{-1}(y)$. (equal dimension + easily seen to be surjective)

Possibly after making ε smaller, F is a diffeomorphism onto $\pi^{-1}(G((-\varepsilon, \varepsilon)^n))$. This uses properness: suppose you start in one fiber, and move this a little bit by the vector field. How do you know that you've caught everything in the nearby fibers? If not, then there would be a missing point in the original fiber, and a subsequence in the other fibers converging to this missing point. So you really need compactness.

If you have a non-submersion, the fiber can be horrible. If I pick a random fiber, then I get a submanifold, but which?

Manifolds with boundary. Unlike other notions in this course like smoothness and transversality which could not possibly be improved upon, the notion of manifolds with boundary is just horrible.

Definition 11.2. A topological n-manifold with boundary is a second-countable Hausdorff space M such that each point $x \in M$ has an open neighborhood $x \in U \subset M$ homeomorphic to an open subset $V \subset [0, \infty) \times \mathbb{R}^{n-1}$ (i.e. it's open in $[0, \infty) \times \mathbb{R}^{n-1}$).

If $\varphi(x) \in (0, \infty) \times \mathbb{R}^{n-1}$ we say that x is an interior point. If $\varphi(x) \in \{0\} \times \mathbb{R}^{n-1}$, x is called a boundary point.

Warning: Needs some work to show it's a well-defined distinction (i.e. you need to check there isn't some homeomorphism that makes a boundary point look like an interior point or vice versa). This property is called "invariance of domain".

Write $M = int(M) \sqcup \partial M$, where $int(M)$ is an n-manifold (without boundary) and ∂M is an $(n-1)$-manifold (without boundary).

Definition 11.3. Let $V \subset [0,\infty) \times \mathbb{R}^{n-1}$ be an open subset. A map $f : V \to \mathbb{R}^m$ is smooth if for each $x \in V$ there is an $x \in \widetilde{V}$ open and a smooth $\widetilde{f} : \widetilde{V} \to \mathbb{R}^m$ such that $\widetilde{f}|_{\widetilde{V}\cap V} = f$.

So smoothness is defined using an extension $\widetilde{f}$ of f that is (1) not unique; (2) defined in a part of the space you don't want to think about. If you just want functions that are C^k, then this is definitely the same as the "usual definition". This is sort of unclear for C^∞ functions.

Definition 11.4. Let M be a topological n-manifold with boundary. A smooth atlas (U_α, g_α) is an open cover (U_α) together with homeomorphisms $g_\alpha : U_\alpha \to V_\alpha \subset [0,\infty) \times \mathbb{R}^{n-1}$, such that the transition maps $h_{\beta\alpha} = g_\beta g_\alpha^{-1}$ are smooth.

Now $M = int(M) \sqcup \partial M$, where $int(M)$ is a smooth n-manifold and ∂M is a smooth $(n-1)$-manifold (take the transition maps and restrict to the boundary).

Theorem 11.5 (Collar neighborhood theorem)**.** Let M be a smooth manifold with boundary. Then there is an open subset $V \subset M$ containing the boundary, and an open subset $U \subset [0,\infty) \times \partial M$ containing $\{0\} \times \partial M$, and a diffeomorphism $U \to V$.

Proof. Same proof as always. Locally, this is certainly true. Find a vector field X on M which points inwards along the boundary. (This property is invariant under linear combinations, so you can construct X using partitions of unity.) Integrate that vector field to get a map C such that $C(0,x) = x$ and $T_{(0,x)}C$ is an isomorphism for all x. The image is an open neighborhood of ∂M. By making U small, you can guarantee that $T_{(t,x)}C$ is an isomorphism for all $(t,x) \in U$. □

Lecture 12: October 4

This definition is due to Thom; he first called it "cobordant" but then "co" acquired other meanings, and so people decided to call it "bordant".

Definition 12.1. Let M_0, M_1 be compact n-dimensional manifolds. We say that M_0 is bordant to M_1 if $M_0 \sqcup M_1$ is the boundary of a compact $(n+1)$-dimensional manifold with boundary. More precisely, if there is a compact $(n+1)$-manifold with boundary N and a diffeomorphism $\partial N \xrightarrow{\cong} M_0 \sqcup M_1$.

This is weaker than diffeomorphism: if M_0 is diffeomorphic to M_1, we can take $N = M_0 \times [0,1]$ (a manifold with boundary) and $\partial N = M_0 \sqcup M_0$ is diffeomorphic to $M_0 \sqcup M_1$. The converse is obviously not true: there is a bordism from S^1 to $S^1 \sqcup S^1$.

Lemma 12.2. Bordism is an equivalence relation.

Proof. Suppose $\partial N_0 = M_0 \sqcup M_1$ and $\partial_1 = M_1 \sqcup M_1$. We have to show that there is some N for which $\partial N = M_0 \sqcup M_2$. The answer is to set $N = N_0 \cup_{M_1} N_1$ (i.e. $N_0 \sqcup N_1$ where the copies of M_1 in each are identified). But there is an issue with the differentiable structure: what is a smooth function on an open set including parts of M_1 and stuff from both sides? Use a collar neighborhood, and that gives charts that are easy to glue. This is well-defined *up to diffeomorphism.* $\square$

Definition 12.3. Let Ω_n be the set of bordism classes of n-dimensional smooth manifolds.

Lemma 12.4. Under disjoint union and Cartesian products, Ω_* is a graded commutative ring.

The operation of disjoint union clearly descends to bordism classes, and similarly for Cartesian products: if $M_1 \sqcup M_2 = \partial N$ then $(M_1 \times P) \sqcup (M_2 \times P) = \partial(N \times P)$. The product is commutative and distributive w.r.t. disjoint unions. The empty set is "0" (and can have any dimension).

Remark 12.5. Note $M \sqcup M$ is the boundary of $M \times I$, and so it is bordant to $\emptyset$. So in Ω_*, $[M] + [M] = [M \sqcup M] = 0$, so Ω_* is an algebra over $\mathbb{F}_2$.

So even though we say "graded commutative" there actually aren't signs to worry about.

What is this ring?

Proposition 12.6. $\Omega_0 \cong \mathbb{Z}/2$

Proof. A compact 0-manifold is a finite set. We've already shown that 2 points are bordant to zero. Why is 1 point not bordant to zero? This depends on the classification of 1-dimensional manifolds (with boundary) up to diffeomorphism: the circle, real line, closed interval, and half-open interval. (We won't do this now; it is trivialized by Riemannian geometry.) The only compact 1-manifold with boundary is the closed interval, and that doesn't make one point bordant to two points. $\square$

Proposition 12.7. $\Omega_1 = 0$

Proof. This is true for the same reasons: the only compact 1-manifold is the circle, and that's bordant to the empty set (since $S^1 = \partial D^2$). $\square$

Proposition 12.8 (Thom or maybe Pontryagin)**.** $\Omega_2 \cong \mathbb{Z}/2$

I will not show this (although it's not all that hard). Later we will show that $[\mathbb{R}P_2] \neq 0$, but there is no completely elementary way to show this. Thom's thesis determines Ω_n for all n. This is a completely hilarious and known way to classify manifolds. It is the natural higher-dimensional analogue of counting mod 2. My 1-year-old can currently count to 2; we will see whether he progresses to counting to higher numbers or counting bordism classes of manifolds.

Theorem 12.9. Let $f : M \to N$ be a smooth proper map, and N connected. Then any two regular fibres $\pi^{-1}(y)$, for regular values $y \in N$, are bordant.

This is the reason we're looking at bordism.

Corollary 12.10. If for some regular value y, $[\pi^{-1}(y)] \in \Omega_*$ is nonzero, f must be surjective.

(If it misses a point, then that point has empty fiber. . .)

Take M, N compact. The maps $f_0, f_1 : M \to N$ are *homotopic* if there is a $h : M \times [0,1] \to N$ such that $h|_{\{0\}\times M} = f_0$, $h|_{\{1\}\times M} = f_1$. Homotopy is an equivalence relation. (This is not totally obvious: due to the differentiable structures you might have to worry about local patching.)

Let $[M, N]$ be the set of homotopy classes. Then if N is connected, define a map $[M, N] \to \Omega_{\dim M - \dim N}$ that sends $[f] \mapsto [f^{-1}(y)]$ for any regular value y. (Apply the theorem to $(t, f(t, x)) : M \times [0, 1] \to N \times [0, 1]$.) It's not just that f must be surjective, but you can't deform it into something that isn't surjective.

It is true, but I'm not going to prove it, that $[M, N]$ is the same thing as the homotopy classes of *continuous* maps (use patching to smooth out a merely continuous map).

Definition 12.11. Let M, N be compact n-manifolds, where N is connected. The degree (mod 2) of a smooth map $f : M \to N$ is

$$\deg f = \# f^{-1}(y) \pmod 2$$

where y is any regular value.

Look at $x = y^2$: there are 2 preimages at $x > 0$, 1 at $x = 0$, and none at $x < 0$. But $x = 0$ isn't a regular value, and so the number of preimages is always $0 \pmod 2$.

Corollary 12.12. Let M be a compact n-manifold. Any smooth map $M \to M$ which is homotopic to the identity is onto.

Let's return to the theorem about bordism classes of fibres. If $\dim N = 0$ then there is nothing to show. If $\dim N = 1$, then $N = S^1$ or $N = \mathbb{R}$. Then the theorem is obvious: in the case $N = \mathbb{R}$ assume $a < b \in \mathbb{R}$ are regular values; then $f^{-1}([a, b])$ is a manifold with boundary: use the charts given by the constant rank theorem gives the desired atlas. A similar argument works for $N = S^1$.

In general, if $a, b \in N$ and N is connected, then take the preimage of a path from a to b. The problem is that it's possible to choose a path where the preimage is bad. But, the preimage should be good for a random path.

Lecture 13: October 7

Let $f : M \to N$ be a smooth proper map, where N is connected. Let $y_0, y_1 \in N$ be regular values. We want to show that $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are bordant: that

$$f^{-1}(y_0) \sqcup f^{-1}(y_1) = \partial B$$

for some B. For a smooth path $c : [0,1] \to N$ (not necessarily embedded) where $c(0) = y_0$, $c(1) = y_1$, consider

$$B = \{(t, x) \in [0,1] \times M \ : \ c(t) = f(x).\}$$

B is a manifold with boundary $\partial B = f^{-1}(y_0) \sqcup f^{-1}(y_1)$ if the map

$$\begin{aligned} [0,1] \times M &\to N \times N \\ (t, x) &\mapsto (c(t), f(x)) \end{aligned}$$

is transverse to the diagonal $\Delta_N \subset N \times N$. (We know in this case that the preimage of the diagonal is a smooth manifold. If you're bothered by the fact we have to use this fact for manifolds with boundary, use $(-\varepsilon, 1+\varepsilon)$ instead of $[0,1]$.)

Example 13.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = (x_1, x_2^2)$. In this case we want our path to cross $\{y_2 = 0\}$ transversely. This gives a cobordism between 2 points and nothing. But if your path doesn't cross the line transversely, then at some point the preimage is 1 point at the bad point, and we don't get a manifold.

In general, you want to "choose a path randomly": make a finite-dimensional space of paths, and then choose randomly within that subset.

Take a family of paths $\gamma : U \times [0,1] \to N$, where $U \subset \mathbb{R}^p$ is an open neighborhood of zero, such that $\gamma(r, 0) = y_0$, $\gamma(r, 1) = y_1$ (i.e. they all start at y_0 and end at y_1), $\gamma(0, t) = c(t)$ (the zero-indexed path is c), and: for every $(r, t) \in U \times (0,1)$,

$$\frac{\partial \gamma}{\partial r_1}(r, t), \cdots, \frac{\partial \gamma}{d r_p}(r, t)$$

generate $TN_{\gamma(r,t)}$. (That is, by varying the p possible parameters of r you're moving the point in all directions.) For sufficiently large p, one can find such a family using partition of unity arguments.

Claim 13.2.

$$\eta : U \times [0,1] \times M \to N \times N$$

$$(r, t, x) \mapsto (\gamma(r, t), f(x))$$

is transverse to Δ_N.

At (r, t, x) such that $\gamma(r, t) = f(x)$, $\operatorname{im} T_{(r,t,x)\eta} \subset T(N \times N) = TN \times TN$:

- contains $T_{f(x)}N \times \{0\}$ if $0 < t < 1$ (because of the way the paths were defined) OR
- contains $\{0\} \times T_{f(x)}N$ if $t = 0, 1$ (these points were regular points to begin with).

In each case once you add $T\Delta_N$ you get everything.

Therefore, $\{(r, t, x) : \gamma(r, t) = f(x)\}$ is a smooth manifold with boundary $U \times \{0\} \times \pi^{-1}(y_0) \sqcup U \times \{1\} \times \pi^{-1}(y_1)$. This is a manifold, but the dimension is huge, and U isn't compact. Now use the idea of "random choice". The projection $C \to U$ is a smooth proper map. Apply Sard's theorem: if $\rho \in U$ is a regular value, then

$$\{(t, x) : \gamma(\rho, t) = f(x)\}$$

is a compact manifold with boundary $\pi^{-1}(y_0) \cap \pi^{-1}(y_1)$.

Theorem 13.3 (Thom transversality theorem)**.** Let $f : M \to N$ be a smooth map, $P \subset N$ a smooth submanifold. Then there is an $\widetilde{f} : M \to N$ "arbitrarily close" to f, such that $\widetilde{f}$ is transverse to P.

What do I mean by "arbitrarily close"? If M is compact, in the sense of uniform convergence; in general, take an open set $U \subset N \times N$ containing Δ_N and ask that $(f(x), \widetilde{f}(x)) \in U$ for all $x \in M$.

Application 13.4. Take smooth maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$, where M_1 and M_2 are compact. Consider $(f_1, f_2) : M_1 \times M_2 \to N \times N$. By Thom's theorem, we can find a map $F : M_1 \times M_2 \to N \times N$ arbitrarily close to $f_1 \times f_2$ which is transverse to the diagonal. Take F sufficiently close (homotopic to (f_1, f_2)). Then, $F^{-1}(\Delta_N)$ is again a compact manifold, where $\dim F^{-1}(\Delta_n) = \dim M_1 + \dim M_2 - \dim N$ (count the number of constraints for lying in the diagonal).

Proposition 13.5. $[F^{-1}(\Delta_N)] \in \Omega_*$ is independent of the choice of F. Also, $[F^{-1}(\Delta_N)]$ depends only on the homotopy classes of f_1, f_2.

(Think about two surfaces in $\mathbb{R}^3$; in general, their intersection might not be nice, but we can perturb them a little so the intersection is a curve. This theorem says that the associated intersection curve is well-defined up to bordism.)

We write $[F^{-1}(\Delta_N)] = [f_1] \cdot [f_2] \in \Omega_*$. In the case $\dim M_1 + \dim M_2 = \dim N$, $[f_1] \cdot [f_2] \in \mathbb{Z}/2$.

Concretely, suppose that (f_1, f_2) already intersects Δ_N transversely. This is the same thing as saying for all $(x_1, x_2) \in M_1 \times M_2$ such that $f_1(x_1) = f_2(x_2)$, we have $\operatorname{im} T_{x_1} f_1 + \operatorname{im} T_{x_2} f_2 = T_{f_1(x_1)N}$. Then

$$[f_1] \cdot [f_2] = [(f_1, f_2)^{-1}(\Delta_N)]$$

and if $\dim M_1 + \dim M_2 = \dim N$,

$$[f_1] \cdot [f_2] = \#\{(x_1, x_2) : f_1(x_1) = f_2(x_2) \pmod 2\}.$$

If this happens to be 1, then no matter how you deform f_1 and f_2 there will always be some intersection point.

$[f_1] \cdot [f_2]$ is called the mod 2 intersection number.

Lecture 14: October 9

Take a compact manifold M, and a smooth map $f : M \to M$. We have embeddings of submanifolds:

$$\begin{aligned} \gamma_f : M &\to M \times M & x &\mapsto (x, f(x)) \\ \delta : M &\to M \times M & x &\mapsto (x, x) \end{aligned}$$

Definition 14.1. The Lefschetz fixed point number (mod 2) of f is $[\gamma_f] \cdot [\delta] \in \mathbb{Z}/2$.

Lemma 14.2. This is a homotopy invariant of f, and vanishes if f has no fixed points. In other words, it's an obstruction to deforming a map to something with no fixed points.

How do you compute this more concretely?

Lemma 14.3. Suppose that every fixed point x of f is nondegenerate, meaning that 1 is not an eigenvalue of $T_x f : T_x M \to T_x M$. Then

$$L(f) = \# \text{ fixed points} \pmod 2.$$

(1 not being an eigenvalue is equivalent to γ_f and δ being transverse.) Algebraic geometry equivalent: the fixed-point scheme is reduced.

So the strategy is to take a map and homotope it to something where the fixed points are nondegenerate (the existence of such a homotopy is an application of Sard's theorem). Then count the fixed points.

Variations 14.4.

- Compactness ensures there are finitely many fixed points. M could be non-compact, as long as $\overline{f(M)} \subset M$ is compact (and similarly for all maps homotopic to M). Then, the fixed points are in $\overline{f(M)}$.

- M can be a compact manifold with boundary, but the image of f has to avoid the boundary (and similarly for all maps homotopic to f)
- Actually, you don't need a manifold at all. Let X be a compact topological space which is ENR, and $f : X \to X$ is a continuous self-map. (ENR = "Euclidean neighborhood retract", which means there is an open set $U \subset \mathbb{R}^n$ and a continuous map $U \to U$ such that $r^2 = r$ and $r(U)$ is homeomorphic to X; e.g. finite CW complexes are ENR's.) Identify $X = r(U)$. Take
$$U \xrightarrow{r} X \xrightarrow{f} X \xhookrightarrow{i} U.$$
Problem: it's not smooth. Approximate $i \circ f \circ r$ by a smooth homotopic map $\widetilde{f}$ (pick your favorite method for smoothing a continuous function) and set
$$L(f) := L(\widetilde{f}).$$
You have to show that this is independent of how the retraction is chosen, etc. This has the same fixed points as the original f.

Definition 14.5. Let M be a compact manifold. The mod 2 Euler characteristic of M is $L(\mathbb{1}_M) = \chi(M) \in \mathbb{Z}/2$.

Example 14.6 ($\chi(T^n) = 0$). Deform $\mathbb{1}_{T^n}$ to $x \mapsto x+\alpha$ for $\alpha \in \mathbb{R}^n \backslash \mathbb{Z}^n$. This is fixed-point free!

Example 14.7 ($\chi(S^n) = 0$). For odd $n = 2m - 1$, think of $S^{2m-1} \subset \mathbb{C}^m$ and deform the identity to a complex rotation $x \mapsto \zeta x$, for $|\zeta| = 1$, $\zeta \neq 1$. This is fixed point free.

For all n, there is a map $f : S^n \to S^n$ homotopic to the identity, which has 2 nondegenerate fixed points. Take the sphere, move it up by $\frac{1}{2}$, and project back to the unit sphere: take
$$f(x) = \frac{x + (0, \cdots, \frac{1}{2})}{\|x + (0, \cdots, \frac{1}{2})\|}.$$
The fixed points are $x_\pm = (0, \cdots, \pm 1)$. A tangent vector around $(0, \cdots, 1)$ ends up getting scaled by $\frac{2}{3}$, and a tangent vector at $(0, \cdots, -1)$ gets scaled by 2.

Example 14.8 ($\chi(\mathbb{R}P^2) = 1$). Recall $\mathbb{R}P^2 = S^2/$ antipodal map. Rotation of S^2 fixes, say, $(0, \cdots, \pm 1)$ and induces a map $\mathbb{R}P^2 \to \mathbb{R}P^2$ with one nondegenerate fixed point $[0 : \cdots : 1]$.

Generalized example 14.9 (χ(figure eight) $= 1$)**.** Embed this in $\mathbb{R}^2$, take the continuous retraction, smooth it out, and count the fixed points. You do it!

Theorem 14.10 (Brouwer)**.** Every continuous map $f : B \to B$ (where $B \subset \mathbb{R}^n$ is a closed ball) has a fixed point.

Proof. Suppose f has no fixed points.

Step 1: without loss of generality assume $f(B) \subset B \backslash \partial B$ (slight radial shrinking). If this has a fixed point, shrink it more, get a convergent subsequence, etc.

Step 2: without loss of generality f is smooth.

Now $L(f) = 0$ but f is homotopic to the constant map $g(x) \equiv 0$ (because the space is contractible), which has $L(g) = 1$. Contradiction. $\square$

Remark 14.11. This generalizes to any compact manifold with boundary and any map homotopic to a constant map (in particular, any map if $\mathbb{1}_M$ can be deformed to a constant).

Example 14.12. Take
$$U = \{x \in \mathbb{C}^n : 1 \le |x| \le 2\}.$$
Then a complex rotation is fixed-point free. So this has nothing to do with fundamental group.

If two maps are homotopy-equivalent, then you can show (just by counting points) that they have the same Lefschetz numbers. Look at Dold's *Lectures on Algebraic Topology.*

Lecture 15: October 11

Introductory remark. Given charts for a manifold
$$M \supset U_\alpha \stackrel{g_\alpha}{\to} V_\alpha \subset \mathbb{R}^n$$
$$M \supset U_\beta \stackrel{g_\beta}{\to} V_\beta \subset \mathbb{R}^n$$
and transition function
$$h_{\beta\alpha} = g_\beta \circ g_\alpha^{-1} : V_\alpha \cap g_\alpha(U_\beta) \to V_\beta \cap g_\beta(U_\alpha),$$
$D(h_{\beta\alpha})$ can be considered as a map
$$D(h_{\beta\alpha}) : V_\alpha \cap g_\alpha(U_\beta) \to GL(n, \mathbb{R}).$$
$GL(n, \mathbb{R})$ has a very rich topology: it has two connected components, is not simply connected if $n > 1$, ... We want to impose restrictions on the image of $D(h_{\beta\alpha})$, but just asking the map to be nullhomotopic isn't very interesting: the domain could just be a ball. The condition has to involve all charts at the same time. The notion of orientation is the first nontrivial topological restriction one can impose on these maps.

Definition 15.1. Let M be a topological manifold. A smooth atlas $(U_\alpha, g_\alpha : U_\alpha \to V_\alpha)$ is *oriented* if all the transition maps satisfy
$$\det D(h_{\beta\alpha}) > 0$$
everywhere.

There is a notion of equivalence of oriented atlases (two are equivalent if their union is an oriented atlas).

Definition 15.2. An oriented smooth manifold is a manifold with an equivalence class of oriented atlases.

Suppose that $M, \widetilde{M}$ are oriented manifolds, and $f : M \to \widetilde{M}$ is a diffeomorphism. We can look at this in local coordinates:

$$V_\alpha \stackrel{\text{chart}}{\leftarrow} U_\alpha \subset M \stackrel{f}{\to} \widetilde{M} \supset \widetilde{U_\beta} \stackrel{\text{chart}}{\to} \widetilde{V}_\beta \subset \mathbb{R}^n.$$

There is a partially defined map $f_{\beta\alpha} : V_\alpha \to \widetilde{V}_\beta$.

Then, the sign of $D(f_{\beta\alpha})$ at a point is independent of these choices of oriented charts (the determinants of the Jacobians of the transition functions are always positive).

These signs give a locally constant function $M \to \{\pm 1\}$; in particular, if M is connected, then the sign is always 1 or -1. In the former case we say that the diffeomorphism is orientation-preserving (and otherwise say it is orientation-reversing). By applying this to $f = \mathbb{1}$, we obtain:

Lemma 15.3. If M admits an orientation, all other choices of orientation are parametrized by $\{\pm 1\}^{\text{connected components}} = \{\text{locally constant } M \to \{\pm 1\}\}$.

You can always reverse the orientation – apply a diffeomorphism-reversing diffeomorphism. Also you can do this just to one connected component.

Remark 15.4. If M is a manifold with boundary (and oriented), then ∂M acquires an orientation.

The convention is that if $M \supset U_\alpha \to V_\alpha \subset (-\infty, 0] \times \mathbb{R}^{n-1}$ is an oriented chart, then $\partial M \supset U_\alpha \cap \partial M \to V_\alpha \cap (\{0\} \times \mathbb{R}^{n-1}) \subset \mathbb{R}^{n-1}$ is an oriented chart.

We're thinking of manifolds with boundary as living in the left half-space. Note that you can have a diffeomorphism that maps the right half space to the left half-space that has positive determinant, but has negative determinant on the boundary. This is bad. So you have to pick a convention for left vs. right half-space.

Warning. These definitions don't work well in the lowest dimensions. Try covering $[0, 1]$ with charts $(a_0, 1]$, (a_1, a_0), etc. with a certain orientation (i.e. "pointing in to the interval"); once you get to $[0, a_n)$ the orientation has to be the opposite one (because now the manifold is on the right hand side as opposed to the left hand side).

An orientation of a 1-dimensional manifold is a function $M \to \{\pm 1\}$.

Example 15.5. Take $M = [0, 1]$, oriented so that $M \subset \mathbb{R}$ is orientation-preserving. Then $\partial M = \{0, 1\}$, where at 0 the orientation is -1, and at 1 the orientation is $+1$.

Define

$$\Omega_*^{SO} = \text{ cobordism ring of oriented manifolds.}$$

Here, we want to say that M_0 and M_1 (oriented compact manifolds) are bordant if there is a compact oriented N such that $\partial N = M_0 \sqcup M_1$. But, this doesn't make sense: M is not bordant to itself because in $M \times [0,1]$ one copy of M is oriented inwards and the other copy is oriented outwards. So, you have to say ask for N such that $\partial N = -M_0 \sqcup M_1$, where $-M_0$ means M_0 with the orientation reversed.

In particular, $\Omega_0^{SO} \cong \mathbb{Z}$ (count points with sign). Also $\Omega_1^{SO} = 0$: the circle bounds a disk so it's bordant to the empty set. It also turns out that $\Omega_2^{SO} = 0$.

Lemma 15.6. If M, N are oriented manifolds and $f : M \to N$ is a smooth map, then the regular fibers $f^{-1}(y)$ acquire preferred orientations.

If $m > n$, in local charts f looks like a map $\mathbb{R}^m \to \mathbb{R}^n$ sending $(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_n)$. Assume the local charts are oriented; the fiber is locally $(0, \cdots, 0) \times R^{m-n}$ and this gives an oriented atlas. For $m = n$, assign to a regular point x the sign of $\det(Df_x)$ (computed in oriented charts).

This leads to the *oriented degree*: if $f : M \to N$ is proper where N is connected, then we get $\deg f = [f^{-1}(y)] \in \Omega^{SO}_{\dim M - \dim N}$. (If $\dim M = \dim N$, then $\deg f \in \mathbb{Z}$.)

(Two fibers are bordant – here, oriented-bordant. Each point in the fiber comes with a sign; as you wander around, two points with opposite signs can cancel out.)

Example 15.7. Let $f : T^2 \to T^2$ be given by $(x, y) \mapsto (my, ny)$ for $m, n \in \mathbb{Z}$ (where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$). Equip both copies of T^2 with the same orientation (coming from $\mathbb{R}^2$). Then $\deg f = mn$. This is because every point is a regular point; each point counts as ± 1, and $Df = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$.

The Lefschetz number $L(f) \in \mathbb{Z}$, where M is a compact manifold, does not require M to be oriented (the same is true for the Euler characteristic $\chi(M) = L(\mathbb{1})$). If you change the orientation on M, you have the opposite orientation on both domain and target, and that cancels out. You could try changing the orientation on one component only, but we're counting fixed points, so it's enough to look at components separately.

Concretely, if f has a nondegenerate fixed point ($f(x) = x$ implies 1 is not an eigenvalue of $Tf_x : TM_x \to TM_x$), then $\det(\mathbb{1} - Tf_x)$ is either positive or negative;

$$L(f) = \sum_{f(x)=x} sign(\det(\mathbb{1} - Tf_x)).$$

Lecture 16: October 16

Let V be a finite-dimensional real vector space of dimension $n > 0$. Two bases $\{e_1, \cdots, e_n\}$, $\{\widetilde{e}_1, \cdots, \widetilde{e}_n\}$ are said to have the same orientation if $\widetilde{e}_i = \sum a_{ij} e_j$, $\det((a_{ij})) > 0$. An orientation of V is an equivalence class of bases under this relation.

Definition 16.1. Let $\mathrm{or}(V)$ be the set of orientations of V. This is a set with two elements, with its nontrivial $\mathbb{Z}/2$ action.

Note: there is a map $\mathrm{or}(V) \times \mathrm{or}(W) \to \mathrm{or}(V \oplus W)$ given by sending $(e_1, \cdots, e_n)$ and $(f_1, \cdots, f_n)$ to $(e_1, \cdots, e_m, f_1, \cdots, f_n)$. This is not bijective (count the elements). But, if you quotient by the $\mathbb{Z}/2$ action on $\mathrm{or}(V) \times \mathrm{or}(W)$ (swapping orientations on both gives the same orientation on $V \oplus W$), then that's isomorphic to $\mathrm{or}(V \oplus W)$. With this in mind, extend this definition to the zero vector space by defining $\mathrm{or}(0) = \{\pm 1\}$.

Definition 16.2. Let M be a smooth manifold (possibly with boundary) An orientation of M is a choice of an element in $\mathrm{or}(TM_x)$ that is locally constant in x (in any chart).

Define
$$M^{\mathrm{or}} = \{(x, o) : x \in M, o \in \mathrm{or}(T_x M)\}$$
with a natural map $\pi : M^{\mathrm{or}} \to M$. This is a 2-1 covering that comes with a free $\mathbb{Z}/2$-action whose quotient is M. One can equip it with the structure of a smooth manifold (with boundary if M has boundary), such that π is a local diffeomorphism (and the $\mathbb{Z}/2$-action is smooth).

Lemma 16.3. M^{or} has a canonical orientation, and the $\mathbb{Z}/2$-action is orientation-preserving.

Proof. $TM^{\mathrm{or}}_{(x,o)_{T\pi}} \cong TM_x$ is oriented by o. $\square$

Lemma 16.4. An orientation of M is the same as a smooth section $\sigma : M \to M^{\mathrm{or}}$ of π.

Corollary 16.5. If M is connected, M admits an orientation iff M^{or} is disconnected.

(There are the points in the smooth section, and the ones that aren't. If M is disconnected, then each component determines a section.)

M^{or} is called the orientation covering of M. If you don't like working with an unorientable manifold M, work with M^{or}, and then M is just the $\mathbb{Z}/2$-quotient of it.

Example 16.6. If n is even, and M is $\mathbb{R}P^n$, one can identify $M^{\mathrm{or}} \to M$ with $S^n \to \mathbb{R}P^n$: a choice of point in S^n determines an orientation at the corresponding point in $\mathbb{R}P^n$ (so we have a map $S^n \to M^{\mathrm{or}}$), and if n is even, the antipodal map is orientation-reversing.

Hence $\mathbb{R}P^n$ for n even is not orientable. (If n is odd, then the antipodal map is orientation-preserving, and $\mathbb{R}P^n$ in this case *is* orientable.)

Lemma 16.7. If
$$0 \to V_1 \to V_2 \to V_3 \to 0$$
is a short exact sequence, an orientation of two of the V_i determines uniquely an orientation of the third.

Proof. Choose a splitting $V_2 \cong V_1 \oplus V_3$ so $\mathrm{or}(V_2) = \mathrm{or}(V_1) \times_{\mathbb{Z}/2} \mathrm{or}(V_3)$ and this is independent of the choice of splitting. (The space of splittings is a connected, linear space.) □

Application 16.8. If M is a manifold with boundary, and $x \in \partial M$ we have

$$0 \to T_x(\partial M) \to T_x M \to \nu_x(\partial M) \to 0$$

where ν_x is the normal space, which has a preferred orientation (outwards). Therefore, an orientation of M determines one of ∂M.

(This depends on our convention that the normal is oriented outwards, and also conflicts with a convention on orientation of the boundary introduced before.)

Application 16.9. Let $f : M \to N$ be a map between oriented manifolds, with y a regular value. Then $f^{-1}(y)$ is oriented.

This comes from the sequence

$$0 \to T_x(f^{-1}(y)) \to T_x M \stackrel{T_x f}{\to} T_y N \to 0.$$

Application 16.10. Suppose $f_1 : M_1 \to N$, $f_2 : M_2 \to N$ are maps of oriented manifolds, where in addition M_1 and M_2 are compact.

We can define $[f_1] \cdot [f_2] \in \Omega^{SO}_{\dim M_1 + \dim M_2 - \dim N}$ (so it's an integer if $\dim M_1 + \dim M_2 = \dim N$). You need to make sure the class of the diagonal is oriented.

Remember for compact M, we have the Euler characteristic $\chi(M) = L(\mathbb{1}) \in \mathbb{Z}$.

Proposition 16.11. Suppose $\dim M = n$ is odd, and M is boundary-less. Then $\chi(M) = 0$.

Corollary 16.12. Suppose M is a compact even-dimensional manifold with $\chi(M) \in \mathbb{Z}/2$ nonzero. Then M is not the boundary of a compact manifold.

Corollary 16.13. $\Omega_{2i} \neq 0$ for all i (take $\mathbb{R}P^{2i}$; then $\chi(\mathbb{R}P^{2i}) = 1 \in \mathbb{Z}/2$).

There are orientable examples, at least in dimensions 4, 8, 12,. . . .

Sketch proof of Corollary 16.12. Suppose $M = \partial N$. Consider its "double" $N \cup_M N$, which is a compact odd-dimensional manifold. Consider a collar neighborhood around the shared glued boundary. Suppose we have a map $f : M \to M$ close to the identity that we're counting fixed points on. Extend it to the rest of the manifold N; then any fixed point in one copy of N also corresponds to a fixed point in the other copy, so you're adding fixed points in pairs. Since we assumed f had an odd number of fixed points, the extension has an odd number of fixed points on $N \cup_M N$. That is, $\chi(N \cup_M N) = \chi(M) \pmod 2$ is nonzero, which contradicts the proposition. $\square$

Sketch proof of proposition. Suppose $f : M \to M$ is a diffeomorphism close to the identity. Then I claim

$$L(f^{-1}) = (-1)^n L(f) = -L(f). \tag{16.1}$$

This, applied to $f = \mathbb{1}$, shows that $\chi(M) = -\chi(M)$, so $\chi(M) = 0$.

(Note that we actually needed the $\mathbb{Z}$-Lefschetz number here.)

Proof of (16.1): If $A : V \to V$ is an invertible linear map not having 1 as an eigenvalue, then

$$\begin{aligned} sign(\det(1 - A)) &= sign(\det(A^{-1} - 1)\det A) \\ &= sign(\det A) \cdot sign(\det(1 - A^{-1}))(-1)^n. \end{aligned}$$

Since f is close to the identity we can assume $sign(\det A) = 1$.

It's not true if M has a boundary; this complicates things about fixed points on the boundary. $\square$

Remember that you don't actually need an orientation to define the Lefschetz number.

Lecture 17: October 21

Let M be a smooth manifold (possibly with boundary). A smooth vector bundle of rank r is:

- a manifold (with boundary if M has boundary) E with a smooth map $\pi : E \to M$
- the structure of an r-dimensional vector space on each set $E_x = \pi^{-1}(x)$ given by:
  addition: $E_x \times E_x \to E_x$
  scalar multiplication $\mathbb{R} \times E_x \to E_x$
  satisfying the usual axioms such that, for each $x \in M$ there is a neighborhood $U \subset M$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to \mathbb{R}^r \times U$

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & \mathbb{R}^r \times U \arrow[dl, "\text{proj}"] \\ & U & \end{tikzcd}$$

  which is compatible with the vector space structure.

Remark 17.1. The neutral element $0_x \in E_x$ is the image of a smooth section $M \to E$, where $x \mapsto 0_x$.

The additive inverses $E_x \to E_x$, $v \mapsto -v$ form a smooth map $E \xrightarrow{-} E$ compatible with the projection to M.

Definition 17.2. Suppose that $E, F \to M$ are vector bundles. A smooth homomorphism $\theta : E \to F$ is a smooth map

$$\begin{tikzcd} E \arrow[rr, "\theta"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\eta"] \\ & M & \end{tikzcd}$$

which is fiberwise compatible with the vector space structure.

We get s smooth map $\mathbb{R}^r \times (U \cap V) \to \mathbb{R}^s \times (U \cap V)$ via

$$\mathbb{R}^r \times (U \cap V) \overset{\cong}{\leftarrow} \pi^{-1}(U \cap V) \xrightarrow{\theta} \eta^{-1}(V \cap U) \xrightarrow{\cong} \mathbb{R}^s \times (V \cap U)$$

which is given by $(v, x) \mapsto (A(x)v, x)$ where $A(x) \in \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^s)$ (and this is a smooth function of x).

Definition 17.3. Let $\pi : E \to M$ be a vector bundle of rank r. A subbundle of rank s is a subset $F \subset E$ such that each $x \in M$ has neighborhood U and a diffeomorphism

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{R}^r \times U \arrow[dl] \\ & U & \end{tikzcd}$$

compatible with the vector space structure such that $\varphi(F \cap \pi^{-1}(U)) = \mathbb{R}^s \times 0^{r-s} \times U$.

That is, the sub-vector spaces in $\mathbb{R}^r \times U$ don't depend on $x \in U$.

Remark 17.4. If $F \subset E$ is a subbundle then we can define the quotient bundle F/E with fibers $(E/F)_x = E_x/F_x$.

Lemma 17.5. Let $A(x) \in \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^s)$ be a family of linear maps smoothly dependent on $x \in U$, where $U \subset \mathbb{R}^n$ is a neighborhood of 0. Suppose rank $A(x)$ is constant. Then, after possibly shrinking U, there are invertible $\varphi(x) \in \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^r)$, $\psi(x) \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^s)$ such that

$$\psi(x) \cdot A(x) \cdot \varphi(x) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

for all x.

The proof is the same as for the straight linear algebra statement.

If $r = s$ then the inverses would smoothly depend on x (Cramer's rule).

Proposition 17.6. Let $\eta : E \to F$ be a homomorphism of vector bundles of constant rank. Then $\ker \eta \subset E$ and $\operatorname{im} \eta \subset F$ are sub-bundles, and η induces an isomorphism $E/\ker \eta \to \operatorname{im} \eta$.

This is trivial using the lemma.

EXAMPLE 17.7. Suppose $\eta : E \to F$ is fiberwise injective. Then $\operatorname{im} \eta$ is a subbundle. (If it's fiberwise injective then the rank is constant.) If η is fiberwise surjective then $\ker \eta$ is a subbundle.

Compare to the definition of an abelian category; once you have the notion of kernel and cokernel there are two ways of defining the image: $E/\ker$, or kernel of projection from F to the cokernel. It is an axiom of abelian categories that these coincide. Unfortunately these here don't form an abelian category – kernels etc. only make sense if we're talking about constant rank maps.

EXAMPLE 17.8. Suppose we have $\eta : E \to E$ such that $\eta^2 = \eta$. The rank is locally constant (there are only two eigenvalues, 0 and 1, so you can't jump randomly between them). Assuming M is connected, the rank is constant. Then $\ker \eta, \operatorname{im} \eta \subset E$ are sub-vector bundles.

EXAMPLE 17.9. Suppose $\eta : E \to E$ is a map where $\eta^2 = \mathbb{1}$ (and M is connected). The only eigenvalues are -1 and 1. Then $\ker(\mathbb{1} - \eta), \ker(\mathbb{1} + \eta) \subset E$ (i.e. the two eigenspaces) are vector sub-bundles.

EXAMPLE 17.10.

(1) Trivial bundle $E = M \times \mathbb{R}^r$
(2) Take $M = S^1$ (thought of as $\mathbb{R}^2/\mathbb{Z}^2$) and $E = S^1 \times \mathbb{R}^2 \supset F$ where $F_x = \mathbb{R}(\cos(\pi x), \sin(\pi x)) \subset \mathbb{R}^2 = E_x$. This is the Moebius strip, and it is not isomorphic to the trivial bundle. You can see this by removing the zero section; the complement is connected. (This does not show that it is not *diffeomorphic* to $\mathbb{R} \times S^1$.)
(3) Tangent bundle $TM \to M$ (you need to equip the union of fibers with a manifold structure and equip it with local trivializations)

LECTURE 18: OCTOBER 23

Examples of vector bundles:

- Trivial bundle $\mathbb{R}^n \times M \to M$
- Möbius band (nontrivial rank-1 vector bundle over S^1)
- Tangent bundle $TM \to M$
- If $N \subset M$ is a submanifold, then we have a sequence
$$0 \to TN \to TM|_N \to \nu N \to 0$$
where $TN \to TM|_N$ is the inclusion and $TM|_N \to \nu N$ is the projection to the quotient. (Here the normal bundle has fibers $\nu N_x = TM_x/TN_x$)

Constructions of vector bundles: (proofs later)

- Direct sum of $E \to M$ and $F \to M$ is a vector bundle $E \oplus F \to M$ whose fibers are $(E \oplus F)_x = E_x \oplus F_x$. There is an obvious isomorphism $E \oplus F \xrightarrow{\cong} F \oplus E$
- Dual vector bundle: if $E \to M$ is a vector bundle then $E^* \to M$ is the vector bundle with fibers $(E^*)_x = (E_x)^*$. There is a canonical isomorphism $E \to (E^*)^*$.
- Tensor product of vector bundles: if $E \to M$ and $F \to M$ are vector bundles then there is a vector bundle $E \otimes F \to M$ whose fibers $(E \otimes F)_x$ are $E_x \otimes F_x$. So every element in $(E \otimes F)_x$ is a *finite sum of* elements of the form $v \otimes w$ for $v \in E_x$, $w \in F_x$
- If $E \to M$ and $F \to M$ are vector bundles then there is a vector bundle $\mathrm{Hom}(E, F) \to M$, which is isomorphic to the bundle $E^* \otimes F$.

Remark 18.1. Let's consider line bundles (vector bundles of rank 1). The isomorphism classes of such bundles form an abelian group under $\otimes$, called $\mathrm{Pic}_{\mathbb{R}}(M)$. The only nontrivial claim here is that there are inverses. The inverse of a line bundle $L \to M$ is $L^* \to M$, because $L^* \otimes L \cong \mathrm{Hom}(L, L)$, and I claim the latter is trivial (i.e. $\mathbb{R} \times M$): fiberwise, there is an isomorphism $\mathbb{R} \to \mathrm{Hom}(L_x, L_x)$ that sends $\lambda \mapsto$ scalar multiplication by λ.

Remark 18.2. Start with $E \to M$. For any d,

$$\underbrace{E \otimes \cdots \otimes E}_{d \text{ times}} = E^{\otimes d} \to M$$

comes with a canonical action of the symmetric group S_d (permuting the copies of E in $E^{\otimes d}$). We can decompose $E^{\otimes d}$ into isotypical parts (decompose a representation into irreducible pieces, and then classes of isomorphic pieces are the isotypical parts). Two such pieces are

$$\begin{aligned} \mathrm{Sym}^d(E) &= (E^{\otimes d})^{S_d} && \text{symmetric product} \\ \Lambda^d(E) &= \text{ alternating product} \end{aligned}$$

The first part behaves like the trivial representation, and Λ behaves like the representation given by the alternating character. So

$$\Lambda^d(E)_x = \left\{ v \in E_x^{\otimes d} \,:\, \begin{cases} \sigma(v) = v & \text{if } \sigma \in A_n \\ \sigma(v) = -v & \text{if } \sigma \notin A_n \end{cases} \right\}$$

Definition 18.3 (Pullback of vector bundles). Let $f : M \to N$ be a smooth map, $\pi : F \to N$ be a vector bundle. Then define

$$f^*F = \{(x, v) \in M \times F \,:\, f(x) = \pi(v)\}$$

where on fibers, $(f^*F)_x = F_{f(x)}$. In particular, we can restrict vector bundles to submanifolds by taking the pullback along an inclusion. So locally, if F looks like $\mathbb{R}^n \times U$ then locally f^*F looks like $\mathbb{R}^n \times f^{-1}(U)$.

Transition maps. Let $E \xrightarrow{\pi} M$ be a vector bundle. Take an open cover $(U_\alpha)_{\alpha \in A}$ of M, such that $E|_{U_\alpha} = \pi^{-1}(U_\alpha)$ is trivial. Fix a local trivialization

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\cong", "\varphi_\alpha"'] \arrow[dr] & & \mathbb{R}^r \times U_\alpha \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

Then the transition maps

$$\mathbb{R}^r \times (U_\alpha \cap U_\beta) \xleftarrow{\varphi_\alpha} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{U_\beta} \mathbb{R}^r \times (U_\alpha \cap U_\beta)$$

are of the form

$$(v, x) \mapsto (\psi_{\beta\alpha}(x)v, x)$$

where $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ is smooth. (By the previous diagram, this takes fibers to fibers. The map is linear because the trivializations are required to respect the vector space structure of the fibers. Because the map is a diffeomorphism, $\psi_{\beta\alpha}$ has to be smooth.)

The transition functions $\psi_{\beta\alpha}$ satisfy the cocycle conditions

$$\psi_{\alpha\alpha} = \mathbb{1}$$
$$\psi_{\gamma\beta} \circ \psi_{\beta\alpha} = \psi_{\gamma\alpha} \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Conversely, the datum of $(U_\alpha)_{\alpha \in A}$, $(\psi_{\beta\alpha})$ satisfying the cocycle conditions recovers a vector bundle up to isomorphism. This is called a non-abelian Čech cocycle.

In particular, any smooth group homomorphism

$$\rho : GL(r, \mathbb{R}) \to GL(s, \mathbb{R})$$

gives you a way to make a rank s vector bundle out of a rank r vector bundle (we just said that, to make a vector bundle, it suffices to give a cover and transition maps, so now just compose the transition maps with ρ to get transition maps of a different rank).

Example 18.4.

$$GL(r, \mathbb{R}) \to GL(r, \mathbb{R})$$
$$A \mapsto (A^t)^{-1}$$

corresponds to dualizing vector bundles. This is also a way to show that the double dual of a vector bundle is itself.

Example 18.5.

$$GL(r, \mathbb{R}) \to GL(\binom{r}{k}, \mathbb{R})$$
$$A \mapsto \Lambda^k(A)$$

where you have to choose an identification $\Lambda^k \xrightarrow{\cong} \mathbb{R}^{\binom{r}{k}}$. Similarly, any smooth group homomorphism

$$GL(r, \mathbb{R}) \times GL(s, \mathbb{R}) \to GL(t, \mathbb{R})$$

gives you a way to make a new vector bundle out of two old ones.

For example, when $r = s = t = 1$ this corresponds to maps $\mathbb{R}^\times \times \mathbb{R}^\times \to \mathbb{R}^\times$. The multiplication map corresponds to tensor product of line bundles. Note that tensor products of 1×1 matrices corresponds to ordinary multiplication: if $\lambda : \mathbb{R} \to \mathbb{R}$, $\mu : \mathbb{R} \to \mathbb{R}$ refer to multiplication maps, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{R} \otimes \mathbb{R} \arrow[r, "\lambda\otimes\mu"] \arrow[d, "\cong"] & \mathbb{R} \otimes \mathbb{R} \arrow[d] \\
\mathbb{R} \arrow[r, "\lambda\mu"] & \mathbb{R}
\end{tikzcd}$$

Example 18.6.

$$GL(r, \mathbb{R}) \times GL(s, \mathbb{R}) \to GL(r + s, \mathbb{R})$$
$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

corresponds to direct sum.

Lecture 19: October 25

Orientation of a vector bundle. There are two equivalent definitions:

Definition 19.1. Let $E \xrightarrow{\pi} M$ be a vector bundle. An orientation of E is an orientation of each fiber $E_x = \pi^{-1}(x)$ which is locally constant in X.

By definition of vector bundles, which locally look like

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{R}^r \times U \arrow[dl] \\
& U &
\end{tikzcd}$$

an orientation of E_x (for $x \in U$) yields an orientation of $\mathbb{R}^r$ as vector spaces. The statement that "these orientations are locally constant in x" needs to be independent of the choice of φ. For a transition map $\psi : U \cap V \to GL(r, \mathbb{R})$, $sign(\det \psi(x))$ is locally constant.

Remark 19.2. Take

$$E^{\text{or}} = \{(x, o_x) : x \in M, o_x \text{ an orientation of } E_x\}.$$

One can make E^{or} into a manifold in a canonical way, so that $E^{\text{or}} \to M$ is a local diffeomorphism (in fact, a 2-1 map). Then, orientations of E correspond bijectively to smooth sections $M \to E^{\text{or}}$.

Remark 19.3. An orientation of M is the same as an orientation of TM (one of our previous definitions of orienting M involved orienting TM_x consistently).

DEFINITION 19.4. An orientation of E is given by a cover $(U_\alpha)_{\alpha\in A}$ of M together with

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & \mathbb{R}^r \times U_\alpha \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

that is fiberwise linear, such that the transition maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ satisfy

$$\det(\psi_{\beta\alpha}(x)) > 0$$

for all x and all pars (α, β).

This is the same as before: fix a standard orientation of $\mathbb{R}^r$ and transfer it to $\pi^{-1}(U_\alpha)$; the condition says that this is independent of choices. This definition is annoying because it's hard to say when orientations are the same, but it's nice because makes it easy to see when maps of vector bundles preserve orientation.

REMARK 19.5. This shows that, if E has an orientation, so does the dual bundle E^*: the transition maps from E^* are the transpose inverse of the original ones.

Proposition 19.6. Every short exact sequence of vector bundles splits.

Fix M and a subbundle F of a smooth vector bundle E; we get a sequence

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} E/F \to 0;$$

we're looking for a splitting $\rho : E \to F$ such that $\rho \circ i = \mathbb{1}_F$. (This has the property that $p = i \circ \rho$ is an idempotent endomorphism of E because it is projection onto F.) Then

$$G = \ker p = \operatorname{im}(\mathbb{1} - p)$$

is again a subbundle of E, and $E_x = F_x \oplus G_x$ for all x. Therefore, any sequence looks like this:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d, equal] & E \arrow[r] & E/F \arrow[r] & 0 \\ 0 \arrow[r] & F \arrow[r] & F \oplus G \arrow[r] \arrow[u, "\cong"] & G \arrow[r] \arrow[u, "\cong"] & 0 \end{tikzcd}$$

Proposition 19.7. By definition of subbundle, E locally looks like $\mathbb{R}^r \times U$ and $F \subset E$ locally looks like $\mathbb{R}^s \times \{0\}^{r-s} \subset \mathbb{R}^r \times U$. Hence we have a local splitting. Take an open cover $(U_\alpha)_{\alpha\in A}$ and splittings $p_\alpha : E|_{U_\alpha} \to F|_{U_\alpha}$, and a partition of unity (f_α) and set

$$p = \sum_{\alpha\in A} f_\alpha p_\alpha : E \to F$$

where $f_\alpha p_\alpha$ means the map that is extended by 0 outside U_α. Now

$$p \circ i = \sum f_\alpha (p_\alpha \circ i) = \sum f_\alpha \cdot \mathbb{1}_F = \mathbb{1}_F.$$

(Justification of the use of partitions of unity: we were trying to solve $p \circ i = \mathbb{1}$. If p_1 and p_2 are solutions, and $f_1, f_2 : M \to \mathbb{R}$ satisfy $f_1 + f_2 = 1$, then $p = f_1 p_1 + f_2 p_2$ is a solution.)

Corollary 19.8. If $N \subset M$ is a submanifold, then

$$0 \to TN \to TM|_N \to \nu_N \to 0.$$

So

$$TM|_N \cong TN \oplus \nu_N$$

non-canonically.

Proposition 19.9. Any oriented line bundle is trivial.

Proof. We have an oriented line bundle (rank 1 bundle) $L \to M$. We want to find an isomorphism $\mathbb{R} \times M \xrightarrow{\cong} L$ which is compatible with orientations. These exist locally, but in general, you can't patch them together because the result might not be an isomorphism. But here, if $(s_\alpha)_{\alpha \in A}$ is a collection of local isomorphisms then (by orientability) we can assume s_α is positive; if $(f_\alpha)_{\alpha \in A}$ is a partition of unity (also positive values by design) then the attempted patching

$$s = \sum f_\alpha s_\alpha$$

is actually an isomorphism because you're adding only positive things and can't get zero that way. $\square$

Corollary 19.10. For any real line bundle L, $L \otimes L \cong \mathbb{R} \times M$.

Idea: you're squaring the transition functions, so they're positive.

Corollary 19.11. For any real line bundle L, we have $L \cong L^*$.

In other words, $\mathrm{Pic}_{\mathbb{R}}(M)$ is actually a vector space over $\mathbb{Z}/2$.

Definition 19.12. A Euclidean metric on a vector bundle is a family of scalar products

$$\langle -, - \rangle_x : E_x \times E_x \to \mathbb{R}$$

which depends smoothly on $x \in M$.

Any Euclidean metric induces an isomorphism

$$\begin{aligned} E &\to E^* \\ v &\mapsto \langle v, - \rangle . \end{aligned}$$

Proposition 19.13. Every vector bundle admits a Euclidean metric.

Corollary 19.14. $E \cong E^*$ for any vector bundle E.

Proof of proposition. By partitions of unity. □

REMARK 19.15. Given a subbundle $F \subset E$ and a Euclidean metric on E, we can look at

$$F^\perp = \{v \in E \,:\, \langle v, w\rangle = 0 \ \ \forall w \in F\}.$$

This is a subbundle, and $E \cong F \oplus R^\perp$.

Back to the splitting of short exact sequences: the Euclidean metric determines a particular splitting. But you also want to be aware of the other ones, which is why we introduced splittings before Euclidean metrics.

LECTURE 20: OCTOBER 28

The topic for today is about "having enough global sections". Let $E \to M$ be a vector bundle. A smooth section is a smooth map $s : M \to E$ such that $\pi(s(x)) = x$ for all x.

Lemma 20.1. For any $\pi : E \to M$, there are smooth sections $(s_1, .., s_m)$ for some $m \in \mathbb{N}$ such that $(s_1(x), \cdots, s_m(x))$ generate E_x for all x.

Equivalently:

Lemma 20.2. For any $\pi : E \to M$ there is a smooth vector bundle homomorphism $\mathbb{R}^m \times M \to E$ (for some m) which is fiberwise surjective.

REMARK 20.3. These lemmas are equivalent. A section is a map $\mathbb{R} \times M \to E$; n sections is a map $\mathbb{R}^n \times M \to E$, and generating the fiber is the same as being fiberwise surjective.

In algebraic geometry, this property is called "being generated by the global sections"; it doesn't always happen.

Lemma 20.4. For any $\pi : E \to M$ there is a smooth vector bundle homomorphism $E \to \mathbb{R}^m \times M$ (for some $m \in \mathbb{N}$), which is fiberwise injective.

REMARK 20.5. Lemma 20.2 for E describes the same thing as Lemma 20.4 for E^*: if the original map is injective then the map on the dual bundle is surjective.

Lemmas 20.2 and 20.4 are also equivalent in a different way: if $\mathbb{R}^m \times M \to E$ is fiberwise onto, we have

$$0 \to F \to \mathbb{R}^m \times M \to E \to 0$$

and a slitting glues an injective map $E \to \mathbb{R}^m \times M$.

Lemma 20.6. In Lemmas 20.1 - 20.4, it is enough to choose $m = \dim M + \operatorname{rank} E$.

It's clear that you need more m than the rank – in that case, fiberwise surjective would imply fiberwise bijective, which would imply the vector bundle is trivial.

Plan:

(1) Prove Lemmas 20.1 - 20.4 for compact M
(2) Prove Lemma 20.6 for compact M
(3) Prove lemmas 20.1 - 20.4 for general M
(4) Prove Lemma 20.6 for general M.

Proof of Lemmas 20.1 - 20.4 in the compact case. Suppose M is compact, and $\pi : E \to M$ is a smooth vector bundle. Take a finite cover $M = \bigcup_{\alpha\in A} U_\alpha$ such that E is trivial on each U_α. Choose sections

$$s_{\alpha,1}, \cdots, s_{\alpha,r} : U_\alpha \to E|_{U_\alpha} =: E_\alpha$$

such that $s_{\alpha,1}(x), \cdots, s_{\alpha,r}(x)$ generate $E_{\alpha x}$ for all $x \in U_\alpha$. Choose a subordinate partition of unity (f_α) and write

$$\mathbb{R}^r \times M \xrightarrow{t_\alpha} E \text{ where } t_\alpha = (f_\alpha s_{\alpha,1}, \cdots, f_\alpha s_{\alpha,r}).$$

This is onto in all fibres where $f_\alpha(x) \neq 0$. But, you can't take the sum, because then that might not be surjective (they might cancel if the supports intersect).

Instead, take

$$\bigoplus_{\alpha\in A} \mathbb{R}^r \times M \xrightarrow{\sum t_\alpha} E$$

where the LHS is the trivial bundle of rank $r \cdot |A|$. This is onto in all fibres, because for any x there is some f_α with $f_\alpha(x) \neq 0$ (i.e. you just need it to be onto from one summand on the left). ☐

Proof of Lemma 20.6 in the compact case. Suppose M is compact, and $\pi : E \to M$ is a smooth vector bundle. Let $E \to \mathbb{R}^m \times M$ be a vector bundle homomorphism which is fiberwise injective. Suppose that $m > \dim M + \operatorname{rank} E$, and consider

$$E \to \mathbb{R}^m \times M \xrightarrow{\text{proj}} \mathbb{R}^m.$$

We have $\dim M + \operatorname{rank} E = \dim E$. By trivial Sard, there is a nonzero $v \in \mathbb{R}^m$ which is not in the image; since the maps are fiberwise linear, nonzero multiples λv are also not in the image.

Hence, $E \to \mathbb{R}^m/\mathbb{R}\cdot v \times M \cong \mathbb{R}^{m-1} \times M$ is still fiberwise injective. ☐

Proof of Lemmas 20.1 - 20.4 for general M. Let $\pi : E \to M$ be a vector bundle. Set $m = \dim M + \operatorname{rank} E$. Choose a bounded below proper smooth function $h : M \to [0, \infty)$ (earlier we proved that these exist). Write $U_i = h^{-1}((u-1, i+1))$. Note:

- $U_i \cap U_j = \emptyset$ if $|i-j| \geq 2$.

Choose a subordinate partition of unity (f_i). Choose also for each i sections

$$s_{i,1}, \cdots, s_{i,m} : U_i \to E|_{U_i}$$

which generate the fiber at each point of U_i. (Here we're using Lemma 20.6 in the compact case; enlarge $(i-1, i+1)$ a little and suppose the lemma works for compact manifolds with boundary.) Take

$$t_i = (f_i s_{i,1}, \cdots, f_i s_{i,m}) : \mathbb{R}^m \times M \to E$$

which is onto for all fibers where $f_i(x) \neq 0$. Problem: there are infinitely many i's.

Take

$$\begin{aligned} t_{\text{even}} &= t_0 + t_2 + t_4 + \cdots \\ t_{\text{odd}} &= t_1 + t_3 + t_5 + \cdots \end{aligned}$$

which is not an infinite sum because they come from partitions of unity. Any two summands in t_{even} or t_{odd} have disjoint supports. So, $t_{\text{even}} : \mathbb{R}^m \times M \to E$ is onto at all $x \in M$ where $f_i(x) \neq 0$ for some even i, and analogously for t_{odd}. Then

$$(t_{\text{odd}}, t_{\text{even}}) : \mathbb{R}^m \times \mathbb{R}^m \times M \to E$$

is a surjective vector bundle homomorphism. □

"Proof" of Lemma 20.6 for general M. Same argument as before. □

Theorem 20.7. Let $\pi : E \to M$ be a smooth vector bundle. There is a smooth section $s : M \to E$ which is transverse to the zero-section.

Corollary 20.8. If $\operatorname{rank} E > \dim M$, then $E \cong E \oplus (\mathbb{R} \times M)$ for some vector bundle F.

The presence of $\mathbb{R} \times M$ here corresponds to the presence of a nowhere-zero section.

Sketch proof of Theorem 20.7. Idea: a "random" section works. But, the space of all sections is too big to work with. Instead, take a reservoir of sections: those that are fiberwise surjective.

Take $F : \mathbb{R}^m \times M \to E$ that is fiberwise surjective. This is automatically transverse to the zero section. If you hit zero, you can add something that comes from the fiber, and move the image in any direction you want. Consider

$$F^{-1}(0_E) \xrightarrow{\text{proj}} \mathbb{R}^m.$$

If $v \in \mathbb{R}^m$ is a regular value of that projection, then $F(v, \cdot) : M \to E$ is transverse to 0_E. This is just linear algebra. Regular values always exist by Sard. $\square$

Lecture 21: October 30

Let M be a compact manifold (without boundary), and $\pi : E \to M$ a vector bundle. Let $s : M \to E$ be a smooth section transverse to the zero section $0_E \subset E$. Then $s^{-1}(0_E) = \{x \in M : s(x) \in E_x \text{ is zero}\}$ is a smooth submanifold, and $\dim s^{-1}(0_E) = \dim M - \operatorname{rank} E$. Last time we outlined why such smooth sections always exist.

Example: in a trivial bundle $\mathbb{R} \times \mathbb{R}$ it's easy to see how you can have one section $s \mapsto (s, 1)$ where $s^{-1}(0_E) = \emptyset$, and a different section $s \mapsto (s, f(s))$ where $s^{-1}(0_E)$ is a discrete set of points. However, the bordism class of $s^{-1}(0_E)$ is independent of the choice of s. We write

$$e(E) = [s^{-1}(0)] \in \Omega_{\dim M - \operatorname{rank} E}.$$

For $\dim M = \operatorname{rank} E$, we get $e(E) \in \Omega_0 \cong \mathbb{Z}/2$; this is called the *Euler number*, and it's an invariant of the vector bundle.

Proof. Every section can be deformed to every other section (think about deforming a section to the zero section by multiplying by t, for $0 < t < 1$). So inside $E \times [0, 1]$ we have a bordism from $s_1(M) \subset E \times \{0\}$ to $s_2(M) \subset E \times \{1\}$. By just paying attention to the zero section $0_E \subset E$, this restricts to a bordism from $0_E \cap s_1(M) \cong s_1^{-1}(0_E)$ to $s_2^{-1}(0_E)$:

$$\begin{tikzcd}
s_1(M) \times \{0\} \arrow[r, hook] \arrow[d, dotted, no head] & E \times [0,1] \arrow[d, dotted, no head] & s_2(M) \times \{1\} \arrow[l, hook'] \arrow[d, dotted, no head] \\
s_1^{-1}(0) \times \{0\} \arrow[r, hook] & M \times [0,1] & s_2^{-1}(0) \times \{1\} \arrow[l, hook']
\end{tikzcd}$$

$\square$

Example 21.1. The Moebius strip $L \to S^1$ is a vector bundle of rank 1. Take $I \times I$ and identify $(0, t) \sim (1, -t)$. There is a nontrivial section: take any path from $(0, t)$ to $(1, -t)$. So $e(L) = 1$.

Example 21.2. Take

$$M = \mathbb{R}P^n = S^n / x \sim -x$$
$$L = \mathbb{R} \times S^n / (t, x) \sim (-t, -x)$$

$L \to M$ is a line bundle (really the "only" line bundle). We define a section $s : M \to L$ taking $x \mapsto (x_1, x)$ where $x = (x_1, \cdots, x_{n+1}) \in S^n$. Then s is transverse to 0_L, and

$$e(L) = [s^{-1}(0)] = [\mathbb{R}P^{n-1}] \in \Omega_{n-1}$$

is nontrivial for all odd n. (You can also see this by removing the zero section and noting that it is connected, so it's not the trivial bundle.)

EXAMPLE 21.3. Take $M = \mathbb{R}P^n$ as before and consider

$$E = \underbrace{L \oplus \cdots \oplus L}_{n \text{ times}} = \mathbb{R}^n \times S^n/(t,x) \sim (-t,-x).$$

A section $M \to E$ is the same as n sections $M \to L$. Take

$$\begin{aligned} s : M &\to E \\ [x] &\mapsto [(x_1, \cdots, x_n, x)] \end{aligned}$$

Then s is transverse to 0_E and $s^{-1}(0_E) = [(0, \cdots, 0, \pm 1)] = pt$. Hence $e(E) = 1 \in \mathbb{Z}/2$, and E is not trivial.

Theorem 21.4.

$$e(TM) = \chi(M) \in \mathbb{Z}/2$$

where $\chi(M)$ is the Euler characteristic.

So $\chi(M)$ is both the Lefschetz number of a map $f : M \to M$ homotopic to the identity, and also the Euler number of $TM \to M$.

Proof. Take a section of TM (i.e. a vector field X) which is transverse to the zero-section. In local coordinates, X is a map $U \to \mathbb{R}^n$. Transversality means that if $X(x) = 0$, then DX_x is an invertible matrix.

In local coordinates near a zero of X, $X : U \to \mathbb{R}^n$ (where $U \subset \mathbb{R}^n$ and the zero is assumed to be the origin) has the form $X(x) = Ax + O(|x|^2)$, where A is an invertible matrix. Without introducing any new fixed points, we can modify our vector field so that in the same local coordinates, $X(x) = Ax$ (multiply by a suitable cutoff function). Then its flow for small times is of the form $\varphi_t(x) = e^{tA}x$ locally (which has no fixed points locally), in particular $(E\varphi_t)_{x=0} = e^{tA}$ does not have 1 as an eigenvalue. It is easy to see that (for small $t \neq 0$), the only fixed points of φ_t are the zeros of X. $\square$

Orientations. If M is oriented and E is an oriented vector bundle, $s^{-1}(0_E)$ inherits an orientation, hence one can define

$$e(E) \in \Omega^{SO}_{\dim M - \operatorname{rank} E}$$

(so this is an element of $\mathbb{Z}$ if $\dim M = \operatorname{rank} E$). Then $e(TM) = \chi(M)$ holds in $\mathbb{Z}$. (This is fairly dull so I won't explain. A is an invertible matrix; the determinant is either positive or negative, and that gives a sign.)

Suppose $\pi : E \to M$ is a vector bundle over a compact space M. For any $0 \leq i \leq \operatorname{rank}(E)$ consider

$$\begin{tikzcd} \mathbb{R}P^{\operatorname{rank}(E)-i} \times M \arrow[r, "p_2"] \arrow[d, "p_1"] & M \\ \mathbb{R}P^{\operatorname{rank}(E)-i} & \end{tikzcd}$$

Define $E^i \to \mathbb{R}P^{\mathrm{rank}(E)-i} \times M$ where $E^i = p_1^*L \otimes p_2^*E$ where L is the line bundle on real projective space. Denote this by $L \boxtimes E$.

Choose a section s^i of E^i transverse to the zero section,

$$\begin{tikzcd} (s^i)^{-1}(0_{E^i}) \arrow[r, hook] \arrow[dr, "f^i"] & \mathbb{R}P^{\mathrm{rank}(E)-i} \times M \arrow[d, "pr_2"] \\ & M \end{tikzcd}$$

where $(s^i)^{-1}(0_{E^i})$ has dimension $\dim M - i$. Suppose that $i_1 + \cdots + i_d = \dim M$. Take

$$\begin{tikzcd} (f^{i_1} \times \cdots \times f^{i_d}) : ((s^{i_n})^{-1}(0) \times \cdots \times (s^{i_d})^{-1}(0)) \arrow[d, "f^{(i_1,\cdots,i_d)}"] \\ M^d \end{tikzcd}$$

where $((s^{i_n})^{-1}(0) \times \cdots \times (s^{i_d})^{-1}(0))$ has dimension $(d-1)\dim M$. Perturb $f^{(i_1,\cdots,i_d)}$ to be transverse to $\Delta^d = \{x_1 = \cdots = x_d\} \subset M$ and define

$$w_{i_1 \cdots i_d}(E) = \#(\widetilde{f}^{(i_1,\cdots,i_d)})^{-1}(\Delta^d) \in \mathbb{Z}/2;$$

this is called the Stiefel-Whitney number. This is an invariant of E (for any $i_1 + \cdots + i_d = \dim M$). If $E = TM$ it is a cobordism invariant of M.

Theorem 21.5 (Thom). $[M] = 0 \in \Omega_*$ iff $w_{i_1,\cdots,i_d}(TM) = 0$ for all $(i_1, \cdots, i_d)$.

This shows that Ω_k is a finite dimensional vector space.

Lecture 22: November 1

A Riemannian metric is a Euclidean metric on TM, so an inner product

$$TM_x \times TM_x \to \mathbb{R}$$

smoothly depending on $x \in M$. For $X, Y \in TM_x$ we write $g(X,Y) \in \mathbb{R}$, or $g_x(X,Y) \in \mathbb{R}$, or $\langle X, Y \rangle \in \mathbb{R}$. This data is not given: you have to define it.

In local coordinates on an open set $U \subset \mathbb{R}^n$, write

$$g = (g_{ij}(x))_{1 \le i,j \le n}$$

meaning that

$$g_x(X,Y) = \sum_{i,j} X_i(x) g_{ij}(x) Y_j(x)$$

for $X, Y : U \to \mathbb{R}^n$, where g_{ij} are functions of $x \in U$.

Lemma 22.1. Any manifold admits a Riemannian metric (special use of a previous result about vector bundles).

Let (M, g) be a Riemannian manifold (a manifold M along with a Riemannian metric g). Let $c : [a, b] \to M$ be a piecewise smooth path.

Definition 22.2. The length of c is

$$L(c) = \int_a^b \left\| \frac{dc}{dt} \right\|_g dt$$

where $\frac{dc}{dt} \in TM_{c(t)}$ is the tangent vector of the path, and

$$\|x\|_g = \sqrt{g(X, X)}.$$

So $\|\frac{dc}{dt}\|_g = \sqrt{g_{c(t)}(c'(t), c'(t))}$.

Lemma 22.3. $L(c)$ is invariant under reparametrization.

(This means that $L(c) = L(\widetilde{c})$ if $\widetilde{c}(t) = c(\psi(t))$ and $\psi : [\widetilde{a}, \widetilde{b}] \to [a, b]$ is nondecreasing and onto.) This is a standard change of variables calculation.

Definition 22.4. Suppose M is connected. For $x, y \in M$ define the Riemannian distance

$$d(x, y) = \inf\{L(c) \,:\, c \text{ is a path from } x \text{ to } y\}.$$

Note that, in general, there is no reason why this length should be attained.

Example 22.5. Let $M = \mathbb{R}^2 \backslash \{(0, 0)\}$, and $g_{ij} = \delta_{ij}$ be the constant metric. Then $d((-1, 0), (1, 0)) = 2$ but the infimum is not attained.

Lemma 22.6. If (M, g) is a connected Riemannian manifold, then d is a metric.

Proof. We need:

- symmetry (obvious – reverse path)
- triangle inequality (obvious – concatenate a path from x to y and from y to z. This makes use of the fact we'd used piecewise smooth paths, but we could have chosen smooth paths, because you can smooth out a path with minimal penalty in length.)
- definiteness (this is the only sticking point).

Suppose $x \neq y$ in M. Choose a local coordinate chart modelled on a ball $U = B_r(0) \subset \mathbb{R}^n$, such that x is the origin and y is outside the chart. I claim there is some $\varepsilon > 0$ such that, for all $p \in \overline{B}_{r/2}(0)$ and all $X \in \mathbb{R}^n$,

$$\sum_{i,j} X_i g_{ij}(p) X_j \geq \varepsilon \sum_i X_i^2.$$

(Take all the vectors of (Euclidean) norm 1; this is a compact set, so there is a minimum g-norm ε. Then the equation above is saying that $\|x\|_g \geq \varepsilon \|x\|_{\text{Euclidean}}$.)

Therefore, if c is a path starting at x and which leaves U, $L(c) \geq \varepsilon^{\frac{1}{2}}\frac{r}{2}$. Therefore, the same is true for paths c from x to y, so $d(x,y) \geq \varepsilon^{\frac{1}{2}}\frac{r}{2} > 0$. $\square$

Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ be a path. Introduce the length

$$L(c) = \int_a^b g_{c(t)}(c'(t), c'(t))^{\frac{1}{2}} dt$$

and the energy

$$E(c) = \frac{1}{2}\int_a^b g_{c(t)}(c'(t), c'(t)) dt.$$

By Cauchy-Schwartz,

$$\begin{aligned} L(c) = \int_a^b 1 \cdot g^{\frac{1}{2}} &\leq (\int_a^b 1)^{\frac{1}{2}} \cdot (\int_a^b g)^{\frac{1}{2}} \\ &= \sqrt{2}(b-a)^{\frac{1}{2}} E(c)^{\frac{1}{2}} \end{aligned}$$

or equivalently

$$E(c) \geq \frac{1}{2(b-a)^{\frac{1}{2}}} L(c)^2.$$

Equality holds if and only if the speed

$$\|c'(t)\| = g_{c(t)}(c'(t), c'(t))^{\frac{1}{2}}$$

is constant in t. So there's a fixed relationship between length and energy, *if the path has constant speed.*

Corollary 22.7. Suppose that $c : [a,b] \to M$, where $c(a) = x$ and $c(b) = y$, is length-minimizing (i.e. $L(c) = d(x,y)$), and has constant speed. Then it minimizes energy among paths with endpoints x and y. Conversely, if it minimizes energy among paths $[a,b] \to M$ with endpoints x and y, then it minimizes length and has constant speed.

With this in mind, we look at paths that minimize energy, or more generally are critical points of the energy.

Definition 22.8. A (smooth) path $c : [a,b] \to M$ is a geodesic if for any smooth family of paths $c_s : [a,b] \to M$ where $c_s(a) = x$, $c_s(b) = y$, and $c_0 = c$, we have

$$\frac{\partial}{\partial s} E(c_s)|_{s=0} = 0.$$

Remark 22.9. If c is energy-minimizing, it's a geodesic. The converse is false generally. For example, take $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and take g to be the constant standard metric (so $g_{ij}(x) = \delta_{ij}$ in standard local coordinates). Then the straight path from $(1,b)$ to $(0,a)$ (for $0 < a < b < \frac{1}{2}$) is *locally* length-minimizing, but a shorter path is the one from $(0,b)$ to $(0,a)$ (recall $(0,b) \sim (0,a)$).

Remark 22.10. A path $c : I \to M$, where $I \subset M$ is any interval, is called a geodesic if its restriction to any closed sub-interval is a geodesic.

Next lecture we will prove the following theorem.

Theorem 22.11. c is a geodesic iff, in local coordinates $c(t) = (c_1(t), \cdots, c_n(t)) \in U \subset \mathbb{R}^n$, it satisfies

$$\frac{d^2 c_k}{dt^2} + \Gamma^k_{ij}(c(t)) \frac{dc_i}{dt}\frac{dc_j}{dt} = 0$$

where

$$\Gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij})$$

(here $\Gamma_{ijk}(x) \in \mathbb{R}$, $x \in U$, $i, j, k \in \{1, \cdots, n\}$), and

$$\Gamma^k_{ij} = \sum_\ell g^{k\ell}\Gamma_{ij\ell}$$

where $g^{k\ell}$ is the inverse matrix to $(g_{k\ell})$.

The symbols Γ^k_{ij} and Γ_{ijk} are called Christoffel symbols.

This looks weird but it's an ODE so we can say a lot about its solutions.

It turns out that geodesics are automatically smooth (because you can bend the corner in a way that makes it shorter).

Lecture 23: November 4

Example 23.1. Let $M \subset \mathbb{R}^n$ be a submanifold. If $x \in M$, then $TM_x \subset \mathbb{R}^n$. Take the standard scalar product on $\mathbb{R}^n$; this induces (by restriction) a Riemannian metric g on M.

Question: what are the geodesics for this metric? Take a smooth path $c : [a, b] \to M \subset \mathbb{R}^n$ from x to y. By definition, c is a geodesic iff the following holds: for any family of paths

$$\widetilde{c} : (-\varepsilon, \varepsilon) \times [a, b] \to M \text{ where } \widetilde{c}(s, a) = x, \widetilde{c}(s, b) = y, \widetilde{c}(0, t) = c(t)$$

we have

$$\frac{d}{ds}(E(\widetilde{c}(s, \cdot)))|_{s=0} = 0.$$

$$\begin{aligned}\frac{d}{ds}E(\widetilde{c}(s,\cdot)) &= \frac{d}{ds}\left(\frac{1}{2}\int_a^b \left\langle \frac{d\widetilde{c}}{dt}, \frac{d\widetilde{c}}{dt}\right\rangle dt\right) \\ &= \int_a^b \left\langle \frac{d}{dt}\frac{d\widetilde{c}}{dt}, \frac{d\widetilde{c}}{dt}\right\rangle dt\end{aligned}$$

integrate by parts

$$\begin{aligned}&= \int_a^b \frac{d}{dt}\left\langle \frac{d\widetilde{c}}{ds}, \frac{d\widetilde{c}}{dt}\right\rangle dt - \int_a^b \left\langle \frac{d\widetilde{c}}{ds}|_{s=0}, \frac{d^2\widetilde{c}}{dt^2}\right\rangle dt \\ &= -\int_a^b \left\langle \frac{d\widetilde{c}}{ds}, \frac{d^2\widetilde{c}}{dt^2}\right\rangle dt\end{aligned}$$

Hence, c is geodesic iff

$$\int_a^b \left\langle X, \frac{d^2c}{dt^2} \right\rangle dt = 0$$

where $X = \frac{d\widetilde{c}}{ds}|_{s=0}$ for any variation $\widetilde{c}$ of c. Note that $X(t) \in TM_{c(t)}$ for all t.

One can show (see tubular neighborhood theorem) that any $X : [a,b] \to \mathbb{R}^n$, $X(t) \in TM_{c(t)}$ can occur. Hence, c is a geodesic iff $\frac{d^2c}{dt^2}$ is orthogonal to $TM_{c(t)}$ for all t. (This is a Lagrange multiplier argument). For example, a great circle on a sphere is a geodesic. For a surface of revolution, the circles of local maximum and local minimum radius are both geodesics, but only the latter are locally length-minimizing (that is, a circle c of maximum radius minimizes length *over all paths between top and bottom points constrained within a small neighborhood of* c, but they do not minimize length *over all paths from a given point to itself, constrained within a neighborhood of* c).

Sanity check: c has constant speed (we know that's necessary for geodesics) because the first derivative is orthogonal to the second derivative.

In the calculation above, $\langle *, * \rangle$ was the standard scalar product in $\mathbb{R}^n$ (so we didn't have to worry about coefficients of the metric). More generally, let $U \subset \mathbb{R}^n$ be an open subset, g a Riemannian metric on U with coordinates $(g_{ij})_{1 \le i,j \le n}$. Let $c : [a,b] \to U$ be a path from x to y. Take a variation

$$\widetilde{c} : (-\varepsilon, \varepsilon) \times [a,b] \to U$$

where $\widetilde{c}(s,a) = x$, $\widetilde{c}(s,b) = y$ and $\widetilde{c}(0,t) = c(t)$.

$$\begin{aligned}
\frac{d}{ds}|_{s=0} E(\widetilde{c}(s,\cdot)) &= \frac{d}{ds} \int_a^b g_{\widetilde{c}(s,t)} \left(\frac{d\widetilde{c}}{\partial t}, \frac{\partial \widetilde{c}}{\partial t} \right) dt \\
&= \frac{d}{ds}|_{s=0} \frac{1}{2} \int_a^b \sum_{ij} g_{ij}(\widetilde{c}(s,t)) \frac{\partial \widetilde{c}_i}{\partial t} \frac{\partial \widetilde{c}_j}{\partial t} dt
\end{aligned}$$

integrate by parts

$$\begin{aligned}
&= \int_a^b \sum_{ij} g_{ij}(\widetilde{c}(s,t)) \frac{\partial \widetilde{c}_i}{\partial t} \frac{\partial^2 \widetilde{c}_j}{\partial s \partial t} dt + \frac{1}{2} \int_a^b \sum_{ijk} \frac{\partial g_{ij}}{\partial x_k} \frac{\partial \widetilde{c}_k}{\partial s} \frac{\partial \widetilde{c}_i}{\partial t} \frac{\partial \widetilde{c}_j}{\partial t} dt \\
&- \int_a^b \sum_{ij} g_{ij}(\widetilde{c}(s,t)) \frac{\partial^2 \widetilde{c}_i}{\partial t^2} \frac{\partial \widetilde{c}_j}{\partial s} dt \\
&- \int_a^b \sum_{ijk} \frac{\partial g_{ij}}{\partial x_k} \frac{\partial \widetilde{c}_k}{\partial t} \frac{\partial \widetilde{c}_i}{\partial t} \frac{\partial \widetilde{c}_j}{\partial s} dt + \text{ same last term as before}
\end{aligned}$$

General cleanup: recall that $c(t) = \widetilde{c}(0,t)$, and let $X(t) = \frac{\partial \widetilde{c}}{\partial s}(0,t)a$.

Change indices so the second term is

$$\sum_{ijk} \frac{\partial g_{ik}}{\partial x_j} \frac{\partial \widetilde{c}_j}{\partial s} \frac{\partial \widetilde{c}_i}{\partial t} \frac{\partial \widetilde{c}_k}{\partial t} dt$$

then we have

$$\cdots = -\int_a^b \sum_{ij} g_{ij}\frac{d^2c_i}{dt^2}X_j dt + \int_a^b \sum_{ijk}\left(\frac{1}{2}\frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right)\frac{dc_i}{dt}\frac{dc_k}{dt}X_j dt$$
$$= -\int_a^b \sum_{ij} g_{ij}\frac{d^2c_i}{dt^2}X_j dt + \int_a^b \sum_{ijk}\left(\frac{1}{2}\frac{\partial g_{ik}}{\partial x_j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x_k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x_i}\right)\frac{dc_i}{dt}\frac{dc_k}{dt}X_j$$

Hence, c is geodesic iff $\sum_i g_{ij}\frac{d^2c_i}{dt^2} + \sum_{ik}\left(-\frac{1}{2}\frac{\partial g_{ik}}{\partial x_j} + \frac{1}{2}\frac{\partial g_{ij}}{\partial x_k} + \frac{1}{2}\frac{\partial g_{jk}}{\partial x_i}\right)\frac{dc_i}{dt}\frac{dc_k}{dt} = 0$ for $j = 1, \cdots, n$.

(g_{ij}) is a matrix, so we are free to multiply by the inverse matrix. So if $(g^{ij}(x))$ is the inverse matrix to $(g_{ij}(x))$ then

$$\frac{d^2c_\ell}{dt^2} + \sum_{ikj} \underbrace{g^{j\ell}\left(-\frac{1}{2}\frac{\partial g_{ik}}{\partial x_j} + \frac{1}{2}\frac{\partial g_{ij}}{\partial x_k} + \frac{1}{2}\frac{\partial g_{jk}}{\partial x_i}\right)}_{\Gamma^\ell_{ik}(x)}\frac{dc_i}{dt}\frac{dc_k}{dt} = 0. \tag{23.1}$$

Theorem 23.2. Let (M, g) be a Riemannian manifold. Then a smooth path $c : I \to M$ is geodesic iff in local coordinates around any point $c(x)$, it satisfies the geodesic equation (23.1):

$$\frac{d^2c_\ell}{dt^2} + \sum_{ik}\Gamma^\ell_{ik}(x)\frac{dc_i}{dt}\frac{dc_k}{dt} = 0$$

This property is certainly necessary. You have to show that it suffices to do this locally. . . we will have better proofs later.

Corollary 23.3. Given $p \in M$ and $X \in TM$ there is some open interval $I \subset \mathbb{R}$ containing 0, and a geodesic $c : I \to M$ where $c(0) = p$ and $c'(0) = X$. If we ask for I to be maximal, c is unique. (This is ODE-theory.)

Remark 23.4. If M is compact then $I = \mathbb{R}$.

This is a second-order ODE: it can stop existing if the first derivative diverges. This cannot happen for a geodesic because geodesics proceed at constant speed.

Lecture 24: November 6

Geodesic normal coordinates. Let (M, g) be a Riemannian manifold, and $p \in M$. There is a map

$$T_pM \supset U \overset{\exp_p}{\to} M$$
$$\exp_p : X \mapsto c(1)$$

where c is the geodesic with $c(0) = p$, $c'(0) = X$. This is well-defined on an open subset $U \subset TM_p$ containing 0. There is some ε such that all geodesics for unit-length vectors are defined for time ε. Then scale these (go slower), so you have geodesics that exist at time 1 for all ε-length vectors X. It is a tautology that

$$T_0(\exp_p) : T_pM \to T_pM$$

is the identity.

By making U smaller, we can achieve that $\exp_p$ is a diffeomorphism onto its image. This gives a preferred chart (modelled on $U \subset TM_p$ instead of $U \subset \mathbb{R}^n$). Alternatively, choose $TM_p \cong \mathbb{R}^n$ such that $g_{ij}(p) = \delta_{ij}$ (Riemannian metric becomes standard); then we get

$$\mathbb{R}^n \supset U \overset{\exp_p}{\to} M$$

where $0 \in U$. The resulting coordinate chart is called a geodesic normal coordinate chart around p.

Lemma 24.1. In geodesic local coordinates, for $x \in U$ (i.e. x close to zero),

$$g_{ij}(x) = \delta_{ij} + O(\|x\|^2).$$

Proof. Clearly, $g_{ij}(0) = \delta_{ij}$, because of the choice of identification $\mathbb{R}^n \cong TM_p$. Note that, in our local coordinates, all the straight lines $c(t) = t \cdot v$ (for $v \in \mathbb{R}^n$) are geodesics. I need to compute Γ_{ijk} in this local coordinate system.

$$\frac{d^2 c_k}{dt^2} + \sum_{ij} \Gamma_{ij}^k(c(t)) \frac{dc_i}{dt} \frac{dc_j}{dt} = 0$$

In our case:

$$\sum_{ij} \Gamma_{ij}^k(tv) v_i v_j = 0 \qquad \text{where } v = (v_1, \cdots, v_n) \in \mathbb{R}^n$$

Set $t = 0$:

$$\sum_{ij} \Gamma_{ij}^k(0) v_i v_j = 0 \text{ for all } (v_1, \cdots, v_n) \in \mathbb{R}^n$$

which implies that

$$\Gamma_{ij}^k(0) = 0$$
$$\Gamma_{ijk}(0) = 0$$

By the formula for Γ_{ijk}:

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 0$$

swap i and k

$$\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk} = 0$$

Now add the previous two equations and watch terms cancel:

$$\partial_j g_{ik} = 0 \text{ at } x = 0.$$

So the first derivative of (g_{jk}) at $x = 0$ is zero. $\square$

So, $g_{ij} = \delta_{ij} + O(\|x\|^2)$. What about the quadratic term? It turns out that it's usually nonzero (as is true generically, as shown in the HW).

Lemma 24.2 (Gauss)**.** In geodesic local coordinates,

$$\sum_{ij} g_{ij}(v)v_i w_j = v \cdot w$$

where $\cdot$ is the standard scalar product.

This does *not* say that g is the standard metric. But, if you fix a point v, and look at the vector v in that direction, the inner product with any arbitrary vector behaves like the standard metric.

Proof. Fix $v, w \in \mathbb{R}^n$. Consider

$$f(t) = \sum_{ij} g_{ij}(tv)v_i w_j - t(v \cdot w).$$

Obviously, $f(0) = 0$.

$$f'(t) = \underbrace{\sum_{ijk} \left(\frac{\partial}{\partial x_k} g_{ij} \right)(tv)tv_k v_i w_j}_{\mathscr{A}} + \underbrace{\sum_{ij} g_{ij}(tv)v_i w_j}_{\mathscr{B}} - \underbrace{v \cdot w}_{\mathscr{D}}.$$

We know that $c(t) = tv$ is a geodesic, so the second derivatives are zero and

$$\sum_{ijk} w_k(\Gamma_{ijk}(tv)v_i v_j) = 0$$

$$\underbrace{\sum_{ijk} t\frac{\partial g_{jk}}{\partial x_i}(tv)v_i v_j w_k}_{\text{corr. to } \mathscr{A}} - \underbrace{\frac{1}{2}t\frac{\partial g_{ij}}{\partial x_k}(tv)v_i v_j w_k}_{\mathscr{C}} = 0$$

Since geodesics go at constant speed

$$\frac{1}{2}\sum_{ij} g_{ij}(tv)v_i v_j = \frac{1}{2}v \cdot v$$

(the LHS is the same for all t, so it's the same as when $t = 0$). Hence, differentiate in the w-direction:

$$\underbrace{\frac{1}{2}\sum_{ijk} \partial_k g_{ij}(tv)tw_k v_i v_j}_{\mathscr{C}} + \underbrace{\sum_{ij} g_{ij}(tv)v_i w_j}_{\mathscr{B}} - \underbrace{v \cdot w}_{\mathscr{D}} = 0$$

Hence, $f'(t) = 0$ so $f(1) = 0$ as well. $\square$

Corollary 24.3. Let $c : I \to M$ be a geodesic, $I \subset \mathbb{R}$ an open neighborhood of 0, and c a path having unit speed. Then there is an $\varepsilon > 0$ such that

$$d(c(0), c(t)) = |t| \text{ if } |t| < \varepsilon$$

where d is the metric.

In words, the geodesic is locally length-minimizing near any of its points. In fact, we will see it's the *unique* local length-minimizer. Note that there are two kinds of "locally" running around: locally as in a short distance from a point, and locally as in smooth variations near the path.

Proof. Let c be a path from $c(0)$ to $c(t)$, where $c(0)$ and $c(t)$ are in one of these little neighborhoods. Suppose there's another path from $c(0)$ to $c(t)$. We may assume that this stays in the neighborhood, or else it's *certainly* longer than c, so we can talk about it in local coordinates. The Gauss lemma says that the metric splits into a radial part, and a part tangent to the sphere. So, given another path γ between $c(0)$ to $c(t)$, γ has the same radial component as c, but more tangential component (because the geodesic has zero tangential component). □

Lecture 25: November 8

Let $N \subset M$ be a submanifold. We have a normal bundle $\nu N = (TM|_N)/TN$.

Theorem 25.1 (Tubular Neighborhood Theorem)**.** There are open subsets $V \subset M$ with V containing N, and $U \subset \nu N$ with U containing the zero-section and a diffeomorphism $\varphi : U \to V$ such that $\varphi|_{\text{zero-section}} = \mathbb{1}_N$.

This shows that a neighborhood of $N \subset M$ depends only on νN. For instance, if νN is trivial, then locally M looks like $N \times \mathbb{R}^{\dim M - \dim N}$. But, there is no unique way of doing this, so it's hard to glue together.

Proof. Choose a Riemannian metric g on M and identify νN with the orthogonal complement of the subbundle $TN \subset TM|_N$. Then for $X \in (\nu N)_p \subset TM_p$, define $\varphi(X) = \exp_p(X)$. This is defined near the zero-section in νN, and $\varphi|_{\text{zero-section}} = \mathbb{1}$, and $T\varphi$ is invertible at each point of the zero-section.

More precisely, if $p \in N \subset \nu N$ is a point in the 0-section, then $T_p(\nu N) = (\nu N)_p \oplus TN_p \stackrel{(T\varphi)_p}{\to} TM_p$ is the map obtained from an embedding $\nu N \subset TM|_N$. □

Corollary 25.2. A closed, connected subset $N \subset M$ is a submanifold iff there is an open $U \subset M$ containing N and a smooth map $r : U \to U$ with $r(U) = N$, $r|_N = \mathbb{1}$.

Moral: submanifolds = smooth neighborhood retracts.

This is useful for finding Lefschetz fixed point numbers.

Lemma 25.3. Suppose (M, g) is a compact, connected Riemannian manifold. Then there is an ε such that any two points $p, q \in M$ with $d(p, q) < \varepsilon$ can be joined by a geodesic of length $d(p, q)$.

(Compactness is used to show that you can get the same ε for all pairs.)

Proposition 25.4. Let (M, g) be a compact, connected Riemannian manifold. For any points $p, q \in M$ there exists a geodesic joining them of length $d(p, q)$.

Corollary 25.5. If M is compact, then $\exp_p : TM_p \to M$ is onto.

Proof of Proposition 25.4. Take a sequence of paths joining p to q, parametrized with unit speed, and whose lengths converge to $d(p, q)$. By looking at the values of those paths at $t_i = i\frac{d(p,q)}{N}$ for $N \gg 0$ fixed (i.e. take a bunch of "evenly spaced" points t_i on the paths), we get points $p = x_0, x_1, \cdots, x_N = q$ such that

$$d(x_{i-1}, x_i) \le \frac{d(p,q)}{N}.$$

Necessarily, this is an equality (if not you'd get a path that was shorter than the infimum), and we can assume that $\frac{d(p,q)}{N} < \varepsilon$. Now connect x_{i-1} to x_i by a geodesic with length $\frac{d(p,q)}{N}$ and consider the composition of these geodesics. A priori this might not be a geodesic – it might have corners. But, the outcome is an absolute length-minimizer, so it must be a geodesic. □

Remark 25.6. The same argument shows that each homotopy class of paths from p to q has a representative that is a geodesic, and that each free homotopy class of loops contains a periodic geodesic. If you want you can define the fundamental group in terms of discretized (broken) geodesics.

Lemma 25.7. Let (M, g) be a connected Riemannian manifold. Suppose that $p \in M$ is a point such that $\exp_p$ is defined on all of TM_p. Then any point q can be joined to p by a geodesic of length $d(p, q)$.

Proof. Take a small $\varepsilon > 0$. $S_\varepsilon(p) = \{x \in M : d(x, p) = \varepsilon\}$ is compact by Gauss' lemma. Choose $x \in S_\varepsilon(p)$ to minimize $d(x, q)$ (travel around the boundary of a neighborhood of x and see which boundary point is closest to q) and let c be the unit speed geodesic with $c(0) = p$, $c(\varepsilon) = x$. Then $d(x, q) = d(p, q) - \varepsilon$ (certainly it can't be $<$, because then you would have made a path shorter than allowed; but if it's bigger, then you know there's a path whose length is as close as you want to $d(p, q)$, and this get a contradiction to the choice of x). Take

$$T = \max\{t \in [0, d(p, q)] : d(c(t), q) = d(p, q) - t\}$$

(t represents how far you went – the furthest you can go before this stops being the right direction to go in). Gauss' lemma shows that $T > 0$. We want to show that $T = d(p, q)$. TBC

Lecture 26: November 13

Proof of Lemma 25.7, con't. For small ε, we claimed that you can find x such that $d(p,x) = \varepsilon$ and $d(x,q) = d(p,q) - \varepsilon$. This uses the Gauss lemma: in normal coordinates around p, x is the point in the sphere of radius ε with least distance to q. The Gauss lemma says that $d(p,x) = \varepsilon$, and the triangle inequality says that $d(x,q) \geq d(p,q) - \varepsilon$. To get the other direction, notice that any path has to leave the ε-ball around x, and the quickest way to do that still takes length ε: if $d(x,q) > d(p,q) - \varepsilon$ we can find a path from p to q of length $< d(x,q) + \varepsilon$. By looking at where that path crosses the sphere of radius ε we get a contradiction to the defining property of x. Hence $d(x,q) = d(p,q) - \varepsilon$ as claimed.

Take the unit speed geodesic c with $c(0) = p$, $c(\varepsilon) = x$. Take

$$T = \max\{t \in [0, d(p,q)] \,:\, d(c(t),q) = d(p,q) - t\}.$$

This set is closed (prove this yourself). We know $T > 0$ and want to show $T = d(p,q)$, since that yields $c(d(p,q)) = q$. Assume $T < d(p,q)$; $c(T)$ is the point at which you stop going in the right direction.

For small $\delta > 0$, the Gauss lemma yields a point y with $d(c(T), y) = \delta$, and

$$\begin{aligned} d(y,q) &= d(c(T),q) - \delta \\ &= d(p,q) - T - \delta \qquad \text{because } T \text{ is in that set above.} \end{aligned} \tag{26.1}$$

By assumption, c exists for all time. So

$$\begin{aligned} d(p,y) &\geq d(p,q) - d(y,q) && \text{triangle inequality} \\ &= T + \delta && \text{by (26.1).} \end{aligned}$$

Hence, $c|_{[0,T]}$, combined with the short geodesic from $c(T)$ to y, is a length-minimizing path from p to y. Hence it's a smooth geodesic, i.e. $y = c(T + \delta)$. Hence, $T + \delta \in \{t : d(c(t),q) = d(p,q) - t\}$, which is a contradiction to the maximality of T. □

Theorem 26.1 (Hopf-Rinow)**.** Suppose (M, g) is a connected Riemannian manifold. The following are equivalent:

(i) Geodesics on M exist for all time
(ii) Bounded subsets are relatively compact
(iii) M is complete as a metric space.

Remark 26.2. By the previous lemma, any of these properties implies that two points p, q can be joined by a geodesic with length $d(p,q)$.

Proof. $(i) \implies (ii)$ Suppose $K \subset M$ is bounded. Take $p \in K$ and $\exp_p : T_pM \to M$. Then $\overline{K} \subset \exp_p(\overline{B}_R(0))$ where $\overline{B}_R(0)$ is the closed ball of radius $R \gg 0$. But $\overline{B}_R(0)$ is compact, hence so is its image, and $\overline{K}$ is a closed subset of it.

$(ii) \implies (iii)$ Elementary topology (Cauchy sequences have bounded subsequences…)

$(iii) \implies (i)$ Take a geodesic $c : [0, T) \to M$, where T is the maximal time of existence. Then $c(T - \frac{1}{2^n}) = x_n$ is a Cauchy sequence. Take its limit; this is a contradiction. $\square$

Let (M, g) be a Riemannian manifold, and $p \in M$. Consider a subset $U \subset TM_p$ containing p, and an arbitrary point $x \in U$. Take $Y, Z \in T_x M$. We want to know what happens to the inner product of Y, Z under the exponential map: do a Taylor expansion around $x = 0$ and get

$$g_{\exp_p(x)}((T_x)\exp_p)(Y), (T_X \exp_p)(Z) = g_p(Y, Z) + \text{ (quadratic term)}.$$

The quadratic term is a multilinear map

$$TM_p \otimes TM_p \otimes TM_p \otimes TM_p \to \mathbb{R}.$$

(The last two terms are symmetric in Y, Z.) This is the obstruction to flatness and is intrinsic. You can actually calculate this; it ends up being a combination of Christoffel symbols and their derivatives.

In modern notation, it equals

$$\frac{1}{12}(R_p(X, Y, Z, X) + R_p(X, Z, X, Y))$$

where $R_p : TM_p \otimes TM_p \otimes TM_p \otimes TM_p \to \mathbb{R}$ is the *Riemannian curvature tensor*.

Lecture 27: November 15

Wednesday is a midterm. There is no final.

Let M be a manifold and $f \in C^\infty(M, \mathbb{R})$ a function. At every point, we have $T_x f : TM_x \to \mathbb{R}$, usually written as $df_x \in TM_x^*$. In fact, df is a smooth section of the cotangent bundle TM^* (write $df \in C^\infty(M, T^*M) =: C^\infty(T^*M)$). Take a smooth vector field X (so $X_x \in TM_x$). Define

$$X.f = df(X) \in C^\infty(M, \mathbb{R}) \text{ and } (X \cdot f)_x = df_x(X_x)$$

(differentiating f in the X direction).

Fact 27.1. $(X.*)$ is a derivation for the algebra $C^\infty(M, \mathbb{R})$:

$$X.(fg) = (X.f)g + f(X.g)$$

(this is just the Leibniz formula).

$$X.(Y.f) - Y.(X.f) = [X, Y].f$$

In local coordinates,

$$X.(Y.f) = (D^2 f)(X, Y) + \text{ lower order terms}.$$

Because the Hessian is symmetric, $X.(Y.f) - Y.(X.f)$ is a first-order differential operator. It is given by the vector field $[X, Y].f$.

Theorem 27.2. Suppose we have a Riemannian metric g. Then there is a unique bilinear (over $\mathbb{R}$) operation

$$\begin{aligned} C^\infty(TM) \times C^\infty(TM) &\to C^\infty(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

satisfying the *connection axioms*:

(1) $\nabla_{fX} Y = f \cdot (\nabla_X Y)$ for $f \in C^\infty(M, \mathbb{R})$ ("taking the derivative with greater intensity")
(2) $\nabla_X(fY) = (X.f) \cdot Y + f \cdot \nabla_X Y$

and also satisfying

(1) (compatibility with the metric)
$$X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
(2) (torsion freeness)
$$\nabla_X Y - \nabla_Y X = [X, Y].$$

This ∇ is called the Levi-Civita connection.

Note: if Y is supported on some subset, then $\nabla_X Y$ is supported on that same subset. So it's OK to work locally.

Proof. In local coordinates, set ∂_i be the i^{th} constant unit vector field, and
$$\nabla_{\partial_i}(\partial_j) = \sum_k A_{ij}^k \partial_k$$
for functions A_{ijk}. Then using the connection axioms,
$$\nabla_X Y = DY \cdot X + \sum_{ijk} A_{ij}^k X_i Y_j \partial_k.$$
Compatibility with the metric says that
$$\begin{aligned} \frac{\partial}{\partial x_k} g_{ij} = \partial_k \cdot g(\partial_i, \partial_j) &= g(\nabla_{\partial_k} \partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k} \partial_j) \\ &= \sum_\ell g_{\ell j} A_{ki}^\ell + g_{\ell i} A_{kj}^\ell \end{aligned} \tag{27.1}$$

The torsion-freeness axiom says that
$$0 = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i$$

$$= \sum_k (A_{ij}^k - A_{ji}^k)\partial_k$$

So far we've only used last two properties in the case of ∂_i etc., but if they are satisfied for the standard vector fields, they are true for all vector fields (by the first few axioms).

Write down the other equations of form (27.1) by permuting the indices:

$$\begin{aligned}
\frac{\partial}{\partial x_k} g_{ij} + \frac{\partial}{\partial x_i} g_{jk} - \frac{\partial}{\partial x_j} g_{ki} &= \sum_\ell g_{\ell j} A_k^\ell + g_{\ell i} A_{kj}^\ell \\
&\quad + g_{\ell k} A_{ij}^\ell + g_{\ell j} A_{ik}^\ell - g_{\ell i} A_{jk}^\ell - g_{\ell k} A_{jk}^\ell \\
&= 2 \sum_\ell g_{\ell i} A_{ki}^\ell \qquad \text{symmetry } A_{ij}^k = A_{ji}^k
\end{aligned}$$

It follows that $A_{ij}^k = \Gamma_{ij}^k$. □

Definition 27.3. The Riemann curvature is a map

$$\begin{aligned}
&R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM) \\
&R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\end{aligned}$$

(The textbook defines the opposite sign convention, which is *awful*. That is an attempt to make the sectional curvature look nicer, but our convention is the standard one.)

Motivation: if Z was just a function, this would be zero (this is the definition of the Lie bracket). We want to see how badly this fails for the Levi-Civita connection. The left two terms are second-order derivatives, and in the Lie bracket case the second-order bit cancels out.

Proposition 27.4. This satisfies

$$\begin{aligned}
R(X,Y)(fZ) &= f \cdot R(X,Y)Z \\
R(fX,Y)(Z) &= f \cdot R(X,Y)Z \\
R(X,fY)(Z) &= f \cdot R(X,Y)Z
\end{aligned}$$

Proof.

$$\begin{aligned}
R(X,Y)(fZ) &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\
&= -f\nabla_{[X,Y]}(Z) - ([X,Y].f)Z + \nabla_X(f \cdot \nabla_Y Z) + \nabla_X((Y.f)Z) \\
&\qquad - \nabla_Y(f \cdot \nabla_X Z) - \nabla_Y((X.f)Z) \\
&= fR(X,Y)Z - ([X,Y].f)Z + (X.f)\nabla_Y Z + (X.Y.f)Z + (Y.f)(\nabla_X Z) \\
&\qquad - (Y.f)(\nabla_X Z) - (Y.X.f)Z - (X.f)(\nabla_Y Z) \\
&= fR(X,Y)Z
\end{aligned}$$

etc. □

Corollary 27.5. R is a pointwise operation:
$$R \in C^\infty(T^*M \otimes T^*M \otimes T^*M \otimes TM).$$

Upshot: you can define differentiation, but it won't be compatible with the Lie bracket.

Lecture 28: November 18

Let (M, g) be a Riemannian manifold; at every point we have a map $R_p : T_pM \otimes T_pM \otimes T_pM \to T_pM$ given by
$$R_p(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$
Equivalently, one considers
$$\begin{aligned} &TM_p \otimes TM_p \otimes TM_p \otimes TM_p \to \mathbb{R} \\ &\qquad R_p(X, Y, Z, W) = g_p(R_p(X, Y)Z, W) \end{aligned}$$
By definition,
$$R(X, Y, Z, W) = -R(Y, X, Z, W).$$

Lemma 28.1. $R(X, Y, Z, W) = -R(X, Y, W, Z)$

Proof. Uses compatibility of ∇ with the metric.
$$\begin{aligned} R(X, Y, W, Z) &= g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W, Z) \\ &= X.g(\nabla_Y W, Z) - g(\nabla_Y W, \nabla_X Z) - Y.g(\nabla_X W, Z) + g(\nabla_X W, \nabla_Y Z) \\ &\quad - [X, Y].g(W, Z) + g(W, \nabla_{[X,Y]} Z) \\ &= -Y.g(W, \nabla_X Z) + g(W, \nabla_Y \nabla_X Z) + X.g(W, \nabla_Y Z) - g(W, \nabla_X \nabla_Y Z) \\ &\quad + g(W, \nabla_{[X,Y]} Z) + X.Y.g(W, Z) - X.g(W, \nabla_Y Z) - Y.X.g(W, Z) \\ &\quad + Y.g(W, \nabla_X Z) - [X, Y].g(W, Z) \\ &= g(W, \nabla_Y \nabla_X Z) - g(W, \nabla_X \nabla_Y Z) + g(W, \nabla_{[X,Y]} Z) \\ &= -R(X, Y, Z, W) \end{aligned}$$
We could have done this computation assuming that $[X, Y] = 0$, because locally everything is a linear combination of the ∂_i's, and $[\partial_i, \partial_j] = 0$. $\square$

In fact, torsion-freeness implies
$$R(X, Y, Z, W) = R(Z, W, X, Y).$$
The proof is not enlightening; see the textbook.

Definition 28.2. Let $P \subset T_pM$ be a two-dimensional tangent subspace. Write
$$P = \mathbb{R}X \oplus \mathbb{R}Y$$

where X, Y are an orthonormal basis: $g(X,X) = 1, g(Y,Y) = 1, g(X,Y) = 0$. Then the *sectional curvature* is

$$K_p(P) = -R(X,Y,X,Y).$$

One can show, using the symmetry properties of R, that K is independent of the choice of (X,Y).

$$\begin{aligned} R(X,Y,\cos(\alpha)X + \sin(\alpha)Y, -\sin(\alpha)X + \cos(\alpha)Y) &= \cos^2(\alpha)R(X,Y,X,Y) - \sin^2(\alpha)R(X,Y,Y,X) \\ &= R(X,Y,X,Y) \end{aligned}$$

To rotate in all coordinates, first rotate in the third and fourth coordinates (above) which doesn't change anything, then rotate in the first and second coordinates, which also doesn't change anything because of the symmetry properties. Also reflections don't change this.

Take a smooth path $\mathbb{R} \supset I \xrightarrow{c} M$. We consider smooth tangent vector fields along that path, $X \in C^\infty(I, c^*TM)$, meaning that $t \in I \mapsto X_t \in TM_{c(t)}$ (you have a tangent vector at every point in the path, but that vector doesn't have to be tangent to the path).

Example 28.3. If $\widetilde{X} \in C^\infty(TM)$ then its pullback then its pullback $X = c^*\widetilde{X}$ is given by $X_t = \widetilde{X}_{c(t)}$, for $X \in C^\infty(c^*TM)$.

But, not every $X \in C^\infty(c^*TM)$ is of the form $X = c^*\widetilde{X}$ (unless c is an embedded path – use the tubular neighborhood theorem to extend a vector field defined on the path to a neighborhood). For example, if c is the constant path, then a vector field along c is just a family of vectors that varies in time; a vector field that is the pullback of something can only have one vector at that point. However, locally in t one can always write

$$X_t = f_1(t)(c^*\widetilde{X}_1)_t + \cdots + f_n(t)(c^*\widetilde{X}_n)_t.$$

(Think about this for the constant path; choose $\widetilde{X_i}$ so that $c^*\widetilde{X_i}$ is the i^{th} basis vector, and choose $f_1, \cdots, f_n$ so that at time t, this produces the correct linear combination of basis vectors.) There is a unique linear operation

$$\frac{\nabla}{\partial t} : C^\infty(c^*TM) \to C^\infty(c^*TM)$$

such that:

- if $X = c^*\widetilde{X}$, $\frac{\nabla}{\partial t}X = c^*(\nabla_{\frac{dc}{dt}}\widetilde{X})$
- $\frac{\nabla}{\partial t}(fX) = \left(\frac{df}{dt}\right)X + f\left(\frac{\nabla}{\partial t}\right)X$ for $f : I \to \mathbb{R}$

In a local coordinate chart on M,

$$c(t) = (c_1(t), \cdots, c_n(t))$$
$$X(t) = (X_1(t), \cdots, X_n(t))$$

we have

$$\left(\frac{\nabla}{\partial t}X\right)_k = \frac{dX_k}{dt} + \sum_{ij}\Gamma^k_{ij}(c(t))c_i'(t)X_j(t).$$

So this is a version of the Levi-Civita connection. It's a first-order ODE, so apply ODE theory. Take $I = [a, b]$. For every $Z \in TM_{c(a)}$, there is a unique $X \in C^\infty(c^*TM)$ which starts at $X_a = Z$ and satisfies $\frac{\nabla}{\partial t}X = 0$. (Basically, start at $c(a)$ and transport the vector Z along the curve.) The map $Z \mapsto Y = X_b$ is a linear map

$$\tau_c : TM_{c(a)} \to TM_{c(b)}$$

called *parallel transport*. If you do this in flat space, the vector really doesn't change when you do this. But in general, having non-flat curvature means that different paths from $c(a)$ to $c(b)$ might result in different parallel transports.

Proposition 28.4. Properties:

(1) For $X, Y \in C^\infty(c^*TM)$,
$$\frac{d}{dt}g(X,Y) = g(\frac{\nabla}{\partial t}X, Y) + g(X, \frac{\nabla}{\partial t}Y).$$
This implies that τ_c is an isometry.

(2) (analogue of torsion-freeness) Let $c : \Omega \to M$ for some open set $\Omega \subset \mathbb{R}^2$. Then
$$\frac{\nabla}{\partial t}\frac{\partial c}{\partial s} = \frac{\nabla}{\partial s}\frac{\partial c}{\partial t}$$
where $\frac{\partial c}{\partial s}$ is a vector field along $t \mapsto c(s,t)$. (On the RHS you're thinking of this as a family of s-paths; on the LHS it's a family of t-paths.) In local coordinates,
$$\left(\frac{\nabla}{\partial s}\frac{\partial c}{\partial t}\right)_k = \frac{\partial^2 c_k}{\partial s \partial t} + \sum_{ij} \Gamma_{ij}^k(c(s,t))\frac{\partial c_i}{\partial s}\frac{\partial c_j}{\partial t}$$

(3) Take $c : \Omega \to M$ as before, and $X \in C^\infty(\Omega, c^*TM)$ (for $X_{s,t} \in TM_{c(s,t)}$). Then
$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}X - \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}X = R\left(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\right)X.$$

First look at a vector field X that is pulled back from the manifold, in which case it becomes a consequence of the definition. Then use the fact that everything can be written as a linear combination of pulled back vector fields.

Corollary 28.5. Suppose (M, g) is flat (i.e. $R \equiv 0$). Then parallel transport $\tau_c : TM_{c(a)} \to TM_{c(b)}$ depends only on the homotopy class of c (relative to the endpoints).

Proof next time.

Corollary 28.6. If M is simply connected and carries a flat metric, then TM is the trivial bundle (because you can trivialize it by parallel transport, and ODE theorems show it's smooth).

Lecture 29: November 22

Lemma 29.1. Let (M, g) be a Riemannian manifold which is flat ($R \equiv 0$). Then parallel transport

$$\tau_c : TM_{c(a)} \to TM_{c(b)}$$

depends only on the homotopy class (rel endpoints).

Suppose (M, g) is flat and simply-connected. Pick $p \in M$ and an orthonormal basis $(X_1, \cdots, X_n)$ of TM_p. By using parallel transport, one finds vector fields $\widetilde{X}, \cdots, \widetilde{X}_n$ with $(\widetilde{X}_i)_p = X_i$, $\nabla X_i = 0$. These satisfy

$$g(\widetilde{X}_i, \widetilde{X}_j) = \text{const} = \delta_{ij}$$

(because ∇ is compatible with the connection) and $[\widetilde{X}_i, \widetilde{X}_j] = 0$ (because ∇ is torsion-free). In particular, (M, g) is locally isometric to $(\mathbb{R}^n, g_{ij} = \delta_{ij})$ by flowing along $(\widetilde{X}_1, \cdots, \widetilde{X}_n)$. If (M, g) is geodesically complete, it is globally isometric to flat space $(\mathbb{R}^n, g_{ij} = \delta_{ij})$.

That's all I'm going to say about flat metrics because that's not the interesting part.

In terms of covariant differentiation, c is a geodesic iff $\frac{\nabla}{\partial t}\frac{\partial c}{\partial t} = 0$. Now look at $c = c(s, t)$ such that for any s, $c(s, \cdot)$ is a geodesic (i.e. a parametrized family of geodesics c_s). Consider $X(s, t) = \frac{\partial c}{\partial s} \in TM_{c(s,t)}$. This satisfies

$$\begin{aligned}
\left(\frac{\nabla}{\partial t}\right)^2 X &= \frac{\nabla}{\partial t}\frac{\nabla}{\partial t}\frac{\partial c}{\partial s} \\
&= \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\frac{\partial c}{\partial t} \\
&= \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial c}{\partial t} - R\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right)\frac{\partial c}{\partial t}
\end{aligned}$$

where the last two equalities come from Proposition 28.4. This equation for $X \in C^\infty(c^*TM)$,

$$\left(\frac{\nabla}{\partial t}\right)^2 X - R(c'(t), X)c'(t) = 0$$

is called the Jacobi equation. Its solutions are called Jacobi fields.

Remark 29.2. The Jacobi equation is a second order linear ODE (with variable coefficients). Hence,

- there are $2n$ linearly independent solutions, which form a basis of the solution space.
- any solution is determined by its value and first derivative at a point, and those can be arbitrary.

Solutions:

$X(t) = c'(t)$ is a solution (R is antisymmetric so $R(c'(t), c'(t)) = 0$)

$X(t) = t \cdot c'(t)$ is a solution (rescaling the parametrization of a geodesic by a constant)

Suppose X is a Jacobi field.

$$\left(\frac{d}{dt}\right)^2 g(X, c'(t)) = \frac{d}{dt} g\left(\frac{\nabla X}{\partial t}, c'(t)\right)$$

(the second term vanishes because c is a geodesic)

$$\begin{aligned} &= g\left(\frac{\nabla^2 X}{dt^2}, c'(t)\right) \\ &= g(R(c'(t), X)c'(t), c'(t)) \\ &= 0 \qquad \text{antisymmetry of } R(-,-,-,-) \end{aligned}$$

Hence, if for some t_0, $X(t_0)$ and $\frac{\nabla X}{\partial t}(t_0)$ are both orthogonal to $c'(t)$, the same will hold for all times. So, there is a $(2n-2)$-dimensional space of Jacobi fields everywhere orthogonal to c'.

Lemma 29.3. Take $p \in M$, $X \in TM_p$. Let c be the geodesic with $c(0) = p$, $c'(0) = X$. For any $Y \in TM_p$, let Z be the unique Jacobi field along c with $Z(0) = 0$, $\frac{\nabla Z}{\partial t}|_0 = Y$. Then $(\exp_p)_{*X}(Y) = Z(1)$.

$\exp_p : TM_p \to M$ sends $X \mapsto c(1)$.

Moral: Jacobi fields give the derivative of the exponential.

Theorem 29.4. Suppose that (M, g) has sectional curvature ≤ 0. Then for any $p \in M$, $\exp_p$ has invertible differential (i.e. it is a local diffeomorphism).

Proof. Take a geodesic c, a Jacobi field Z along c with $Z(0) = 0$, but $Z \not\equiv 0$. We have to show that $Z(t) \neq 0$ for all $t \neq 0$ (and hence the derivative of the exponential map is injective).

$$\begin{aligned} \left(\frac{d}{dt}\right)^2 \cdot \frac{1}{2} g(Z(t), Z(t)) &= \frac{d}{dt} g\left(\frac{\nabla Z}{\partial t}, Z\right) \\ &= g\left(\frac{\nabla Z}{\partial t}, \frac{\nabla Z}{\partial t}\right) + g\left(\frac{\nabla^2 Z}{\partial t^2}, Z\right) \\ &= \underbrace{g\left(\frac{\nabla Z}{\partial t}, \frac{\nabla Z}{\partial t}\right)}_{\geq 0} + \underbrace{R(c', Z, c', Z)}_{\substack{\geq 0 \text{ because} \\ \text{sectional curvature } \leq 0}} \end{aligned}$$

Hence, $\psi(t) = \frac{1}{2} g(Z(t), Z(t))$ is convex. But $\psi(0) = 0$, $\psi'(0) = 0$ (it's a quadratic function), $\psi''(0) > 0$ which implies $\psi(t) \neq 0$ for all t. $\square$

Theorem 29.5 (Hadamard-Cartan)**.** If (M, g) is simply-connected, complete, and has sectional curvature ≤ 0, then for all p, $\exp_p : TM_p \to M$ is a diffeomorphism (i.e. M is diffeomorphic to $\mathbb{R}^n$).

By completeness, we know that $\exp_p$ is onto, and we know it's a local diffeomorphism. We need to show that it's a covering map. This involves some work.

If this is compact but not simply-connected, then you can apply this to the universal cover. Any space with a contractible universal cover is a $K(G, 1)$, so these conditions are pretty restrictive, homotopy-theory-wise.

Lecture 30: November 25

Let (M, g) be a Riemannian manifold. Recall

$$E(c) = \int_a^b \frac{1}{2} g(c'(t), c'(t))dt.$$

Consider this as a function on the space of paths $c : [a, b] \to M$, with fixed endpoints $c(a)$, $c(b)$. Let (c_s) be a family of such paths, depending on $s \in (-\varepsilon, \varepsilon)$, and with $c_0 = c$. So

$$\begin{aligned}
(-\varepsilon, \varepsilon) \times [a, b] &\to M \\
(s, t) &\mapsto c_s(t) \\
(s, a) &\mapsto \text{ constant in } s \\
(s, b) &\mapsto \text{ constant in } s
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{ds} E(c_s)|_{s=0} &= \int_a^b \frac{\partial}{\partial s} \frac{1}{2} g\left(\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t}\right) dt \\
&= \int_a^b g\left(\frac{\nabla}{\partial s}\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t}\right) dt \\
&= \int_a^b g\left(\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \frac{\partial c_s}{\partial t}\right) dt \\
&= \int_a^b \frac{d}{dt} g\left(\frac{\partial c_s}{\partial s}, \frac{\partial c_s}{\partial t}\right) dt - g\left(\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial t}\right) dt \\
&= -\int_a^b g\left(\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial t}\right) dt
\end{aligned}$$

We write $X = \frac{\partial c_s}{\partial s}|_{s=0} \in C^\infty(c^*TM)$, $X(a) = X(b) = 0$.

The *first variation* of the energy in the X direction is

$$X \mapsto -\int_a^b g(X, \frac{\nabla}{\partial t}\frac{\partial c}{\partial t})dt$$

This vanishes iff c is a geodesic.

Assuming c is a geodesic,

$$\left(\frac{d}{ds}\right)^2 E(c_s)|_{s=0} = -\int_a^b \frac{\partial}{\partial s} g\left(\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial t}\right) dt$$

the first term in the compatibility with the metric expansion vanishes

$$\begin{aligned}
&= -\int_a^b g\left(\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial t}\right) dt \\
&= -\int_a^b g\left(\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\frac{\partial c_s}{\partial t}\right) dt + \int_a^b g\left(\frac{\partial c_s}{\partial s}, R\left(\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial s}\right)\frac{\partial c_s}{\partial t}\right) dt \\
&= \int_a^b g\left(\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}\right) dt + \int_a^b g\left(R\left(\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial s}\right)\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial s}\right) dt
\end{aligned}$$

On the space

$$\{X \in C^\infty(c^*TM) \,:\, X(a) = 0, X(b) = 0\}$$

we have the symmetric bilinear form

$$Q(X,Y) = \int_a^b g\left(\frac{\nabla}{\partial t}X, \frac{\nabla}{\partial t}Y\right) + g(R(c'(t),X)c'(t),Y)dt$$

called the *second variation form*. Then our computation says that for $X = \left(\frac{\partial c_s}{\partial s}\right)|_{s=0}$,

$$\left(\frac{d}{ds}\right)^2 E(c_s)|_{s=0} = Q(X,X).$$

If $Q(X,X)$ is negative, then that means you can find a nearby shorter path with the same endpoints.

Lemma 30.1. The null space of Q is precisely the space of Jacobi fields X along c with $X(a) = X(b) = 0$. The null space has dimension $\leq \dim M - 1$.

Proof. Integration by parts:

$$Q(X,Y) = \int_a^b -g\left(\frac{\nabla^2 X}{\partial t^2}, Y\right) + g\left(R(c',X)c',X\right) dt$$

□

Idea 30.2. There is a decomposition

$$\begin{aligned}
\mathcal{H} &= \{X \in C^\infty(c^*TM) \,:\, X(a) = X(b) = 0\} \\
\mathcal{H} &= \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+
\end{aligned}$$

which is orthogonal for X, $\mathcal{H}_0$ = null space of Q, and $Q|_{\mathcal{H}_-}$ is negative definite, $Q|_{\mathcal{H}_+}$ is positive definite, and $\dim \mathcal{H}_- < \infty$.

Definition 30.3. $\dim \mathcal{H}_-$ is called the *Morse index* of a geodesic c.

$$Q(X,Y) = \int_a^b g\left(\frac{\nabla X}{\partial t}, \frac{\nabla Y}{\partial t}\right) + g(R(c', X)c', Y)dt.$$

This can be made rigorous in two ways (leading to the notion of Morse index).

(1) *(Analytically):* Complete $\mathcal{H}$ in the Sobolev $W^{1,2}$ norm (the one that has $g(\frac{\nabla X}{\partial t}, \frac{\nabla Y}{\partial t})$ as inner product). This yields a Hilbert space, and
$$Q(X,Y) = \langle X, Y\rangle_{W^{1,2}} + \langle X, \mathscr{R}Y\rangle_{W^{1,2}}$$
where $\mathscr{R}$ is a compact self-adjoint operator. Apply the spectral theorem to $\mathbb{1}+\mathscr{R}$. Then $\mathcal{H}_-$ is the eigenspace associated to the negative eigenvalues of $\mathbb{1}+\mathscr{R}$.

(2) *(By piecewise extension):* Take a decomposition of $[a,b]$ into small sub-intervals, and allow piecewise-smooth X. See Milnor's *Morse Theory*.

Lemma 30.4. If (M,g) has non-positive sectional curvature, the variation form of any geodesic is positive definite.

Proof. The sectional curvature is $-g(R(c', X)c', Y)$, so if it's non-positive, then
$$Q(X,X) \geq \int_a^b g\left(\frac{\nabla X}{\partial t}, \frac{\nabla X}{\partial t}\right) dt > 0$$
for all $X \neq 0$. □

Make the ansatz[1]
$$X(t) = f(t)\overline{X}(t)$$
where $\frac{\nabla}{\partial t}\overline{X} = 0$, $g(\overline{X}, \overline{X}) = 1$, $g(\overline{X}(t), \frac{dc}{dt}) = 0$. Then
$$Q(X,X) = \int_a^b f'(t)^2 + f(t)^2 \cdot g(R(c', X)c', X)dt.$$
Without loss of generality, suppose $a = 0$, $g(c', c') = 1$. Suppose that all sectional curvatures are $\geq \kappa > 0$. Then
$$Q(X,X) \leq \int_0^b f'(t)^2 - \kappa f(t)^2 dt.$$
Set $f(t) = \sin\left(\frac{\pi}{b}t\right)$. Note that $f(0) = 0$ and $f(b) = 0$. Then
$$Q(X,X) \leq \int_a^b \frac{\pi^2}{b^2}\cos\left(\frac{\pi}{b}t\right)^2 - \kappa \sin\left(\frac{\pi}{b}t\right)^2 dt$$
$\int_0^b \cos^2\left(\frac{\pi}{b}t\right) + \sin^2\left(\frac{\pi}{b}t\right) dt = b$, and the pieces are equal by change of coordinates:
$$= \left(\frac{\pi^2}{b^2} - \kappa\right) \cdot \frac{b}{2}.$$

[1] German word for "start" – instead of considering the general case we will consider something of this form

Lemma 30.5. If (M, g) has sectional curvature $\geq \kappa > 0$, any geodesic of length $> \frac{\pi}{\sqrt{\kappa}}$ has indefinite (i.e. neither positive semi-definite nor negative semi-definite) second variation pairing.

Upshot: there exists X such that $Q(X, X) < 0$, provided the geodesic is long enough.

Corollary 30.6. If (M, g) is compact and has sectional curvature $\geq \kappa > 0$, its diameter is $\leq \frac{\pi}{\sqrt{\kappa}}$.

Any two points are joined by a geodesic; if the geodesic is longer than $\frac{\pi}{\sqrt{\kappa}}$, then that's not the shortest geodesic between them.

This is a sharp bound (take a sphere).

Corollary 30.7. If (M, g) is compact and has sectional curvature > 0, then $\pi_1(M)$ is finite.

Represent every homotopy class by a minimal-length geodesic. These have an upper bound for their length. This takes more work. . .

In this case, the universal cover is finite.

Lecture 31: November 27

Lemma 31.1. Let $E \to [a, b] \times M$ be a differentiable vector bundle. Then the restrictions $E|_{\{t\}\times M}$ are all mutually isomorphic vector bundles over M (for $t \in [a, b]$).

So the isomorphism type doesn't suddenly jump. However, there is no canonical choice of isomorphism. This is false in algebraic geometry.

This implies:

Lemma 31.2. Let $f_0, f_1 : M \to N$ be two maps which are smoothly homotopic. Then for any vector bundle $E \to N$, the pullback $f_0^* E \to M$, $f_1^* E \to M$ are isomorphic.

By a smooth homotopy, I mean a smooth map $F : [0, 1] \times M \to N$ where $F(0, x) = f_0(x)$, $F(1, x) = f_1(x)$. Apply Lemma 31.1 to show that $F^* E|_{\{0\}\times M} \cong F^* E|_{\{1\}\times M}$.

Corollary 31.3. Any vector bundle on $\mathbb{R}^n$ is trivial.

$\mathbb{R}^n$ is contractible, so the identity is homotopic to the constant map to a point.

DEFINITION 31.4. Let $E \to M$ be a smooth vector bundle. A connection is a $\mathbb{R}$-bilinear map

$$C^\infty(TM) \times C^\infty(E) \to C^\infty(E) \text{ notated } (X, \xi) \mapsto \nabla_X \xi$$

such that for $f \in C^\infty(M, \mathbb{R})$,

- $\nabla_{fX}\xi = f \cdot \nabla_X \xi$
- $\nabla_X(f\xi) = f \cdot \nabla_X \xi + (X.f)\xi$

EXAMPLE 31.5. If (M, g) is a Riemannian manifold, then the Levi-Civita connection ∇ is an example of a connection in the above sense.

EXAMPLE 31.6. Let $E = \mathbb{R}^r \times M$ be the trivial bundle. Then we have the trivial connection

$$\nabla_X^{\text{triv}} \xi = X.\xi$$

for $\xi \in C^\infty(M, \mathbb{R}^r)$.

In the case of Riemannian manifolds, we had a preferred connection (the Levi-Civita connection). But in this more general setting, we don't. (Note that torsion-freeness doesn't even make sense when $TM \neq E$.)

Lemma 31.7. Let ∇ be a connection on E and A a vector bundle homomorphism $TM \to E \otimes E^* = \operatorname{End} E$. Then

$$\widetilde{\nabla} = \nabla + A, \text{ defined as } \widetilde{\nabla}_X \xi = \nabla_X \xi + A(X)\xi$$

defines a connection.

Proof. We need to verify the axioms in Definition 31.4.

$$\begin{aligned}
\widetilde{\nabla}_X(f\xi) &= \nabla_X(f\xi) + fA(X)\xi \\
&= f \cdot \nabla_X \xi + (X.f)\xi + fA(X)\xi \\
&= f(\widetilde{X}_X \xi) + (X.f)\xi
\end{aligned}$$

Similarly for the other property. $\square$

Lemma 31.8. If ∇ and $\widetilde{\nabla}$ are connections on E, then

$$\widetilde{\nabla}_X \xi - \nabla_X \xi = A(X)\xi$$

for some vector bundle homomorphism $A : TM \to \operatorname{End}(E)$.

Proof.

$$\begin{aligned}
\widetilde{\nabla}_X(f\xi) - \nabla_X(f\xi) &= f(\widetilde{\nabla}_X \xi - \nabla_X \xi) \\
\widetilde{\nabla}_{fX}\xi - \nabla_{fX}\xi &= f(\widetilde{\nabla}_X \xi - \nabla_X \xi)
\end{aligned}$$

so $\widetilde{\nabla} - \nabla$ is a pointwise operation, hence a vector bundle homomorphism $TM \otimes E \to E$ (equivalently, it's a vector bundle homomorphism $TM \to E \otimes E^* = \operatorname{End}(E)$). $\square$

Proposition 31.9. Any smooth vector bundle has a connection.

Proof. Take a covering $M = \bigcup_{\beta \in B} U_\beta$ such that $E|_{U_\beta}$ is trivial, and fix a trivialization with its trivial connection $\nabla_\beta^{\mathrm{triv}}$ (on $E|_{U_\beta}$).

Let (ψ_β) be a subordinate partition of unity. Then

$$\nabla_X \xi = \sum_{\beta \in B} \psi(\nabla_{\beta, X|_{U_\beta}} \xi|_{U_\beta})$$

is a connection:

$$\begin{aligned} \nabla_X(f\xi) &= f \sum_{\beta \in B} \psi_\beta(\nabla_{\beta, X|_{U_\beta}} \xi|_{U_\beta}) + (\sum_{\beta \in B} \psi_B)(X, f)\xi \\ &= f \cdot \nabla_X \xi + (X.f)\xi. \end{aligned}$$

□

So the space of connections is an affine space over the space of maps $TM \to \mathrm{End}(E)$.

Definition 31.10. Let ∇ be a connection on E. The curvature

$$\begin{aligned} C^\infty(TM) \times C^\infty(TM) \times C^\infty(E) &\to C^\infty(E) \\ (X, Y, \xi) &\mapsto F_\nabla(X, Y)\xi \end{aligned}$$

is given by $F_\nabla(X, Y)\xi = \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi$.

Proposition 31.11. F_∇ is given by a vector bundle homomorphism

$$\Lambda^2(TM) \to \mathrm{End}(E)$$

(there is a part where it's symmetric, not antisymmetric, but on that part the curvature dies).

Sketch of proof. Show $F_\nabla(X, Y)\xi$ is a pointwise operation. □

Example 31.12. If $E = TM$ and ∇ is the Levi-Civita connection of g, then $F_\nabla = R$.

Example 31.13. If E is trivial and $\nabla = \nabla^{\mathrm{triv}}$, then $F_\nabla = 0$.

Proof. If $\dim M = 1$ then $F_\nabla = 0$ for any ∇. □

These ideas are important in topology (cf. Chern), among other places.

Connections on the trivial bundle. Let $E = \mathbb{R}^r \times M$ be a trivial bundle, and $\nabla = \nabla^{\mathrm{triv}} + A$ (we proved that any connection has this form) where $A : TM \to \mathrm{End}(E) = \mathrm{Mat}_{r \times r}(\mathbb{R}) \times M$ (so if $X \in TM_x$ then $A(X)$ is an $r \times r$ matrix).

Let Φ be an automorphism of the trivial bundle $\Phi : M \to GL_r(\mathbb{R})$. Then we can apply this automorphism to the connection and obtain another connection

$$\widetilde{\nabla}_X \xi = \Phi \nabla_X (\Phi^{-1} \cdot \xi).$$

Explicitly,

$$\begin{aligned}
\widetilde{\nabla}_X \xi &= \Phi(\nabla_X^{\text{triv}}(\Phi^{-1}\xi)) + \Phi A(X)\Phi^{-1}\xi \\
&= \Phi(X.\Phi^{-1})\xi + \nabla_X^{\text{triv}}\xi + \Phi A(X)\Phi^{-1}\xi \\
&= \nabla_X^{\text{triv}}\xi + (-(X.\Phi)\Phi^{-1} + \Phi A(X)\Phi^{-1})\xi && \text{see note below} \\
&= \nabla_X^{\text{triv}}\xi + \widetilde{A}(X)\xi && \text{this defines } \widetilde{A}
\end{aligned}$$

(in the penultimate step, differentiate $\Phi(x)\Phi^{-1}(x) = \mathbb{1}$ and get the fact that $\Phi(X.\Phi^{-1})\xi + (X.\Phi)\Phi^{-1}\xi = 0$). That is, we have defined

$$\widetilde{A}(X) := \Phi A(X)\Phi^{-1} - (X.\Phi)\Phi^{-1},$$

which is called the *gauge transformation formula.* (Without the second term, connections would be the same as vector bundle homomorphisms.) Physicists consider equivalence classes of A's, where $A \sim \widetilde{A}$ in the formula.

Example 31.14. Suppose $r = 1$. Then $\nabla = \nabla^{\text{triv}} + A$ where A is a vector bundle homomorphism $TM \to \mathbb{R}$, i.e. a section of TM^*. This is also called a 1-form, and the gauge transformation formula for $\Phi : M \to \mathbb{R}^*$ is

$$\widetilde{A} = A - d(\log|\Phi|)$$

(Now Φ is a 1×1 matrix, and hence the elements in the first term commute; the log comes in because $d \log|t| = \frac{1}{t}$.)

Lecture 32: December 2

Let $E = \mathbb{R}^r \times M$ be a trivial bundle. Having a section ξ of E is the same as having an r-tuple $\xi = (\xi_1, \cdots, \xi_r)$ of functions. The trivial connection is $(\nabla_X^{\text{triv}}\xi)_i = d\xi_i(X) = X.\xi_i$. A general connection looks like $\nabla = \nabla^{\text{triv}} + A$, where A is a vector bundle homomorphism $TM \to \text{End}(E) = \text{Mat}_r(\mathbb{R}) \times M$. Equivalently, $A = (A_{ij})_{1 \le i,j \le r}$ where each A_{ij} is a section of TM^*, and

$$(\nabla_X \xi)_i = X.\xi_i + \sum_j A_{ij}(X)\xi_j.$$

Let φ be an automorphism of the bundle E, i.e. $\varphi : M \to GL_r(\mathbb{R})$. If ∇ is a connection, so is

$$\widetilde{\nabla}_X \xi = \varphi \nabla_X(\varphi^{-1}\xi).$$

Explicitly, $\widetilde{\nabla} = \nabla^{\text{triv}} + \widetilde{A}$, where

$$\widetilde{A}(X) = \varphi A(X)\varphi^{-1} - (X.\varphi)\varphi^{-1}.$$

Given $\nabla = \nabla^{\text{triv}} + A$, what is the curvature? Define

$$\begin{aligned}
F_\nabla(X,Y)\xi &= \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi \\
&= \nabla_X(Y.\xi + A(Y)\xi) - \nabla_Y(X.\xi + A(X)\xi) - [X,Y]].\xi - A([X,Y])\xi
\end{aligned}$$

$$
\begin{aligned}
&= X.Y.\xi + X.(A(Y)\xi) + A(X)(Y.\xi) + A(X)A(Y)\xi - Y.X.\xi - Y.(A(X)\xi) \\
&- A(Y)(X.\xi) - A(Y)A(X)\xi - [X,Y].\xi - A([X,Y])\xi
\end{aligned}
$$

Using the Jacobi formula for $[X,Y]$, and the Leibniz rule which shows that $X.A(Y)\xi = -A(Y)(X.\xi)$ etc,

$$
= A(X)A(Y)\xi - A(Y)A(X)\xi + X.(A(Y))\xi - Y.(A(X))\xi - A([X,Y])\xi
$$

So

$$
F_\nabla(X,Y) = A(X)A(Y) - A(Y)A(X) + X.A(Y) - Y.A(X) - A([X,Y]).
$$

The first two terms are the Lie bracket of functions, and the last three terms are the exterior derivative of a 1-form. Note that $X.A(Y)$ contains derivatives of Y and $Y.A(X)$ contains derivatives of X, but these cancel out with $A([X,Y])$. Even more explicitly,

$$
F_\nabla(X,Y) = \sum_k A_{ik}(X)A_{kj}(Y) - \sum_k A_{ik}(Y)A_{kj}(X) + X.A_{ij}(Y) - Y.A_{ij}(X) - A_{ij}([X,Y]).
$$

Remark 32.1. If $\widetilde{\nabla} = \varphi\nabla\varphi^{-1}$, then

$$
F_{\widetilde{\nabla}}(X,Y)\xi = \varphi F_\nabla(X,Y)\varphi^{-1}\xi.
$$

Let $M = [a,b]$, and consider a vector bundle $E \to M$ with connection ∇.

Lemma 32.2. For any $\eta \in E_a$, there is a unique section ξ of E with $\nabla_{\frac{\partial}{\partial t}}\xi = 0$ and $\xi(a) = \eta$.

Proof. In a local trivialization, the equation $\nabla_{\frac{\partial}{\partial t}} \times$ has the form

$$
\frac{\partial \xi_i}{\partial t} + \sum A_{ij}\left(\tfrac{\partial}{\partial t}\right)\xi_j(t) = 0
$$

which is a first order ODE. The rest of the argument is standard: ask what is the maximum interval $[a,?]$ where you have a solution; local considerations will show that $? = b$. □

Corollary 32.3. Any vector bundle over $[a,b]$ is trivial.

Proof. Choose a basis $\eta_1, \cdots, \eta_r \in E_a$, and continue them uniquely to the parallel sections given in the lemma. If you think about the uniqueness statement, you'll see that these generate a basis in any other fiber. □

Consider a vector bundle $E \to M$ with connection ∇, where $M = [a,b] \times N$.

Lemma 32.4. For any section η of $E|_{\{a\}\times N}$, there is a unique section ξ of E with $\xi|_{\{a\}\times N} = \eta$ and $\nabla_{\frac{\partial}{\partial t}}\xi = 0$ (where the t variable is the position in $[a,b]$).

(Now you have an ODE that depends on another parameter.) This shows that if the vector bundle is trivial over $E|_{\{a\}\times N}$ then it is trivial overall. You need a bit more thinking to get:

Corollary 32.5. Any vector bundle $E \to [a,b] \times N$ is isomorphic to $p^*(E|_{\{a\}\times N})$ where $p : [a,b] \times N \to N$ is the projection.

This implies Lemma 31.1.

If $E \to N$ is a vector bundle with connection ∇, and $f : M \to N$ is a smooth map, there is a unique induced connection $\widetilde{\nabla} = f^*\nabla$ on f^*E characterized by

$$(f^*\nabla)_X(f^*\xi)_x = f^*(\nabla_{Tf(X)}\xi)_{f(x)}$$

for a section $\xi \in C^\infty(E)$ (so $f^*\xi \in C^\infty(f^*E)$), and $X \in TM_x$ (so $Tf(X) \in TM_{f(x)}$). Do this in local trivializations. In particular, given a path $c : [a,b] \to N$, we can use sections of c^*E which satisfy $(c^*\nabla)_{\frac{\partial}{\partial t}}\xi = 0$ to define parallel transport maps $\tau_{\nabla,c} : E_{c(a)} \to E_{c(b)}$. (We're not going to talk about details because it's just the generalization of $\frac{\nabla}{\partial t}$.)

Proposition 32.6. If ∇ is flat (i.e. $F_\nabla = 0$), then $\tau_c : E_{c(a)} \to E_{c(b)}$ is unchanged under homotopies of c which leave the endpoints fixed.

The curvature describes what happens to the parallel transport when you try to wiggle the path.

Corollary 32.7. If M is simply-connected, any vector bundle admitting a flat connection is trivial.

(Take a basis at one fiber, and transport that basis around. By simply-connected-ness, these are all homotopic, so there is only one way to move the basis around; this gives a trivialization.)

Suppose we have a general connected M with a flat connection. The homotopy class of a loop determines an automorphism of the fiber: take a basis of the fiber, and transport it around that loop; it will come back to (probably) a different basis. So we have a correspondence

$$\left\{ \begin{array}{c} \text{vector bundles } E \to M \\ \text{with flat connection } \nabla \end{array} \right\} \Big/ \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{representations} \\ \pi_1(M) \to GL_r(\mathbb{R}) \end{array} \right\} \Big/ \cong .$$

To go in the backwards direction, look at the universal cover and introduce a flat connection. We won't talk about this.

In algebraic geometry, there is a good notion of "vector bundles with flat connections", but fundamental groups are hard.

REMARK 32.8. Let (E, ∇) be a vector bundle with a flat connection. Then a section ξ with $\nabla_X \xi = 0$ for all X forms a locally trivial sheaf of rank r vector spaces ("a local system").

Question: given a vector bundle, when does there exist a connection inducing flat curvature?

No class on Wednesday.

LECTURE 33: DECEMBER 6

For a general bundle, what obstructions are there to the existence of a flat connection?

Reminder 33.1 (Differential forms)**.** Differential forms

$$\Omega^k(M) = C^\infty(\Lambda^k TM^*)$$

are sections of the k^{th} exterior power of the cotangent bundle. So $\Omega^0(M) = C^\infty(M, \mathbb{R})$, $\Omega^1(M) = C^\infty(T^*M)$. Equivalently, a k-form is a map

$$\begin{aligned}
\omega : C^\infty(TM) \otimes \cdots \otimes C^\infty(TM) &\to C^\infty(M, \mathbb{R}) \\
\omega(X_1, \cdots, X_i, \cdots, X_j, \cdots) &= -\omega(\cdots, X_j, \cdots, X_i, \cdots) \qquad \text{(antisymmetry)} \\
\omega(X_1, \cdots, fX_i, \cdots, X_k) &= f \cdot \omega(X_1, \cdots, X_k)
\end{aligned}$$

for $f \in C^\infty(M, \mathbb{R})$. Differential forms are a graded symmetric algebra. There is a *wedge product*

$$\wedge : \Omega^k(M) \otimes \Omega^\ell(M) \to \Omega^{k+\ell}(M).$$

So far, all of these things hold for *any* bundle, not just TM. But because the bundle is TM, this comes with an *exterior differential* (de Rham differential)

$$d : \Omega^i(M) \to \Omega^{i+1}(M), \;\; d^2 = 0.$$

$d : \Omega^0(M) \to \Omega^1(M)$ is just the ordinary derivative $df(X) = X.f$. For functions, you have a second derivative that is a symmetric bilinear form, but you can't define this in a coordinate-free way without a metric. But, there is an *antisymmetric* form that can be written in a coordinate-free way, and this is

$$\begin{aligned}
&\Omega^1(M) \xrightarrow{d} \Omega^2(M) \\
&(d\alpha)(X_1, X_2) = X_1 \cdot \alpha(X_2) - X_2 \cdot \alpha(X_1) - \alpha([X_1, X_2]). \qquad (33.1)
\end{aligned}$$

More generally,

$$\begin{aligned}
(d\alpha)(X_1, \cdots, X_{k+1}) = &\sum_i (-1)^{i+1} X_i \cdot \alpha(\cdots, \widehat{X}_i, \cdots) \\
&+ \sum_{i<j} (-1)^{i+j} ([X_i, X_j], \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots)
\end{aligned}$$

Alternatively, use

$$d(\omega \wedge \eta) = d\omega\eta + (-1)^k \omega \wedge d\eta$$

for $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ to describe d in general, based on its behaviour for functions. For instance, in local coordinates

$$d(f(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Define de Rham cohomology:

$$H^k_{dR}(M) = \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) \Big/ \operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M)).$$

If M is compact and oriented (possibly with boundary), there is a canonical integration map

$$\int_M : \Omega^n(M) \to \mathbb{R}$$

defined by using partitions of unity and integrating in local charts. This satisfies the Stokes formula

$$\int_M d\theta + (-1)^n \int_{\partial M} \theta = 0$$

for $\theta \in \Omega^{n-1}(M)$, $n = \dim M$. (The sign $(-1)^n$ depends on the choice of a sign rule for the integration map; here we're using the Koszul sign rule.) If M is closed (i.e. the boundary is empty), this induces a map $H^n_{dR}(M) \to \mathbb{R}$.

Lemma 33.2. Let $E \to M$ be a linear bundle (a vector bundle of rank 1) with connection ∇. F_∇ is a vector bundle map $\Lambda^2(TM) \to \operatorname{End}(E) \cong \mathbb{R} \times M$ hence a 2-form. This 2-form is closed (i.e. $dF_\nabla = 0$) and its cohomology class is independent of ∇.

Proof. Prove the second part fist. Take two connections $\widetilde{\nabla} = \nabla + \alpha$, for $\alpha \in \Omega^1(M)$. This means

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \alpha(X)\xi$$

(here we're thinking of α as multiplication by some scalar).

$$\begin{aligned}
F_{\widetilde{\nabla}}(X,Y)\xi &= \widetilde{\nabla}_X \widetilde{\nabla}_Y \xi - \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi - \widetilde{\nabla}_{[X,Y]}\xi \\
&= \widetilde{\nabla}_X (\nabla_Y \xi + \alpha(Y)\xi) - \widetilde{\nabla}_Y(\nabla_X \xi + \alpha(X)\xi) - \nabla_{[X,Y]}\xi - \alpha([X,Y])\xi \\
&= F_\nabla(X,Y)\xi + \alpha(X)\nabla_Y\xi + (X.\alpha(Y))\xi + \alpha(Y)\nabla_X\xi + \alpha(Y)\alpha(X)\xi \\
&\quad - \alpha(Y)\nabla_X\xi - (Y.\alpha(X))\xi - \alpha(X)\nabla_Y\xi - \alpha(X)\alpha(Y)\xi - \alpha([X,Y])\xi \\
&= F_\nabla(X,Y)\xi + (X.\alpha(Y))\xi - (Y.\alpha(X))\xi \\
&= (F_\nabla(X,Y) + d\alpha(X,Y))\xi
\end{aligned}$$

In other words, $F_{\widetilde{\nabla}} = F_\nabla + d\alpha$. This would prove the second part if we knew the form was closed. But it also proves the first part: locally, $\nabla = \nabla^{\text{triv}} + \alpha$, so $F_\nabla = d\alpha$. Hence it's closed (since $d^2 = 0$), and the rest follows as well. (If it's closed for one connection, it's closed for every connection, and we show it's closed for the trivial connection.) □

Lemma 33.3. For any line bundle $E \to M$ and connection ∇, $[F_\nabla] \in H^2_{dR}(M)$ is zero.

In fact, every line bundle admits a flat connection. Take $E \to M$ with a Euclidean metric $\langle -, - \rangle_x : E_x \otimes E_x \to \mathbb{R}$. A connection ∇ is compatible with that metric if

$$X. \langle \xi, \eta \rangle = \langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle .$$

Theorem 33.4. Let $(E, \langle -, - \rangle)$ be a vector bundle with a Euclidean metric. Then

(1) Connections compatible with that metric exist.

(2) Suppose ∇ is compatible with $\langle -, - \rangle$. Then $\widetilde{\nabla} = \nabla + A$ is compatible with $\langle -, - \rangle$ iff $A : TM \to \mathrm{End}(E)$ lands in skew-symmetric endomorphisms.

(3) If ∇ is compatible with a metric, parallel transport preserves that metric. (If you have a flat connection, then you get a representation into $O(n)$.)

(4) If ∇ is compatible with a metric, then $F_\nabla : \Lambda^2 TM \to \mathrm{End}(E)$ lands in skew-symmetric endomorphisms.

We won't prove this; we've had enough theory of connections.

Corollary 33.5. If $E \to M$ is a line bundle, and ∇ is compatible with a metric, then ∇ is flat.

We failed to extract cohomological information.

Let $E \to M$ be an oriented plane bundle (i.e. rank 2). The curvature is some endomorphism of a 2-dimensional vector space. If you put a metric (and require compatibility with that metric), then you get a skew-symmetric endomorphism, and those look like $\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$.

Lemma 33.6. There is a unique vector bundle homomorphism $I : E \to E$ which preserves $\langle -, - \rangle$, satisfies $I^2 = -\mathbb{1}$, and such that for $X \in E_x$ nonzero (X, IX) is an oriented basis of E_x.

Note: I is automatically skew-symmetric (it generates the 1-dimensional space of skew-symmetric automorphisms).

Sketch of proof. If X_1, X_2 is an orthonormal basis of sections which is positively oriented, then $IX_1 = X_2$, $IX_2 = X_1$. Such bases exist locally, and that ensures existence as well as uniqueness. $\square$

Let ∇ be a connection compatible with $\langle -, - \rangle$. Then (by 1-dimensionality of the space of skew-symmetric automorphisms) there exists a unique $\lambda \in \Omega^2(M)$ such that

$$F_\nabla(X,Y)\xi = \lambda(X,Y)I\xi.$$

Lemma 33.7. $\lambda \in \Omega^2(M)$ is closed ($d\lambda = 0$) and $[\lambda] \in H^2_{dR}(M)$ is independent of ∇ and $\langle -, - \rangle$ (an invariant of the oriented plane bundle E).

This is a genuine nontrivial cohomology class, and is called the Euler class (up to some normalization to be described later).

Lecture 34: December 9

18.966: Emmy Murphy on Symplectic/ Contact geometry

Let $E \to M$ be an oriented plane bundle with a Euclidean metric. Then there is a unique way of making the fibers E_x into complex vector spaces by introducing $I_x : E_x \to E_x$ that satisfies $I_x^2 = -\mathbb{1}$, where $(\xi, I_x\xi)$ is positively oriented for all $\xi \neq 0$, and I_x preserves the inner product. (Note: I_x is then also skew adjoint.)

Recall that ∇ is compatible with $\langle -, - \rangle$ if

$$X.\langle \xi, \eta \rangle = \langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle .$$

I claim one can choose a local trivialization of E (i.e. $E|_U \cong \mathbb{R}^2 \times U$ for a small neighborhood U of x_0) which is compatible with the orientation, and in which $\langle -, - \rangle$ is the standard inner product. Choose everywhere linearly independent sections s_1, s_2 of $E|_U$ such that $(s_1(x), s_2(x))$ is positively oriented; apply Gram-Schmidt to make them orthonormal. This yields the desired trivialization.

In such a local trivialization,

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and if ∇ is compatible with the Euclidean metric, then

$$\nabla_X = \nabla_X^{\text{triv}} + \alpha(X)I$$

for some $\alpha \in \Omega^1(U)$. (By compatibility with the metric, the second term is skew-adjoint.) The curvature is

$$\begin{aligned} F_\nabla(X,Y)\xi &= X.A(Y)\xi - Y.A(X)\xi - A([X,Y])\xi + A(X)A(Y)\xi - A(Y)A(X)\xi \\ &= X.\alpha(Y)I\xi - Y.\alpha(X)I\xi - \alpha([X,Y])I\xi = d\alpha(X,Y)I\xi. \end{aligned}$$

Lemma 34.1. If $E \to M$ is an oriented plane bundle with a Euclidean metric, and ∇ a connection compatible with that metric, then

$$F_\nabla(X,Y)\xi = \beta(X,Y)I\xi \tag{34.1}$$

where $\beta \in \Omega^2(M)$ is closed, $d\beta = 0$. The class $[\beta] \in H^2_{dR}(M)$ is independent of ∇.

Proof. β is just $d\alpha$ so of course it's closed. The local computation shows that (34.1) holds with $d\beta = 0$. It also shows that if we replace ∇ by $\widetilde{\nabla} = \nabla + \alpha I$ for $\alpha \in \Omega^1(M)$ then $\beta^2 = \beta + d\alpha$. $\square$

Remark 34.2. In fact, $E \to M$ admits a flat connection compatible with $\langle -, - \rangle$ if and only if $[\beta] \in H^2_{dR}(M)$ is zero.

(If it's zero, β can be written as $d\alpha$.)

Addendum 34.3. $[\beta] \in H^2_{dR}(M)$ is an invariant of the oriented vector bundle E (independent of the metric).

Proof. This follows from the fact that, given two metrics $\langle -, - \rangle_0$ and $\langle -, - \rangle_1$, one can find a metric on the pullback bundle $\pi^* E \to [0,1] \times M$ (where $\pi : [0,1] \times M \to M$ is the projection) which restricts to $\langle -, - \rangle_k$ on $\{k\} \times M$ for $k = 0, 1$. If ∇_k is compatible with $\langle -, - \rangle_k$, one can find a connection ∇ on $\pi^* E$ compatible with $\langle -, - \rangle$ such that $\nabla|_{\{k\} \times M} = \nabla_k$ (partitions of unity). Therefore, if $F_{\nabla_k} = \beta_k \cdot I_k$ then $\beta_k = \beta|_{\{k\} \times M}$ for some closed $\beta \in \Omega^2([0,1] \times M)$. This implies $[\beta_0] = [\beta_1] \in H^2_{dR}(M)$. $\square$

Definition 34.4. Let $E \to M$ be an oriented plane bundle. Choose a Euclidean meric and compatible connection, write $F_\nabla = \beta \cdot I$ (for a 2-form β). Then

$$e(E) = -\frac{1}{2\pi}[\beta] \in H^2_{dR}(M)$$

is called the *Euler class* of E.

This is the obstruction to admitting a flat connection.

So we've addressed, for $n = 1, 2$, the question of when an n-bundle admits a metric with a flat connection. What about the general case?

Let $E \to M$ be a rank r vector bundle, ∇ a connection. In a local trivialization,

$$\nabla_X \xi = \nabla_X^{\text{triv}} + A(X)\xi$$

where $A = (A_{ij})$ is a matrix of 1-forms. Similarly, the curvature $F = (F_{ij})$ is a matrix of 2-forms

$$F = dA + A \wedge A$$

where dA means you apply d to each entry and $A \wedge A$ means you multiply the matrices, and on the level of entries, "multiplying" individual 1-forms means wedging them together. The second term takes care of the $A(X)A(Y)$ and $A(Y)A(X)$ terms: $(A \wedge A)(X, Y) = A(X)A(Y) - A(Y)A(X)$.

Lemma 34.5 (Bianchi identity)**.**

$$dF + A \wedge F - F \wedge A = 0$$

Proof. Using the fact that $d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$ (and getting the sign from the fact that these are 1-forms), we have

$$\begin{aligned} dF &= d(dA) + dA \wedge A - A \wedge dA \\ &= dA \wedge A - A \wedge dA \\ A \wedge F - F \wedge A &= A \wedge dA = dA \wedge A + A \wedge A \wedge A - A \wedge A \wedge A. \end{aligned}$$

Add these up and you get zero. □

For any k, consider $\mathrm{Tr}(F^k) = \mathrm{Tr}(F \wedge \cdots \wedge F) \in \Omega^{2k}$ which is a $2k$-form.

Lemma 34.6. $\mathrm{Tr}(F^k)$ is closed.

Proof.

$$\begin{aligned} d\,\mathrm{Tr}(F^k) &= \mathrm{Tr}(dF^k) = \mathrm{Tr}(d(F \wedge \cdots \wedge F)) \\ &= \mathrm{Tr}(dF \wedge F \wedge \cdots \wedge F) + \mathrm{Tr}(F \wedge dF \wedge \cdots) + \cdots \\ &= -\mathrm{Tr}(A \wedge F \wedge \cdots \wedge F) + \mathrm{Tr}(F \wedge A \wedge F \wedge \cdots \wedge F) - \mathrm{Tr}(F \wedge A \wedge \cdots \wedge F) \\ &\quad + \mathrm{Tr}(F \wedge F \wedge A \wedge \cdots) + \cdots \end{aligned}$$

But $\mathrm{Tr}(BA) = \mathrm{Tr}(AB)$, and the same is true if the entries are differential forms. You have to worry about signs, but it's OK because F is a 2-form, so whatever you commute it with doesn't introduce new signs. □

Let $P \in \mathbb{R}[X_{11}, \cdots, X_{1r}, \cdots, X_{r1}, \cdots, X_{rr}]$ be a polynomial in r^2 variables, thought of as the coefficients of a matrix X.

Lecture 35: December 11

Let $E \to M$ be a rank r vector bundle, and ∇ a connection. Choose a local trivialization $E|_U \cong \mathbb{R}^r \times U$. In that trivialization $\nabla = \nabla^{\mathrm{triv}} + A$, where A is a matrix of 1-forms. Then $F = F_\nabla = dA + A \wedge A$ satisfies the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0.$$

Last time, we saw that for any $k \geq 1$, $\mathrm{Tr}(\underbrace{F \wedge \cdots \wedge F}_{k}) \in \Omega^{2k}(U)$ is a closed form. Let $P(X) \in \mathbb{R}[(X_{ij})_{1 \leq i,j \leq r}]$ be a homogeneous degree d polynomial thought of as the entries of a matrix. We say that P is *invariant* if $P(XY) = P(YX)$, or equivalently that $P(YXY^{-1}) = P(X)$ for all invertible Y. Note that invertible matrices are dense in the set of all matrices.

Examples: $P(X) = \mathrm{tr}(X)$, $P(X) = \mathrm{tr}(X^k)$, $P(X) = \det(X)$.

We can write the determinant as

$$\det(t \cdot \mathbb{1} - X) = t^r - t^{r-1}S_1(X) + t^{r-2}S_2(X) + \cdots .$$

Note S_1 is the trace, and in general S_d is the d-symmetric product of the eigenvalues (but in general we're trying to think of this as a function of the entries of the matrix, not the eigenvalues).

Lemma 35.1. Let $P(X)$ be an invariant polynomial. Then

$$DP(X)[Y, X] = 0$$

for all X, Y.

($[Y, X]$ is just the commutator.)

Proof. Take $X_s = e^{sY}Xe^{-sY}$. Then $X_0 = X$, $\frac{\partial}{\partial s}X_s|_{s=0} = [Y, X]$. $P(X_s) = P(X)$ so $\frac{\partial}{\partial s}P(X_s)|_{s=0} = DP(X)[Y, X]$. □

Or, an easier argument: from before we saw that $P(YXY^{-1}) = P(X)$, so P is constant along conjugacy classes. $DP(X)[Y, X]$ is its derivative along a vector that is tangent to the conjugacy class.

Proposition 35.2. Suppose $P(X)$ is an invariant polynomial of degree d. Then $P(F) \in \Omega^{2d}(U)$ is closed.

Proof. $d(P(F)) = DP(F)dF = -DP(F)[A, F]$ by the Bianchi identity, and this is 0 by the lemma. The first equality is kind of sketchy. But F is an even form (it's a matrix of 2-forms), and there is no sign when trying to commute d across $F_{12} \wedge F_{23} \wedge \cdots$. □

Proposition 35.3. $P(F)$ is independent of the local trivialization (hence extends smoothly to all of M).

Proof. Change the trivialization by $\Phi : U \to GL_r(\mathbb{R})$ sending $F \mapsto \Phi F \Phi^{-1}$. By conjugacy-invariance, $P(F)$ remains the same. □

Proposition 35.4. $[P(F)] \in H^{2d}_{dR}(M)$ is independent of ∇ (invariant of E).

Proof. Consider $M \times [0, 1]$, where you have used one ∇ on one side and the other one on the other side (we did this before). Then this induces a closed 2-form on $M \times [0, 1]$. □

Definition 35.5.

$$p_{\frac{d}{2}}(E) = \frac{1}{(2\pi)^d}[S_d(F)] \in H^{2d}_{dR}(M)$$

is the $\left(\frac{d}{2}\right)^{\text{th}}$ Pontryagin class of E.

REMARK 35.6.

$$p_{\frac{d}{2}}(E) = 0 \text{ for } \frac{d}{2} \notin \mathbb{Z}.$$

To see that, take ∇ compatible with a Euclidean metric. Then F is skew-symmetric so $S_d(F) = 0$ for all odd d. (The trace of a skew-symmetric matrix is 0; more specifically, the eigenvalues are arranged into λ and $-\lambda$ pairs. This also applies to powers.)

The theory I've been explaining is called Chern-Weil theory. The best place to read about it is the appendix of Milnor/Stasheff's *Characteristic Classes*.

Take a homogeneous degree d polynomial $P(X)$ in $X = (X_{ij})_{1\le i,j\le r}$ where $X_{ii} = 0$ and $X_{ij} = -X_{ji}$. Say P is invariant if $P(YXY^{-1}) = P(X)$ where X is skew-symmetric and $Y \in SO(r)$ (if you just ask for $Y \in O(r)$ you don't get anything different from what you already had). Question: are there more invariant polynomials under this restriction?

Lemma 35.7. For r even, there is a unique invariant polynomial $Pf(X)$ (Pfaffian) of degree $\frac{r}{2}$ such that

$$Pf(X)^2 = \det(X), \qquad Pf\begin{pmatrix} 0 & 1 & & & & 0 \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ 0 & & & & & \end{pmatrix} = 1$$

Proof. Suppose $X = (X_{ij})$ is skew-symmetric. Write $\omega = \sum X_{ij} e_i \wedge de_j \in \Lambda^2(\mathbb{R}^r)$. Then

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{\frac{r}{2}} = \left(\tfrac{r}{2}\right)! \cdot (-1)^{\frac{r}{2}} Pf(X) \cdot de_1 \wedge \cdots \wedge de_r \in \Lambda^r(\mathbb{R}^r)$$

(this is because $\Lambda^r(\mathbb{R}^r)$ is 1-dimensional, and this is how we are defining the Pfaffian). More work to do. $\square$

Given an oriented bundle $E \to M$ of even rank r, and a connection ∇ compatible with a Euclidean metric,

$$e(E) = [\tfrac{1}{(2\pi)^{\frac{r}{2}}} Pf(F)] \in H^{2r}_{dR}(M)$$

is called the Euler class of E. You can show it's closed. Also, it is independent of the local trivialization. When you change trivializations, you have $\Phi F \Phi^{-1}$, and you have to show $Pf(\Phi F \Phi^{-1}) = Pf(F)$. (You need to choose charts that are compatible with the orientation, or else the signs of the Pfaffians won't agree.)

Theorem 35.8 (Chern-Gauss-Bonnet)**.** If M is compact, closed, oriented of dimension r, then

$$\int_M \tfrac{1}{(2\pi)^{\frac{r}{2}}} Pf(E) = \text{ integral Euler number of } E\ .$$

This is not that hard to prove. The key point is is to realize that this vanishes when there is a nowhere-zero section. ω must be nondegenerate; if you had a nowhere zero section you could choose your connection carefully so that the matrix had less than full rank.