A course in
Algebraic Geometry

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Contents

1. September 1

1.1. Algebraic sets. We start with an algebraically closed field k. First define the affine space

DEFINITION 1.1. $\mathbb{A}_k^n = \{(a_1 \cdots a_n) : a_i \in k\}$

DEFINITION 1.2. An *algebraic set* $X \subset \mathbb{A}^n$ is the set of common zeroes of a collection of polynomials $f_1 \cdots f_m \in k[x_1 \cdots x_n]$. We denote this set $V(f_1 \cdots f_n)$. That is, $V(f_1 \cdots f_n) = \{(a_1 \cdots a_n) : f_1(a_1 \cdots a_n) = \cdots = f_m(a_1 \cdots a_n) = 0\}$.

But there might be some other polynomials $g_1 \cdots g_n$ such that $V(f_1 \cdots f_n) = V(g_1 \cdots g_n)$. In general, this set is determined by the *ideal* generated by the polynomials f_i, or g_i. So if $I = (f_1 \cdots f_n)$, we can write $V(f_1 \cdots f_n) = V(I) = \{(a_1..a_n) \in \mathbb{A}^n : f(a_1 \cdots a_n) = 0 \ \ \forall f \in I\}$. Since the polynomial ring is Noetherian, every ideal is finitely generated, and so every $V(I)$ can be written as $V(f_1 \cdots f_n)$ for some finite collection of f_i.

The following relations are really useful, and are not hard to check.

LEMMA 1.3. *The sets $V(I)$ act like closed sets in a topological space, in the following sense:*

(1) *If $I_1 \subset I_2$, then $V(I_1) \supset V(I_2)$.*
(2) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$.
(3) $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$
(4) $V(\{0\}) = \mathbb{A}^n$ *and* $V(k[x_1 \cdots x_n]) = \emptyset$

Therefore, we can define a topology on this affine space.

DEFINITION 1.4. The Zariski topology on $\mathbb{A}^n$ is given as follows:

The closed subsets are algebraic sets (i.e. sets of the form $V(I)$ for some ideal I defined as above).

EXAMPLE 1.5. Consider $\mathbb{A}_k^1$. I claim that the algebraic sets are finite sets, as well as all of $\mathbb{A}^1$ and the empty set. Algebraic sets are zeroes of polynomials in some ideal. In this case we are considering ideals in $k[x]$, which is a PID, so those ideals all look like (f). Our field is algebraically closed, so write $f = a(x - \alpha_1) \cdots (x - \alpha_n)$. So all the closed sets look like $V(f) = \{\alpha_1 \cdots \alpha_n\}$. (And, of course, for every finite set we can find a polynomial where those are the only roots, so every finite set is closed.)

EXAMPLE 1.6. Now consider $\mathbb{A}^2$. Obviously, we have $\mathbb{A}^2$ and $\emptyset$ are algebraic sets. Also, we have things that are plane curves: $V(f)$. However, not every ideal is generated by one polynomial: we also have the ideals $V(f_1, f_2)$. Suppose f_1 is irreducible, and f_2 is not divisible by f_1. Then $V(f_1, f_2)$ is a finite set. Why? (Homework.) But every other set is some union of sets of the form $V(f')$ and $V(f, g)$.

Of course, we can write $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$. But this does not work topologically. Why? (Homework, again.)

DEFINITION 1.7. $\sqrt{I} = \{f \in k[x_1 \cdots x_n] : f^r \in I\}$ is the radical of the ideal I. A *radical ideal* is a set whose radical is itself.

EXAMPLE 1.8. Consider $I = (f)$ where $f = (x-1)^3$. Then $x - 1 \notin I$ but $x - 1 \in I$. However, if $I = (f)$ where f does not have any repeated roots, then $\sqrt{I} = I$.

THEOREM 1.9 (Hilbert's Nullstellensatz). *Let $X = V(I)$. Then*

$$\{f \in k[x_1 \cdots x_n] : f(p) = 0 \quad \forall p \in X\} = \sqrt{I}.$$

We already had a mapping $I \mapsto V(I)$ from ideals to sets. The theorem motivates defining a map from sets to ideals:

$$X \mapsto I(X) := \{f : f(p) = 0 \quad \forall p \in X\}$$

COROLLARY 1.10. *There is an order-reversing bijection between radical ideals in $k[x_1 \cdots x_n]$ and algebraic sets in $\mathbb{A}^n$.*

Let $f = xy$ and consider $V(f) \subset \mathbb{A}^2$; this is the union of the two coordinate axes, and so we can write $V(xy) = V(x) \cup V(y)$. This suggests a question: what algebraic sets *cannot* be written as the (nontrivial) union of two algebraic sets? More precisely,

DEFINITION 1.11. Let $X \neq \emptyset$ be a topological space. We say X is *irreducible* if X cannot be written as the union of two proper closed subsets.

Zariski-land is really different from what we're used to in $\mathbb{R}^n$, where the only irreducible closed sets are single points. Here, something as big as $\mathbb{A}^1$ is irreducible: we just said that closed subsets consist of finitely many points, and we were assuming that k is infinite.

PROPOSITION 1.12. *There is a 1-1 correspondence between irreducible algebraic sets (with the induced topology) and the prime ideals in $k[x_1 \cdots x_n]$. In addition, every algebraic set can be written as the union of its* irreducible components*, by which we mean irreducible closed subsets. (For example, $V(xy)$ is the union of two irreducible components.)*

PROOF. We have already shown a bijection between radical ideals and irreducible algebraic sets. We want to show that $V(I)$ is irreducible iff I is prime.

($\Longrightarrow$) Assume that I is not the whole thing, or empty. We want to show that if $fg \in I$, then either f or g is in I. Consider the ideal generated by fg; this is in I. By the first lemma, $V(fg) \supset V(I)$. Now, $V(I) \subset V(f) \cup V(g)$. $V(I) = (V(f) \cap V(I)) \cup (V(g) \cap V(I))$. By irreducibility, either $V(I) = V(f) \cap V(I)$ or $V(I) = V(g) \cap V(I)$. So $V(I) \subset V(f)$ or $V(I) \subset V(g)$, which implies $f \in I$ or $g \in I$.

($\Longleftarrow$) By contradiction. Write $V(I) = V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$. The intersection of radical ideals is radical. So $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$. ⊟

Any radical ideal can be written as the intersection of finitely many prime ideals (by commutative algebra): $I = B_1 \cap \cdots \cap B_r$ with B_i minimal prime ideals containing I. Therefore, $V(I) = V(B_1) \cup \cdots \cup V(B_v)$ for B_i prime. So each piece is an irreducible component of $V(I)$.

DEFINITION 1.13. An *affine algebraic variety* is an irreducible algebraic set in $\mathbb{A}^n$, with its induced topology. (So it's an irreducible closed space). A *quasi-affine variety* is an open subset of an affine variety.

EXAMPLE 1.14. Examples of affine varieties.

(1) We have seen that $\mathbb{A}^1$ is an affine variety. In fact, so is every $\mathbb{A}^n$: use the fact that $\mathbb{A}^n = V(0)$, and zero is prime in the polynomial ring.
(2) Linear varieties: let $\ell_1 \cdots \ell_m$ be independent linear forms of $x_1 \cdots x_n$. (These are linearly independent homogeneous polynomials of degree 1.) Let $a_1 \cdots a_m \in k$. Then $V(\ell_1 - a_1, \ell_2 - a_2, \cdots, \ell_m - a_m) \in \mathbb{A}^n$ is an affine variety, which we call a *linear variety* of dimension $n - m$. For example, if $\ell = ax + by$ then $V(\ell - c)$ is just the line $ax + by = c$.
(3) Points are affine varieties. Use $\ell_1 = x_1, \ell_2 = x_2, \cdots, \ell_n = x_n$ in the previous example. Then $V(x_1 - a_1, \cdots, x_n - a_n)$ is exactly the point $(a_1 \cdots a_n)$. (Two points are an algebraic set, but are not irreducible.)
(4) Consider $f \in k[x_1 \cdots x_n]$ such that (f) is prime. Then $V(f)$ is usually called a *hypersurface* in $\mathbb{A}^n$, and is an algebraic variety.
(5) If $f \in k[x, y]$ then $V(f)$ is usually called a plane curve: for example, think about the graph of $y^2 = x^3$ (this has a cusp at the origin). $y^2 - x^3 - x^2$ passes the origin twice, and is called *nodal.* $y^2 - x^3 - x$ is called an elliptic curve (it has two disconnected components). If k is the complex numbers, this curve is a torus.
(6) *Parametrized curves.* Given $f_1(t) \cdots f_n(t) \in k[t]$, then $x_i - f_i \in k[t, x_1 \cdots x_n]$ and we can consider $V(x_1 - f_1 \cdots x_n - f_n) \subset \mathbb{A}^{n+1}$. For example, we have $V(x - t^2, y - t^3) \subset \mathbb{A}^3$. It looks like $y^2 - x^3$ if you project to the xy plane.

1.2. Morphisms between quasi-affine varieties. Recall that an affine variety is a special case of a quasi-affine variety.

DEFINITION 1.15. Let $X \subset \mathbb{A}^n$ be a quasi-affine variety. A function $f : X \to k$ is considered *regular* at $p \in X$ if there is some set $U \ni p$ open in X, and $f_1, f_2 \in k[x_1 \cdots x_n]$ such that

(1) $f_2(q) \neq 0$ for $q \notin U$
(2) $f = \frac{f_1}{f_2}$ on U

(Basically, it agrees with a well-defined rational function on a neighborhood.)

$f : X \to k$ is called *regular* if it is regular at every point in X.

Observe that the set of regular functions on X form a ring, denoted by $\mathcal{O}(X)$. For example, $k[x_1 \cdots x_n] \subset \mathcal{O}(\mathbb{A}^n)$, but you can actually prove equality.

So now we have a notion of functions on $\mathbb{A}^n$. Are they continuous?

LEMMA 1.16. *Let $f : X \to k$ be a regular function. Regard it as map $f : X \to \mathbb{A}^1_k$ naturally. Then f is continuous.*

COROLLARY 1.17. *Let f be a regular function on a quasi-affine variety X. If $f = 0$ on some open subset $U \subset X$, then $f = 0$ on X. (So there are no bump functions.)*

PROOF OF COROLLARY. Consider $Z = \{p \in X : f(p) = 0\}$. This is closed, because f is continuous and points are closed. We can write $X = Z \cup (X \backslash U)$. These are two closed subsets, and X is irreducible, so one of them must be all of X. But $X \backslash U$ is not X, which implies that $X = Z$. ⊟

PROOF OF LEMMA. In fact, it is enough to show that $f^{-1}(0)$ is closed. Why? We want to show that, for every closed subset of $\mathbb{A}^n$, the preimage is closed. But the closed subsets of $\mathbb{A}^n$ are either the whole set, the empty set, or finite sets. But since this is a ring, to show that $f^{-1}(a)$ is closed, it suffices to show that $(f - a)^{-1}(0)$ is closed.

For every p, we need to find some neighborhood $U \ni p$ such that $f^{-1}(0) \cap U$ is closed. By definition of regular functions, we can choose U such that $f = \frac{f_1}{f_2}$ on U, with f_1 and f_2 polynomials. But $f^{-1}(0) \cap U = f_1^{-1}(0) \cap U = V(f_1) \cap U$, which is closed in U. ⊟

2. September 6

RECALL that a set is quasi affine if it is open in an affine variety (irreducible algebraic set). Let X be a quasi-affine variety over a field k, and let $\mathcal{O}(X)$ be the ring of regular functions on X. (A function is regular at a point if it agrees with a well-defined rational function on a neighborhood.)

PROPOSITION 2.1. *If $f : X \to k$ is regular, then $f : X \to \mathbb{A}^1$ is continuous with respect to the Zariski topology.*

COROLLARY 2.2. *If $f = 0$ on $U \in X$ open, then $f = 0$ on X.*

COROLLARY 2.3. *$\mathcal{O}(X)$ is an integral domain.*

PROOF. If $fg = 0$, then if $Z_1 = \{x : f(x) = 0\}$ and $Z_2 = \{x : g(x) = 0\}$, then $Z_1 \cup Z_2 = X$. Since X is irreducible, then one of the $Z_i = X$, and so one of f or g is zero. ⊟

DEFINITION 2.4.

(1) The field of rational functions on X is the fraction field of $\mathcal{O}(X)$, denoted by $K = k(X)$.

(2) Let $p \in X$. The *local ring of X at p* is given by $\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}(U)$. (For every $V \subset U$ there is a map $\mathcal{O}(U) \to \mathcal{O}(V)$, and that is where the direct limit is taken.) *For every p define an equivalence relation on $\mathcal{O}(X)$, where two regular functions are equivalent if they agree on some neighborhood of a point. Then $\mathcal{O}_{X,p} = \mathcal{O}/\sim$. Our ring is local because it has a unique maximal ideal: $\mathfrak{m}$ is the set of functions that vanish at 0.*

REMARK 2.5. Let $f \in k(X)$ be a rational function $f = \frac{f_1}{f_2}$, for $f_1, f_2 \in \mathcal{O}(X)$. Therefore, f defines a regular function on $U = X - \{f_2 = 0\}$. Conversely, if $f \in \mathcal{O}(U)$ then we claim that f can be regarded as a regular function. Why? Choose $p \in U$. There is some open neighborhood $V \subset U$ such that $f = \frac{f_1}{f_2}$ on V, where f_1 and f_2 are polynomials. In particular, $f_i \in \mathcal{O}(U)$. So $f \in K$, at least in this neighborhood. However, we want this to be well-defined over all p. If you choose any $p' \in V' \subset U$, and write $f = \frac{f_1'}{f_2'}$ on V', then $\frac{f_1}{f_2} = \frac{f_1'}{f_2'}$ on $V \cap V'$. So $f_1 f_2' = f_1' f_2$ on $V \cap V'$. But since all the f_i and f_i' were well-defined everywhere, we can write $\frac{f_1}{f_2} = \frac{f_1'}{f_2'} \in K$. So we've just gotten an injective map $\mathcal{O}(U) \hookrightarrow K$ which is in fact a ring homomorphism. You end up getting a sequence

$$\mathcal{O}(X) \hookrightarrow \mathcal{O}(U) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow K = k(X) = k(U)$$

To clarify the first inclusion, note that $\mathcal{O}(X)$ contains functions that agree with rational functions on X, whereas in $\mathcal{O}(U)$ they only have to agree on the smaller set U.

If X is affine, it is defined by some prime ideal: $X = V(\mathfrak{P})$. Can we determine the regular functions in terms of $\mathfrak{P}$?

THEOREM 2.6. *Let $X = V(\mathfrak{P}) \subset \mathbb{A}^n$ be affine. Then*

(1) *There is a natural isomorphism $A(X) := k[x_1 \cdots x_n]/\mathfrak{P} \to \mathcal{O}(X)$ (all regular functions are restrictions of polynomials).*
(2) *There is a 1-1 correspondence between points on X and maximal ideals of $A(X)$. We will denote the maximal ideal corresponding to $\{p\}$ as $\mathfrak{m}_p$.* These are the functions that vanish at the point p.
(3) *The localization $A(X)_{\mathfrak{m}_p}$ is isomorphic to $\mathcal{O}_{X,p}$.*

We call $A(X)$ the coordinate ring *of X.*

PROOF. (2) Obvious. All the nonempty irreducible algebraic sets correspond bijectively to prime ideals. If $Y \subset X$ is a nonempty irreducible algebraic set, it corresponds bijectively to a containment $I(Y) \supset I(X) = \mathfrak{P}$. In particular, points in X correspond to maximal ideals that contain $\mathfrak{P}$. [Points are closed by Example 1.14, and are certainly irreducible.] These are the same as the maximal ideals of $A(X)$.

(3) There is a natural map $k[x_1 \cdots x_n] \to \mathcal{O}(X)$ that factors through $k[x_1 \cdots x_n]/\mathfrak{P}$: all the polynomials that vanish on X are exactly the elements of $\mathfrak{P}$. On the other hand, we have a natural inclusion $\mathcal{O}(X) \to Frac(A(X))$, by the same argument as we saw earlier:

any regular function on some open subset, can be written as $f = \frac{f_1}{f_2}$. (We showed that this is well-defined.) So we have inclusions

$$A(X) \hookrightarrow \mathcal{O}(X) \hookrightarrow Frac(A(X)) = K$$

(K is the fractional field of $\mathcal{O}(X)$, but $\mathcal{O}(X) \subset Frac(A(X))$.)

$$\begin{tikzcd} A(X) \arrow[r, hook] \arrow[d, hook] & \mathcal{O}(X) \arrow[r, hook] \arrow[d, hook] & Frac(A(X)) \arrow[d, "="] \\ A(X)_{\mathfrak{m}_p} \arrow[r, hook] & \mathcal{O}_{X,p} \arrow[r, hook] & K \end{tikzcd}$$

The map $A(X)_{\mathfrak{m}_p} \to \mathcal{O}_{X,p}$ is given by writing things in $A(X)$ as fractions on an appropriate open set. However, we claim it is an isomorphism. It is a fact from commutative algebra that every integral domain is the intersection of its localizations. We have

$$A(X) = \bigcap_{\mathfrak{m}} A(X)_{\mathfrak{m}} \hookrightarrow \bigcap_{p} \mathcal{O}(x,p)$$

$\mathcal{O}(X)$ is contained in any local ring, so it is contained in the last intersection. Since $\mathcal{O}_{X,p} = \varinjlim \mathcal{O}(U)$, we can write these as rational functions and shrink the open set, to get rational functions that do not vanish at p.

So we get an isomorphism for the infinite intersections, which makes $A(X) = \mathcal{O}(X)$. $\boxminus$

Definition 2.7. Let X and Y be two quasi-affine varieties. A *morphism* $\varphi : X \to Y$ is a continuous map such that for any $V \subset Y$ open and $f : V \to k$ regular,

$$f \circ \varphi : \varphi^{-1}(V) \to k$$

is regular.

The collection of quasi-affine varieties, with morphisms defined as above, forms a category.

Definition 2.8. A morphism $\varphi : X \to Y$ is called an isomorphism if there is some $\psi : Y \to X$ such that $\psi\varphi = \varphi\psi = Id$.

Definition 2.9. A quasi-affine variety is also said to be affine if it is isomorphic to an affine variety.

Example 2.10. $\mathbb{A}^1 - \{0\}$ is a quasi-affine variety. But later we will see that it is isomorphic to $V(xy-1) \subset \mathbb{A}^2$. So it is also an affine variety.

Here is an easier way to see what things are morphisms.

Lemma 2.11. *Let $\varphi : X \to Y \subset \mathbb{A}^n$ be a map with X quasi-affine and Y affine. Then φ is a morphism iff $x_i \circ \varphi$ is regular (where x_i is any coordinate function on Y: a restriction to the i^{th} coordinate).*

Proof. x_i is certainly a regular function, so if φ is a morphism, then so is $x_i \circ \varphi$.

Conversely, if $x_i \circ \varphi$ is regular for any x_i, we need to show

(1) φ is continuous;
(2) For any open $V \subset Y$ and $f \in \mathcal{O}(V)$, the composition $f \circ \varphi$ is regular on $\varphi^{-1}(V)$.

Now let's prove these things.

(1) Let $f \in k[x_1 \cdots x_n]$. We can write a generic closed set as $Z = V(f_1) \cap \cdots \cap V(f_n)$. So the goal is to show that $\varphi^{-1}(V(f))$ is closed. This is the set of x such that $f(\varphi(x)) = 0$. So it suffices to show that $f \circ \varphi$ is continuous. In fact, we claim that it is regular. Write
$$f \circ \varphi = f(x_1 \circ \varphi, \cdots, x_n \circ \varphi)$$
and recall that, on an affine space, regular functions are rational functions. Substituting the regular ($\implies$ rational) functions $x_i \circ \varphi$ into the variables of the regular ($\implies$ rational) function f still produces a rational function.
(2) (Now since V need not be affine, we cannot directly use the trick that regular = rational. But we can still do this if we say things are rational *in a neighborhood*.) If $V \subset Y$ and $p \in U \subset V$ we want to show that φ is regular at p. At $\varphi(p)$ we can write $f = \frac{f_1}{f_2}$, where f_1 and f_2 are polynomials and $f_2(p) \neq 0$. Around p, $f \circ \varphi = \frac{f_1 \circ \varphi}{f_2 \circ \varphi}$; numerator and denominator are regular functions, so $f \circ \varphi$ is also regular at p.

$\boxminus$

EXAMPLE 2.12. $G_m := \mathbb{A}^1 - 0 \to V(xy - 1) \subset \mathbb{A}^2$ where $t \mapsto (t, t^{-1})$. Composing with x_1 gives a regular function, as does composition with x_2. Conversely, you can give a map $(x, y) \to xy$ backwards. This gives an isomorphism of quasi-affine varieties; and now we can say that $\mathbb{A}^1 - 0$ is an affine variety.

PROPOSITION 2.13. *Let X be quasi-affine and Y be affine. Then there is a natural isomorphism*
$$Hom(X, Y) \xrightarrow{\sim} Hom(\mathcal{O}(Y), \mathcal{O}(X))$$
where $Hom(X, Y)$ is the set of all morphisms $X \to Y$. (Note that $\mathcal{O}(Y) = A(Y)$.)

PROOF. There is clearly a forwards map. Any $\varphi : X \to Y$ induces a pullback map $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, where $f \mapsto f \circ \varphi$. We want to construct an inverse map:
$$Hom_{alg}(A(Y), \mathcal{O}(X)) \to Hom(X, Y)$$
Let $Y \subset \mathbb{A}^n$. Start with a map $\psi : A(Y) \to \mathcal{O}(X)$, construct $\varphi : X \to \mathbb{A}^n$, where $\varphi : p \mapsto (\psi(x_1)(p), \psi(x_2)(p), \cdots)$ sends any function in the ideal defining Y to zero. So the previous map sends p into Y. So send $p \mapsto p(\varphi(x))$. To check it is a morphism, we have to look at the coordinate functions. But we constructed it so that we knew what they were: $x_i \circ \varphi = \psi(x_i)$. $\boxminus$

COROLLARY 2.14. *There is a contravariant equivalence of categories between affine varieties and finitely-generated integral k-algebras. A morphism between X and Y is the same as a morphism between their coordinate rings, by the previous proposition.*

Given any finitely generated k-algebra $k[x_1 \cdots x_n]/\mathfrak{P}$, since A is integral $\mathfrak{P}$ is a prime ideal, and A is exactly the coordinate ring of $V(\mathfrak{P})$.

In the homework, you will see that $\mathbb{A}^2 - \{(0,0)\}$ is not an affine variety, but its coordinate ring is the same as $k[x,y]$. So the coordinate ring is not really enough to determine the variety.

DEFINITION 2.15. Let $A \to B$ be a homomorphism of commutative algebras, and M a B-module. We define the derivations

$$Der_A(B,M) = \{D \,:\, D : B \to M \text{ satisfying (1) and (2)}\}$$

(1) $D(b_1 b_2) = b_1 D(b_2) + b_2 D(b_1)$
(2) $D(a) = 0 \ \forall a \in A$

EXAMPLE 2.16. If $A = k$ and $B = k[x_1 \cdots x_n]$. Then $M = k$ is a B-module, defined by the map $B \to k$ taking every $x_i \mapsto 0$.

We have:

$$Der_k(B,k) = \{D = \sum \lambda_i \frac{\partial}{\partial x_i} \,:\, \lambda_i \in k\}$$

By definition, $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$.

Also note that $k \cong \mathcal{O}_{X,p}/\mathfrak{m}$. This is because the map $A(X) \to k$ given by evaluation at p has, by definition, kernel $\mathfrak{m}_p$... and so does the map $\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p} \to k$.

DEFINITION 2.17. The Zariski tangent space of X at p is defined as

$$T_pX = Der_k(\mathcal{O}_{X,p}, k)$$

Observe that the space of all derivations $Der_A(B,M)$ is a B-module. (Adding and scalar multiplication by b also makes a derivation.) Therefore, T_pX is a $\mathcal{O}_{X,p}$-module. But this factors through $\mathfrak{m}$, making it a k-vector space.

LEMMA 2.18.

$$T_pX \cong (\mathfrak{m}/\mathfrak{m}^2)^\times$$

Note that the $\times$ means dual space*!*

PROOF. Given a derivation $D : \mathcal{O}_{X,p} \to k$, we can restrict to $\mathfrak{m}$ and get a map $\mathfrak{m} \to k$. But this vanishes on $\mathfrak{m}^2$: $D(fg) = fD(g) + gD(f)$ but because $f, g \in \mathfrak{m}$, when you mod out by $\mathfrak{m}$ you have $D(fg) = 0 \in \mathfrak{m}/\mathfrak{m}^2$. Given such a map, the derivation is uniquely determined. As a k-vector space, $\mathcal{O}_{X,p} \cong k \oplus \mathfrak{m}$. By definition this tangent space is the k-derivations $Der_k(\mathcal{O}_{X,p}, k)$. You can easily check that this gives a bijective correspondence. ⊟

Exercise: Show $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional k-vector space.

We will prove the following theorem next time:

THEOREM 2.19. *Let X be quasi-affine. Then there is some non-empty Zariski-open subset $U \subset X$ such that* $\dim_K Der_k(K,K) = \dim_k T_pX$ *for all $p \in U$.*

This number is called the dimension *of X.*

3. September 8

Next week no class. After that, class is at 2:30 - 4:00.

3.1. Smooth points.

THEOREM 3.1. *Let X be a quasi-affine variety over k. Let $K = k(X)$ be the function field.*

(1) *For all $p \in X$,* $\dim Der_k(K,K) \leq \dim_k T_pX$
(2) *There is a nonempty open subset $U \subset X$ such that*
$$\dim_k T_pX = \dim_K Der_k(K,K) \quad \forall p \in U$$

This motivates a definition:

DEFINITION 3.2.
$$\dim X = \dim_K Der_k(K,K) = \min_{p\in X} \dim_k T_pX$$

DEFINITION 3.3. $p \in X$ is called *non-singular* (or smooth) if $\dim T_pX = \dim X$. Otherwise, p is called a *singular point* of X. X is called smooth or nonsingular if every point on X is smooth.

On a smooth manifold, the dimension of the tangent space is always the same. But here, we can have points that are not smooth.

PROOF OF THEOREM. We can assume that X is affine. (Otherwise take the Zariski closure, and the function field does not change.) There is a natural map obtained by restriction:
$$Der_k(K,K) \to Der_k(\mathcal{O}(X),K)$$
Because the fraction field is in $\mathcal{O}(X)$, this is an isomorphism. If $X = V(\mathfrak{P}) \subset \mathbb{A}^n$, then
$$Der_k(\mathcal{O}(X),K) = \{D = \sum \lambda_i \frac{\partial}{\partial x_i}, \lambda_i \in K \,:\, D(f) = 0 \;\forall f \in \mathfrak{P}\}$$
If we choose a set of generators $f_1 \cdots f_m$ for I, we can rewrite the above as
$$\{D = \sum \lambda_i \frac{\partial}{\partial x_i} \,:\, \lambda_i \in K, \sum \lambda_i \frac{\partial f_j}{\partial x_j} = 0 \;\forall j\}$$

If it vanishes on x_j, then it vanishes on the whole ideal: $D(fg) = fD(g) + gD(f)$. This is the kernel of the map $K^n \to K^m$ by the derivation $\frac{\partial f_j}{\partial x_i}$. So the dimension of the derivations is just $n - rk_K J$, where J is the matrix of partials described above.

All the entries of this matrix are polynomials, so you can evaluate at p. By the same reasoning, $\dim_k T_pX = n - rk_K J(p)$. We want to show that

- $rk_k J(p) \leq rk_K J$
- there is some nonempty subset U such that $rk_k J(p) = rk_K J$

Assume $rk_K J = r$. Then there are invertible $n \times n$ matrices A, B with entries in K such that you can write AJB as a block matrix with I_r in the upper left and zeroes elsewhere. Write $A = A_0/\alpha$, $B = B_0/\beta$ such that α and $\beta \in \mathcal{O}(X)$ and A_0 and B_0 have entries in $\mathcal{O}(X)$. Let $f = \alpha\beta \det A_0 \cdot \det B_0$. This is something in $\mathcal{O}(X)$ which is not zero (since none of the terms is zero). Just consider $U = X - V(f)$. This is nonempty. You can write $A_0 J B_0 = \alpha\beta \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. So for all $p \in U$, $rk_k J(p) = r$. ⊟

LEMMA 3.4. *The function $p \mapsto \dim_k T_pX$ is upper semicontinuous. That is,*

$$X_\ell = \{p \in X \,:\, \dim T_pX \geq \ell\}$$

is closed. So $\dim T_pX \geq n - r$ for all $p \in X$.

An open subset of X has dimension exactly $n - r$. Others have dimension $\geq$ this.

PROOF. $\dim T_pX = n - rk_k(\frac{\partial f_i}{\partial x_j}(p))$.

$$X_\ell = V(\mathfrak{P} + \text{ the ideal generated by all } (n-\ell+1) \times (n-\ell+1) \text{ minors of } J)$$

So this is a closed subset. ⊟

EXAMPLE 3.5. Assume k does not have characteristic 2. Remember $y^2 = x^3$ has a cusp, $y^2 = x^3 + x^2$ intersects itself at the origin, and $y^2 = x^3 + x$ has two connected components. Since these are plane curves, we hope they have dimension one.

In the first case, take $f = y^2 - x^3$. Taking the partials, we find $f_x = -3x^2$, $f_y = 2y \neq 0$. So $J = (-3x^2, 2y)$. Consider $J(p)$. We know $J(0,0) = 0$. The Zariski tangent space at this point is two-dimensional, so it is a singular point. You can check that any other point on the curve is smooth.

In the case of $f = y^2 - x^3 - x^2$, we find $J = (2x, 2y)$, and again $J(0,0) = 0$ and the origin is singular.

In the case of $f = y^2 - x^3 - x$, we have $J = (-3x^2 - 1, 2y)$. We find that the rank of $J(p)$ is always 1. This comes from solving the equations:

$$\begin{aligned} y^2 &= x^3 + x \\ -3x^2 - 1 &= 0 \\ 2y &= 0 \end{aligned}$$

There are no solutions. So all the points are smooth.

EXAMPLE 3.6. $\dim \mathbb{A}^n = n$ and $\dim * = 0$

3.2. Facts from commutative algebra.

DEFINITION 3.7. Let K/k be a field extension. We say that K is separably generated over k if there is some L with $k \subset L \subset K$ such that L/k is purely transcendental and K/L is a finite separable algebraic extension.

THEOREM 3.8. *If K/k is finitely separably generated, then*

$$\dim_K Der_k(K, K) = tr.d._k K$$

(where tr.d. is the transcendental degree).

THEOREM 3.9. *Let k be a perfect field (e.g. $k = \overline{k}$). Then every finitely-generated K/k is separably generated.*

For example, in $\mathbb{A}^n$, if $K = k[x_1 \cdots x_n]$ we know that the transcendence degree is n. But we could also find this by calculating the dimension of the Zariski tangent space.

PROPOSITION 3.10. *Let X be an affine variety. Then $\dim X = n - 1$ if and only if $X = V(f)$ for some irreducible $f \in k[x_1 \cdots x_n]$. (We call such $V(f)$ a* hypersurface.*)*

REMARK 3.11. If $\dim X = n - 2$, then it is not necessarily true that $I(X)$ is generated by two elements.

PROOF. Suppose $X = V(f)$ is a hypersurface. We want to show that $\dim X = n - 1$. The rank of $\frac{\partial f}{\partial x_i}$ is at most 1, so $\dim X = n - rk(\frac{\partial f}{\partial x_i}) \geq n - 1$. We want to show that $rk(\frac{\partial f}{\partial x_i}) \neq 0$. *Remember that this is going on in a space in which we have modded out by f.* Otherwise, $\frac{\partial f}{\partial x_i}$ lands in the ideal generated by f (i.e. f divides this); but the degree is smaller, which can only happen if $\frac{\partial f}{\partial x_i} = 0$ (*here zero is in k*). This can happen if the field has characteristic $p > 0$, and you can write $f = g(x_1^p, x_2^p, \cdots, x_n^p)$. (Note that in characteristic p, x^p has zero derivative.) Since k is algebraically closed, we can write $g = g_1(x_1 \cdots x_n)^p$ (we're using the formula $(a+b)^p = a^p + b^p$ mod p; write $ax^p + by^p$). We assumed X was irreducible, so this is a contradiction.

Now assume $\dim X = n - 1$. We want to conclude that X is a hypersurface. Let $g \in \mathfrak{P} = I(X)$. We have $X \subset V(g)$. We could assume that g is irreducible. If g is a product of things, at least one is in $\mathfrak{P}$ because $\mathfrak{P}$ is prime. We already know that $\dim V(g) = n - 1$. So now the proposition will follow from a lemma:

LEMMA 3.12. *Let A be an integral domain over k, and $\mathfrak{P} \subset A$ be a prime ideal. Then*

$$tr.d._k A/\mathfrak{P} \le tr.d._k A$$

with equality iff $\mathfrak{P} = 0$ or both are infinity.

REMARK 3.13. By definition, $tr.d._k A/\mathfrak{P} = tr.d._k Frac(A/\mathfrak{P})$.

From this lemma, we can conclude that hypersurfaces have dimension 1.

PROOF. Assume the contrary: that $n = tr.d._k A$. Then there is some $x_1 \cdots x_n \in A$ such that $\overline{x_1} \cdots \overline{x_n}$ are algebraically independent over $A/\mathfrak{P}$. Let $0 \neq y \in \mathfrak{P}$. Since we assumed the transcendental degree of A is n, there exists a polynomial $P \in k[Y, X_1 \cdots X_n]$ such that $P(y, x_1 \cdots x_n) = 0$. But A is an integral domain, so we can assume that P is irreducible. If P factors as the product of two polynomials, then this relation still holds for one of them (this is by integrality). Note that P is not a multiple of Y: if you plug in y, it is not zero. Therefore, $P(0, \overline{x}_1 \cdots \overline{x}_n)$ is an algebraic relation in $A/\mathfrak{P}$. Contradiction.

$\square$

Here is a theorem that says how to construct affine varieties containing certain smooth points.

THEOREM 3.14. *Let $f_1 \cdots f_r \in k[x_1 \cdots x_n]$ have no constant term (i.e. all the polynomials vanish at the origin), and have independent linear terms. (That is, if you write $f_i = \sum a_{ij}x_j + \cdots$, then $rk(a_{ij}) = r$.) Let $\mathfrak{P} = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 \cdots x_n]$. Then*

1. *$\mathfrak{P}$ is a prime ideal*
2. *$X = V(\mathfrak{P})$ has dimension $n - r$, and $0 \in X$ is smooth*
3. *$V(f_1 \ldots f_n) = X \cup Y$ for some algebraic set Y not containing zero*

COROLLARY 3.15. *Let X be an affine variety of dimension $n - r$ in which 0 is a smooth point. Then there is some $f_1 \cdots f_r \in I(X)$ such that*

$$I(X) = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1..x_n]$$

Of course, we can transform any point to be the origin. Then this says that our variety X is defined by r equations (even if the original variety is not defined by r equations: it is $X \cup Y$ for Y not containing the chosen point).

PROOF OF THE COROLLARY. We need to find r polynomials with independent linear terms; then we can apply the theorem. Let $\mathfrak{m} = (x_1 \cdots x_n) \subset k[x_1 \cdots x_n]$ and $\mathfrak{m}_0$ be the maximal ideal of $\mathcal{O}(X)$ corresponding to zero. We have a sequence

$$0 \to \mathfrak{P} \to \mathfrak{m} \to \mathfrak{m}_0 \to 0$$

$$0 \to (\mathfrak{P} + \mathfrak{m}^2)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_0/\mathfrak{m}_0^2 \to 0$$

$\mathfrak{m}/\mathfrak{m}^2$ has dimension n, $(\mathfrak{P}+\mathfrak{m}^2)/\mathfrak{m}^2$ has dimension r, and $\mathfrak{m}_0/\mathfrak{m}_0^2$ has dimension $n-r$. Choose an arbitrary basis $(f_1 \dots f_r)$ for $\mathfrak{P}+\mathfrak{m}^2/\mathfrak{m}^2$, and that lifts to a basis of $\mathfrak{P}$. So $f_1 \cdots f_r$ has independent linear terms. ($\mathfrak{m}^2$ has a basis, which means that there are linearly independent terms.) From the theorem,

$$\mathfrak{P} = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 \cdots x_n]$$

is a prime ideal.

$$\mathfrak{P} = \{\sum \frac{f_i g_i}{k_i} : k_i(0) \neq 0, \sum \frac{g_i f_i}{k_i} \in k[x_1 \cdots x_n]\}$$

This is inside of $I(X)$, because it vanishes on X. (Note that $\frac{g_i f_i}{k_i}$ is a regular function, except where the denominator vanishes.)

We have $Y = V(\mathfrak{P}) \supset X$. By the theorem, $\dim Y = n-r$. By assumption, $\dim X = n-r$. But if they have the same dimension, they have to be equal. So, locally around a point the ideal is always generated by r polynomials. ⊟

Proof of the theorem. Let $(A, \mathfrak{m}, k)$ be a local ring. Then $\widehat{A} := \varprojlim A/\mathfrak{m}^n$. Recall that we have an inverse system

$$.. \to A/\mathfrak{m}^3 \to A/\mathfrak{m}^2 \to A/\mathfrak{m} = k$$

We have $\mathcal{O}_{\mathbb{A}^n,0} = k[x_1 \cdots x_n]_{(x_1 \cdots x_n)}$. So then

$$\widehat{\mathcal{O}}_{\mathbb{A}^n,0} = k[[x_1 \cdots x_n]]$$

where we now allow infinite polynomials (a.k.a. power series).

$$\mathcal{O}(\mathbb{A}^n) = k[x_1 \cdots x_n] \subset \mathcal{O}_{\mathbb{A}^n,0} = k[x_1 \cdots x_n]_{(x_1 \cdots x_n)} \subset k[[x_1 \cdots x_n]]$$

(Rational functions like $\frac{1}{1-f}$ have power series.) Remember $\mathfrak{P} = (f_1 \cdots f_v)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 \cdots x_n]$ was defined as before. Now define

$$\mathfrak{P}' = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0}$$

and

$$\mathfrak{P}'' = (f_1 \cdots f_r)k[[x_1 \cdots x_n]]$$

The ring homomorphism of the pullback of a prime ideal is always prime.

It is enough to show two things:

(1) $\mathfrak{P}''$ is prime in $k[[x_1 \cdots x_n]]$
(2) $\mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0} = \mathfrak{P}'$

The second fact comes from commutative algebra. By definition,

$$\begin{aligned}\mathfrak{P}' \subset \mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0} &= \{\sum g_i f_i : g_i \in k[[x_1 \cdots x_n]], \sum g_i f_i \in k[x_1 \cdots x_n]_{(x_1 \cdots x_n)}\} \\ &= \bigcap_{N=0}^{\infty} (\mathfrak{P}' + \mathfrak{m}^N)\end{aligned}$$

If we just truncate this, then the remaining terms will be in $\mathfrak{P}' + \mathfrak{m}^N$. That is, if $g_i = \sum_{\alpha<N} x_i^\alpha + R$ then $Rf_i \in \mathfrak{m}^N$ and $f_i \sum x_i^\alpha \in \mathfrak{P}'$.

The second part is proven by the following theorem from commutative algebra

THEOREM 3.16 (Krull Theorem). *If $(A, \mathfrak{m}, k)$ is Noetherian local, then for any ideal I,*

$$I = \bigcap_{N=0}^{\infty} (I + \mathfrak{m}^N).$$

$\boxminus$

4. September 20

4.1. Finishing things from last time.

THEOREM 4.1. *Let $f_1 \cdots f_r \in k[x_1 \cdots x_n]$ be polynomials with no constant term and independent linear terms. Then the ideal of the local ring*

$$\mathfrak{P} = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 \cdots x_n]$$

is prime. $\dim V(\mathfrak{P}) = n-r$ *and zero is a smooth point on $V(\mathfrak{P})$. Moreover, $V(f_1 \cdots f_r) = V(\mathfrak{P}) \cup Y$ when $- \notin Y$ is some algebraic set.*

Last time, we prove a corollary: If we have a smooth point, then locally this variety is cut out by r equations.

PROOF. $k[x_1 \cdots x_n] \subset k[x_1 \cdots x_n]_{(x_1 \cdots x_n)} = \mathcal{O}_{\mathbb{A}^n,0} \subset k[[x_1 \cdots x_n]] = \widehat{O}_{\mathbb{A}^n,0}$. Consider

$$\mathfrak{P}' = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0}$$
$$\mathfrak{P}'' = (f_1 \cdots f_r)k[[x_1 \cdots x_n]]$$

It is enough to prove that $\mathfrak{P}''$ is prime, and that $\mathfrak{P}' = \mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0}$.

To prove the second thing, note that

$$\mathfrak{P}' \subset \mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0} = \{n\mathcal{O}_{\mathbb{A}^n,0} \,:\, f = \sum h_i f_i,\ \ h_i \in k[[x_1 \cdots x_n]]\}.$$

Suppose $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,0}$.

$$0 \to \mathfrak{m}^N \to \mathcal{O}_{\mathbb{A}^n,0} \to \mathcal{O}_{\mathbb{A}^n,0}/\mathfrak{m}^N \to 0$$

We can show that the last term is just $k[x_1 \cdots x_n]/(x_1 \cdots x_n)^N$. So $\mathcal{O}_{\mathbb{A}^n,0} = \mathfrak{m}^N \oplus \{$polynomials of degree ¡ N$\}$ as a k-vector space. (Every element in the local ring can be written as a polynomial of small degree plus something in a power of the local ring.)

We can write $f = f' +$ remainders, where f' is a polynomial of degree $< N$. So going back to the expression of $\mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0}$ we have

$$h_i = h_i' + \text{ degree} \geq N \text{ terms}$$

Then $f - \sum h_i' f_i = (f - \sum h_i' f_i) + \mathfrak{m}^N \mathcal{O}_{\mathbb{A}^n,0})$ Look at the first term; it has degree $\geq N$. All of this is in $\mathfrak{m}^N \mathcal{O}_{\mathbb{A}^n,0}$. So

$$\mathfrak{P}'' \cap \mathcal{O}_{\mathbb{A}^n,0} \subset \bigcap_{N \geq 0} (\mathfrak{P}' + \mathfrak{m}^N)$$

which is equal to $\mathfrak{P}'$ by Krull's theorem.

Now we want to show that $\mathfrak{P}''$ is prime. We need:

PROPOSITION 4.2. *(Formal inverse function theorem) Let $f = \sum a_i x_i +$ higher terms $\in k[[x_1 \cdots x_n]]$ and $a_i \neq 0$. Then any $g \in k[[x_1 \cdots x_n]]$ can be uniquely written as $g = u \cdot f + h(x_2 \cdots x_n)$*

PROOF. Easy. Expand u and h and show you can solve for all the coefficients. ⊟

COROLLARY 4.3.

$$k[[x_1 \cdots x_n]]/(f) \xleftarrow{\sim} k[[x_2 \cdots x_n]]$$

COROLLARY 4.4. *Let $f_1 \cdots f_r$ be r power series. Write*

$$f_i = \sum a_{ij} x_j + \text{ higher terms}$$

If $\det((a_{ij})_{1 \leq i,j \leq r}) \neq 0$, then

$$k[[x_1 \cdots x_n]]/(f_1 \cdots f_r) \xleftarrow{\sim} k[[x_{r+1 \cdots x_n}]]$$

PROOF. Induction. ⊟

Back to the theorem. If $f_1 \cdots f_r$ have independent linear terms, then

$$k[[x_1 \cdots x_n]]/(f_1 \cdots f_r) \xleftarrow{\sim} k[[x_{r+1 \cdots x_n}]]$$

This shows that $\mathfrak{P}''$ is prime, which in turn shows that $\mathfrak{P}$ is prime.

$$A(V(\mathfrak{P})) = k[x_1 \cdots x_n]/\mathfrak{P} \hookrightarrow k[[x_1 \cdots x_n]]/\mathfrak{P} \cong k[[x_{n+1 \cdots x_n}]]$$

They are algebraically independent because they are algebraically independent in the power series map.

We used commutative algebra to relate that dimension of a variety to the transcendental degree. Therefore, $\dim V(\mathfrak{P}) \geq n - r$. By definition

$$T_p V(\mathfrak{P}) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

$$0 \to \mathfrak{P}/(\mathfrak{P} + (x_1 \cdots x_n)^2) \to (x_1 \cdots x_n)/(x_1 \cdots x_n)^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2 \to 0$$

$T_p V(\mathfrak{P})$ is cut out by the linear terms of $f_1 \cdots f_r$ in $T_p \mathbb{A}^n$. Because the linear terms are independent, we have that $\dim T_p V(\mathfrak{P}) = n - r$. We know that the dimension of the tangent space is $\geq$ the dimension of the variety. But this tangent space is exactly $n - r$, which is the upper bound anyway, so $\dim V(\mathfrak{P}) = n - r$ and p is smooth. We want to show that

$$V(f_1 \cdots f_r) = V(\mathfrak{P}) \cup Y$$

where Y does not contain the origin. $\mathfrak{P}$ is not necessarily generated by $f_1 \cdots f_r$; this is only true of the local ring $\mathcal{O}_{X,p}$. So let $g_1 \cdots g_s$ be a set of generators of $\mathfrak{P}$ (recall this is finite because everything is Noetherian).

$$g_i \in (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 \cdots x_n]$$

Equivalently, there are some h_i not vanishing at the origin such that $h_i g_i \in (f_1 \cdots f_r)$.

Let $h = \prod h_i$. Then $h\mathfrak{P} \subset (f_1 \cdots f_r)$ and $V(f_1 \cdots f_r) \subset V(h\mathfrak{P}) = V(h) \int V(\mathfrak{P})$. In other words, $V(f_1 \cdots f_r) = V(f_1 \cdots f_r, h) \cup V(\mathfrak{P})$. Since $h(0) \neq 0$, $0 \notin Y$.

$\boxminus$

COROLLARY 4.5. *Let $p = (0, \cdots, 0) \in X = V(\mathfrak{P})$ be a smooth point of X with $\dim X = n - r$. Choose $f_1 \cdots f_r$ with independent linear terms. You can always do this because of the short exact sequence*

$$0 \to \mathfrak{P}/(\mathfrak{P} + (x_1 ... x_n)^2) \to (x_1 ... x_n)/(x_1 ... x_n)^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2 \to 0$$

Then $\mathfrak{P} = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 ... x_n]$. (The idea is that locally the ideal is generated by r functions.)

COROLLARY 4.6. *Let X be quasi-affine with dimension r. Let $p \in X$ be a smooth point . Then*

$$\widehat{\mathcal{O}_{X,p}} \cong k[[x_1 ... x_r]]$$

If we are in this situation $\mathfrak{P} = (f_1 \cdots f_r)\mathcal{O}_{\mathbb{A}^n,0} \cap k[x_1 ... x_n]$ then the complete local ring is just the power series ring.

Note that this is false if we consider the local ring instead of its completion.

4.2. More general definitions.

DEFINITION 4.7. Let $k = \overline{k}$ be a field. A (pre)variety over k is a connected topological space together with a covering $\mathcal{U} = \{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to X_\alpha$ with X_α quasi-affine, that plays well on overlaps:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a morphism of quasi-affine varieties. In addition, we require this covering $\mathcal{U}$ to be maximal; i.e., if Y is quasi-affine $V \subset X$ is open and $\psi : V \xrightarrow{\sim} Y$ quasi-affine such that

$$\varphi_\alpha \circ \psi^{-1} : \psi(U_\alpha \cap V) \to \varphi_\alpha(U_\alpha \cap V)$$

is a morphism, then $V \in \mathcal{U}$.

EXERCISE 4.8. Prove that this is irreducible.

DEFINITION 4.9. A function $f : X \to k$ is called regular if for every α,

$$f \circ \varphi_\alpha^{-1} : X_\alpha \to k$$

is regular. We denote $\mathcal{O}(X)$ to be the ring of regular functions on X. We can also define the local ring at the point p to be

$$\mathcal{O}_{X,p} = \varinjlim_{U \ni p \text{ open}} \mathcal{O}(U).$$

DEFINITION 4.10. $k(X)$ is defined to be any $k(X_\alpha)$: you have a covering by quasi affine varieties, so you can talk about the rational functions. But for every piece, there might be a different set of all such. But these are canonically identified.

REMARK 4.11. $k(X)$ is not necessarily the fractional field of $\mathcal{O}(X)$.

We also have $\dim X$, T_pX, smoothness, ...

DEFINITION 4.12. Let X be a (pre)variety. An irreducible closed subset $Y \subset X$ with the canonical variety structure is called a closed subvariety of X. If you have a closed irreducible subset, it is connected. Restrict the covering to this subset, and you get a covering if this closed subset; restrict the homeomorphism and it will map to a closed subset of a quasi-affine variety. You can check that the transition maps work out.

EXERCISE 4.13.

$$\begin{tikzcd} X \arrow[r] & Y \\ Z \arrow[u, hook] \arrow[r] & V \arrow[u, hook] \end{tikzcd}$$

If $Z \subset X$ is closed the bottom map is a morphism.

DEFINITION 4.14. Let X, Y be two varieties. A continuous map $\varphi : X \to Y$ is called a morphism if for any $p \in X$, $U \ni \varphi(p)$, $f \in \mathcal{O}(U)$, we have

$$f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U))$$

(Continuity is needed, because we want $\varphi^{-1}(U)$ to be an open set.)

4.3. Projective varieties.

DEFINITION 4.15. The projective n-space $\mathbb{P}^n$ is the set

$$k^{n+1} - \{0, \cdots, 0\}/\sim$$

where $(a_0 \cdots a_n) \sim (\lambda a_0, \cdots, \lambda a_n)$ where $\lambda \in k^*$.

REMARK 4.16. $\mathbb{P}^n$ can also be regarded as the set of 1-dimensional subspaces in k^{n+1}.

An element $p \in \mathbb{P}^n$ is called a point (even though it's just an equivalence class). Any $(a_0 \cdots a_n) \in p$ is called a set of homogeneous coordinates of p. We need to give a natural variety structure; first we define the topology.

DEFINITION 4.17. An algebraic set in $\mathbb{P}^n$ is the set Z of zeroes for a set of homogeneous polynomials $f_1 \cdots f_m$:

$$Z = V(f_1 \cdots f_r) = \{p \in \mathbb{P}^n \ : \ * \text{ holds:}\}$$

(*) If $(a_1 \cdots a_n)$ are homogeneous coordinates of p
then $f_1(a_0 \cdots a_n) = \cdots . = f_m(a_0, \cdots, a_m) = 0$.

Let $I = (f_1 \cdots f_r)$. Then I is a homogeneous ideal of $k[x_0 \cdots x_n]$; i.e. if $f \in I$ and $f = \sum_d f_d$ then $f_d \in I$. There is a result of commutative algebra that said that any ideal of homogeneous polynomials has a finite set of homogeneous generators.

So an algebraic set in $\mathbb{P}^n$ is given by homogeneous polynomials.

LEMMA 4.18.

- $V(k[x_1...x_n]) = V(x_1...x_n) = 0$, *and* $V(0) = \mathbb{P}^n$
- $V(\cup_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$
- $V(I_1 \cap I_2) = V(I_1 \cup V(I_2))$

Unlike the affine case, you can have two radical ideals corresponding to one set.

DEFINITION 4.19. (Zariski topology on $\mathbb{P}^n$) We need to choose a set of subsets that are closed: these are the algebraic sets.

THEOREM 4.20 (Hilbert's Nullstellensatz). *For all homogeneous I and all homogeneous f with $deg(f) \geq 1$, then*

$$f \in \sqrt{I} \iff f \text{ vanishes on } V(I)$$

COROLLARY 4.21. *Algebraic sets in $\mathbb{P}^n$ are in 1-1 correspondence with homogeneous radical ideals. contained in $(X_0 \cdots X_n) = S_+$.*

PROPOSITION 4.22. *Let I be a homogeneous ideal in $S = k[x_0...x_n]$ (this will denote a ring with a grading; the un-graded ring will be denoted $k[x_0...x_n]$). From commutative algebra, $\sqrt{I}$ can be written as an intersection of minimal prime ideals containing this radical:*

$$\sqrt{I} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r$$

Then the $\mathfrak{P}_i$ are homogeneous.

PROOF. Exercise. ⊟

COROLLARY 4.23. *$V(\mathfrak{P})$ is irreducible iff $\mathfrak{P}$ is a homogeneous prime (in S_+). Furthermore, every algebraic set in $\mathbb{P}^n$ can be uniquely written as the union of its irreducible components.*

DEFINITION 4.24. A projective variety is an irreducible algebraic set in $\mathbb{P}^n$. A quasi projective variety is an open subset of a projective variety.

PROPOSITION 4.25. *$\mathbb{P}^n$ together with the Zariski topology has a natural variety structure over k. (So every projective variety has a natural variety structure.)*

5. September 22

RECALL that we had introduced the projective n-space

$$\mathbb{P}^n = \{(a_0 \cdots a_n) \in k^{n+1} - 0\}/\sim$$

We can define the Zariski topology on $\mathbb{P}^n$, where the closed subsets are $V(I)$ where I is a homogeneous ideal.

PROPOSITION 5.1. *There is a natural variety structure on $\mathbb{P}^n$.*

PROOF. The idea is to cover it with some cover, each of which is isomorphic to an affine variety (not just a quasi-affine). Let $H_i = \{(a_0 \cdots a_n) : a_i = 0\}/\sim = V(x_i)$. This is closed. Let $U_i = \mathbb{P}^n - H$; this is open, and the collection of these cover the whole projective space. Now define $\varphi_i : U_i \to \mathbb{A}^n$ where $(a_0 \cdots a_n) \to (\frac{a_0}{a_i}, \cdots, \frac{\widehat{a_i}}{a_i}, \cdots, \frac{a_n}{a_i})$. This is a bijection. We need to show

(1) φ_i is a homeomorphism
(2) $\varphi_i \circ \varphi_j^{-1}$ is a morphism

Because it's bijective, we just need to show that closed subsets go to closed subsets. Let $I \subset S$ be a homogeneous ideal (remember S is the polynomial ring of $n+1$ variables, but regarded as a graded ring). Without loss of generality take $i = 0$. Then we need to show $\varphi_0 : U_0 \cap V(I) \to \mathbb{A}^n$ goes to a closed thing. Define

$$I^{ih} = \{f(1, \cdots x_n) : f \in I\} \subset k[x_1 ... x_n]$$

But

$$U_0 \cap V(I) = \{(a_0 \cdots a_n) : f(a_0 \cdots a_n) = 0 \, \forall f \in I\} \mapsto \{(a_0 \cdots a_n) : g(\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0}) = 0 \, \forall g \in I^{ih}\}$$

Note that $(\frac{a_1}{a_0} \cdots \frac{a_n}{a_0}) = (1, \frac{a_1}{a_n} \cdots \frac{a_n}{a_0})$ because things are homogeneous. On the other hand, let $I \subset k[x_1 ... x_n]$ be any ideal. We define $I^h \subset S$ where

$$I^h = \{x_0^{\deg f} f(\frac{x_1}{x_0} \cdots \frac{x_n}{x_0} \ : \in I)\}$$

$$\varphi_i(U_0 \cap V(I^h)) = V(I)$$

Now assume $i = n$, $j = 0$.

$$\varphi_n \circ \varphi_0^{-1} : \mathbb{A}^n \backslash \{x + n \neq 0\} \to \mathbb{A}^n \backslash \{x_1 \neq 0\}$$

$$(a_1 \cdots a_n) \mapsto (\frac{1}{a_n}, \frac{a_1}{a_n} \cdots \frac{a_{n-1}}{a_n})$$

This pulls back coordinate functions to regular functions, by definition:

$$(\varphi_n \circ \varphi_0^{-1})^*(x_0) = \frac{x_{i+1}}{x_n}$$

This is the ratio of two polynomials, so it is a regular function.

$\square$

In particular, every "projective variety" (i.e. $V(\mathfrak{P}) \subset \mathbb{P}^n$) is actually a variety.

EXAMPLE 5.2. $V(xy - z^2) \subset \mathbb{P}^2$.

$$U_x = \{(x, y, z) \,:\, x \neq 0\} \cong \mathbb{A}^2$$
$$V(xy - z^2) \cap U_x = V(y - z^2)$$

This is a parabola. *I think that you can take $x = 1$ because $x \neq 0$ and we're working with homogeneous coordinates. So choosing which "thing to set to 1" is basically like choosing charts.*

$$U_z = \{(x, y, z) \,:\, z \neq 0\} \cong \mathbb{A}^2$$
$$V(xy - z^2) \cap U_z = V(xy - 1) \subset \mathbb{A}^2$$

This looks like a hyperbola. So in projective geometry, there is no difference between a parabola and a hyperbola.

Let $Y = V(\mathfrak{P}) \subset \mathbb{P}^n$ be a projective variety. We denote $S(y) = \frac{S}{\mathfrak{P}}$ the homogeneous coordinate ring of Y. So $S(Y)$ is a graded ring given by the degree of S. (The ideal is a homogeneous ideal, so the grading descends.)

$$S(Y) = \oplus_d S(Y)_d$$

By the equivalence of categories in the affine case, the coordinate ring is the same as the variety. But this does not work in the projective case. You could have isomorphic projective varieties with non-isomorphic coordinate rings.

This coordinate ring determines Y, but you could have different $S(Y)$ for the same Y (???). Let $\mathfrak{P}'$ be a homogeneous prime ideal of $S(Y)$. In other words, you have a homogeneous prime ideal containing $\mathfrak{P}$ and its quotient is $\mathfrak{P}'$. We denote

$$S(Y)_{\mathfrak{P}'} = \{\frac{f}{g} \,:\, f, g,\ \text{homogeneous}, \deg g = \deg f, g \notin \mathfrak{P}'\}$$

PROPOSITION 5.3. *Let $Y = V(\mathfrak{P})$ be a projective variety, $p \in Y$ and $\mathfrak{m}_p$ the homogeneous maximal ideal corresponding to p. Then*

(1) $\mathcal{O}_{Y,p} \cong S(Y)_{(\mathfrak{m}_p)}$
(2) $K = k(Y) = S(Y)_{(0)} = \{\frac{f \pmod{\mathfrak{P}}}{g \pmod{\mathfrak{P}}} \,:\, f, g \in S, \deg f = \deg g, g \notin \mathfrak{P}\}$
(3) $\mathcal{O}(Y) = k$

Rational functions can always be written as a quotient of fractions where the denominator does not vanish in the localization. This is the same as requiring $g \notin \mathfrak{m}_p$.

PROOF. Let $Y_i = Y \cap U_i$ where U_i was the open subset $\{(a_0 \cdots a_n) \,:\, a_i \neq 0\}$ from before. Observation: there is a natural isomorphism

$$\varphi_i^* : A(Y_i) \to S(Y)_{(x_i)}$$

Assume $i = 0$. There is a map $k[x_1 \cdots x_n] \to S_{(x_i)}$ given by $f \mapsto f(\frac{x_1}{x_0} \cdots \frac{x_n}{x_0})$. *Wait... aren't we supposed to be dividing by everything that's* not x_0*?* Via this map, $\mathfrak{P}^{ih} \iff \mathfrak{P}S_{(x_0)}$. So $Y_0 = V(\mathfrak{P}^{ih})$.

Now (1) and (2) folow from the corresponding facts for affine varieties. That is, we can assume $p \in U_0 \cap Y = Y_0$. So

$$\mathcal{O}_{Y,p} = \mathcal{O}_{Y_0,p} = A(Y_0)_{m_p^{ih}} \xrightarrow{\sim} (S(Y)_{(x_0)})$$

$$\varphi_0^*(\mathfrak{m}_p^{ih}) = \mathfrak{m}_p S(Y)_{(x_0)}$$

The localization is transitive, so this is $S(Y)_{\mathfrak{m}_p}$.

Now we will do part (3): we want to show that $\mathcal{O}(Y) = k$. Let $f \in \mathcal{O}(Y) \hookrightarrow k(Y) = S(Y)_{(0)} \subset L$, where L is the fractional field of $S(Y)$. Recall that Y is covered by Y_i (which are each $Y \cap U_i$). Each regular function on Y gives a regular function on Y_i:

$$\mathcal{O}(Y) \to \prod_{i=0}^{n} \mathcal{O}(Y_i)$$

But $\mathcal{O}(Y_i) = A(Y_i) \cong S(Y)_{(x_i)}$. So for each i there is some N_i such that

$$\chi_i^{N_i} f \in S(Y)$$

Now let $N \geq \sum N_i$. Then $S(Y)_N \cdot f \subset S(Y)_N$. This is because every element in $S(Y)$ is spanned by monomials: $x_0^{r_0}, \cdots$ Now watch the degrees (?) So for all m, $S(Y)_N \cdot f^m \subset S(Y)_N$. Each $S(Y)_N$ is a finite-dimensional vector spaces. f is a map $S(Y)_N \to S(Y)_N$. There is some m such that

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0 : S(Y)_N \to S(Y)_N$$

implies $f^m + a_1 f^{m-1} + \cdots + a_m = 0$ in L.

($S(Y)_{(0)}$ requires that the denominator is always homogeneous; this is not a requirement in L.) k is algebraically closed in L, which implies $f \in k$. This proves that every regular function on a projective variety is a constant. $\boxminus$

5.1. Product of two varieties.

DEFINITION 5.4. Let X, Y be two varieties. The product $X \times_k Y$ is a variety together with two morphisms $X \times Y \to X$, $X \times Y \to Y$ (projections), such that for any other variety Z the natural map $\mathrm{Hom}(Z, X \times Y) \to \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$ is a bijection. So we have a diagram

$$\begin{tikzcd}
Z = \widetilde{X \times Y} \arrow[rr] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl, "pr_X"] \arrow[dr, "pr_Y"] & \\
& X & & Y
\end{tikzcd}$$

THEOREM 5.5. *The product of X and Y exists, and is unique up to a unique isomorphism.*

PROOF. Uniqueness is clear.

Existence: there is a natural variety structure on the set $X \times Y$, making it the product $X \times_k Y$.

(1) If X and Y are affine, then $U_i \times_k Y$ exists.
(2) Let $\{U_i\}$ be a cover of X, and each $U_i \times_k Y$ exists, then $X \times_k Y$ exists.

This proves the theorem, because you can cover X by affines $X = \cup U_i$ and $Y = \cup V_i$. Then each $U_i \times_k Y$ exists, because (by (2)) each $U_i \times_k V_j$ exists (by (1)). Applying (2) again says that $X \times_k Y$ exists.

$$\{\text{ affine varieties over } k\} \overset{\sim}{\to} \{\text{f.g. integral } k\text{-algebras}\}$$

In the forwards direction, $X \mapsto \mathcal{O}(X) = A(X)$. In the backwards direction, write $Spec(A) \leftarrow A$.

LEMMA 5.6. *Let Z be any variety, and X be affine.*

$$\operatorname{Hom}(Z, X) \to \operatorname{Hom}_{k-alg}(\mathcal{O}(X), \mathcal{O}(Z))$$

is bijective. (We proved this for quasi-affine varieties, but now we claim it for all varietes.)

PROOF. Cover $Z = \cup U_i$ by quasi-affines. Then $\operatorname{Hom}(Z, X) \to \prod_i \operatorname{Hom}(U_i, X)$ by restricting the elements of $\operatorname{Hom}(Z, X)$. We can further restrict them to the intersection of two quasi-affines:

$$\operatorname{Hom}(Z, X) \to \prod \operatorname{Hom}(U_i, X) \rightrightarrows \prod \operatorname{Hom}(U_i \cap U_j, X)$$

The middle set is the equalizer. $\operatorname{Hom}(Z, X)$ is the equalizer of

$$\prod_i \operatorname{Hom}(U_i, X) \rightrightarrows \prod \operatorname{Hom}(U_i \cap U_j, X)$$

Concretely, this means: a morphism $\varphi : Z \to X$ is an equivalence to $\varphi_i : U_i \to X$ such that

$$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$$

$$\mathcal{O}(Z) \to \prod \mathcal{O}(U_i) \rightrightarrows \prod \mathcal{O}(U_i \cap U_j)$$

(A function on Z is the same as a function on U_i that coincides on the intersection.)

$$\begin{tikzcd}
\operatorname{Hom}(Z, X) \arrow[r] \arrow[d, "\sim"] & \prod \operatorname{Hom}(U_i, X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\sim"] & \prod \operatorname{Hom}(U_i \cap U_j, X) \arrow[d, "\sim"] \\
\operatorname{Hom}(O(X), \mathcal{O}(Z)) \arrow[r] & \prod \operatorname{Hom}(\mathcal{O}(X), \mathcal{O}(U_i)) \arrow[r, shift left] \arrow[r, shift right] & \prod \operatorname{Hom}(\mathcal{O}(X), \mathcal{O}(U_i \cap U_j))
\end{tikzcd}$$

Check the injectivity/surjectivity by diagram chasing. (Exercise.)

Back to (1). Let X, Y be affine. $A = \mathcal{O}(X)$, $B = \mathcal{O}(Y)$. Then $A \otimes_k B$ is a finitely generated integral k-algebra.

$$A = k[x_1 \cdots x_n]/(f_1 \cdots f_m)$$
$$B = k[y_1 \cdots y_s]/(g_1 \cdots g_t)$$

Then

$$A \otimes_k B = k[x_1 \cdots x_n, y_1 \cdots y_s]/(f_1 \cdots f_m, g_1 \cdots g_t)$$

You can show that

$$Spec(A \otimes_k B) = V(f_1 \cdots f_m, g_1 \cdots g_t) \subset \mathbb{A}^{n+s}$$

The underlying set is the underlying set of X times the underlying set of Y.

CLAIM 5.7. *$Spec(A \otimes_k B) = X \times_k Y$*

$\mathrm{Hom}(Z, Spec(A \otimes B)) = \mathrm{Hom}(A \otimes_k B, \mathcal{O}(Z))$ (by the lemma) as k-algebras. By the definition of tensor products,

$$\cdots = \mathrm{Hom}(A, \mathcal{O}(Z)) \times \mathrm{Hom}(B, \mathcal{O}(Z)) = \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$$

□

EXAMPLE 5.8. $\mathbb{A}^n \times_k \mathbb{A}^m = \mathbb{A}^{n+m}$. But the topology on $\mathbb{A}^n \times \mathbb{A}^m$ is not the same as the product topology on $\mathbb{A}^n \times \mathbb{A}^m$.

Rest of the proof: homework.

□

EXAMPLE 5.9. $\mathbb{P}^n \times_k \mathbb{P}^m$ exists. What is it? We know that $\mathbb{P}^n = \cup_i U_i$, $\mathbb{P}^m = \cup_j V_j$. We have a lemma, which is proven in the homework.

LEMMA 5.10. *If $U \subset X$ open, then $U \times_k Y$ is equal to $U \times Y \subset X \times_k Y$ with the induced topology.*

$$\begin{aligned} \mathbb{P}^n \times_k \mathbb{P}^m &= \cup(U_i \times_k \mathbb{P}^m) \\ &= \cup_{i,j}(U_i \times_k U_j) \end{aligned}$$

where each piece is isomorphic to $\mathbb{A}^{n_m}$.

6. September 27

6.1. Products of projective varieties. Last time we defined the product of two varieties. Consider $\mathbb{P}^n \times_k \mathbb{P}^m$. If we write $\mathbb{P}^n = \cup U_i$ and $\mathbb{P}^m = \cup V_j$ where each U_i and V_j is isomorphic to the affine space, then we have an open cover

$$\mathbb{P}^n \times_k \mathbb{P}^m = \cup_{i,j} U_i \times_k V_j$$

where $U_i \times V_j \cong \mathbb{A}^n \times_k \mathbb{A}^m \cong \mathbb{A}^{n+m}$.

LEMMA 6.1. *The topology on $\mathbb{P}^n \times_k \mathbb{P}^m$ can be described as follows: closed subsets are of the form $V(f_1 \cdots f_N)$ where $f_i \in k[x_0 \cdots x_n, y_0 \cdots y_n]$ that are bi-homogeneous; i.e. we can write*

$$f_i = \sum_{\substack{a_0+\cdots+a_n=d \\ b_0+\cdots+b_m=e}} (coeff) x_0^{a_0} \cdots x_n^{a_n} y_0^{b_0} \cdots y_m^{b_m}$$

We call (d, e) the bi-degree of f_i.

PROOF. We know the topology on $U_i \times V_j$, because we have an isomorphism

$$U_i \times V_j \overset{\varphi_i \times \varphi_j}{\to} \mathbb{A}^n \times_k \mathbb{A}^m \cong \mathbb{A}^{n+m}$$

where the LHS has the Zariski topology. (When we write $\mathbb{A}^n \times_k \mathbb{A}^m$ we mean the topology inherited from this particular construction of product, not the usual product topology.) It suffices to show that

$$(\varphi_i \times \varphi_j)((U_i \times V_j) \cap V(f_1 \cdots f_N))$$

are the algebraic sets in $\mathbb{A}^{n+m}$. This is the same as when we showed that $\varphi_i : U_i \to \mathbb{A}^n$ is a homeomorphism.

$\boxminus$

Our next goal is to show that $\mathbb{P}^n \times \mathbb{P}^m$ is a *projective* variety: it is isomorphic to some projective variety (a closed subvariety of some $\mathbb{P}^k$).

DEFINITION 6.2. Let $\varphi : X \to Y$ be a morphism of (pre)varieties. (So far when we say "variety" we mean "prevariety"; we will remove this later by imposing an additional condition.) We say that φ is a closed embedding if φ is one-to-one on to an irreducible closed subset $\varphi(X) \subset Y$ (therefore $\varphi(X)$ has a variety structure) and $\varphi : X \to \varphi(X)$ is an isomorphism.

THEOREM 6.3. *Let $N = mn + m + n$. Then there is a natural closed embedding*

$$s : \mathbb{P}^n \times_k \mathbb{P}^m \to \mathbb{P}^N,$$

called the Segre embedding.

COROLLARY 6.4. *Let X, Y be projective. Then $X \times_k Y$ is projective.*

PROOF OF COROLLARY. This follows from the theorem and the following lemma:

LEMMA 6.5. *Let $\varphi : X' \to X$, $\psi : Y' \to Y$ be closed embeddings. Then $\varphi \times \psi : X' \times_k Y' \to X \times_k Y$.*

$\boxminus$

PROOF OF THE THEOREM. We will define the map $(a_0 \cdots a_n), (b_0 \cdots b_m) \to (a_i b_j)$ where $1 \leq i \leq n+1, 1 \leq b \leq m+1$. First, we show that it is a morphism. This is a local condition, so we can check on each piece of an affine open cover. Suppose $\mathbb{P}^n$ has coordinate functions $x_0 \cdots x_n$, $\mathbb{P}^m$ has coordinate functions $y_0 \cdots y_m$, and $\mathbb{P}^N$ has coordinate functions z_{ij}. Let $U_{ij} = \mathbb{P}^N - H_{ij}$ where $H_{ij} = \{z_{ij} = 0\}$.

$$\begin{tikzcd} U_i \times V_j \arrow[r, "s"] \arrow[d, "\varphi_i \times \varphi_j"] & U_{ij} \arrow[d, "\varphi_{ij}"] \\ \mathbb{A}^n \times \mathbb{A}^m \arrow[r] & \mathbb{A}^{mn+m+n} \end{tikzcd}$$

where the left vertical map takes $(a_0 \cdots a_n)(b_0 \cdots b_m) \mapsto (\frac{a_0}{a_i} \cdots, \widehat{\frac{a_i}{a_i}} \cdots \frac{a_n}{a_i}), (\frac{b_0}{b_j} \cdots \frac{b_m}{b_j})$. The right vertical map takes $(c_{ij}) \mapsto (\frac{c_{00}}{c_{i_0 j_0}} \cdots \frac{c_{mn}}{c_{i_0 j_0}})$ Then the bottom map is $\varphi_{i_0 j_0} \circ s \circ (\varphi_{i_0} \times \varphi_{j_0})^{-1}$. When you restrict s to $U_{i_0} \times V_{j_0}$ it's not hard to see that it goes to the right place.

$$\varphi_{00} \circ s \circ (\varphi_0 \times \varphi_0)^{-1} = (x_1 \cdots x_n)(y_1 \cdots y_n) \to (x_1 \cdots x_n, y_1 \cdots y_m, x_i y_i)$$

is a morphism.

We want to show that s is injective. Suppose $s(a,b) = s(a',b')$. This means that $a_i b_j = \lambda a'_i b'_j$ (they represent the same point in projective space), for any i, j. So there are some i_0, j_0 such that $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$. So $(a,b) \in U_{i_0} \times V_{j_0}$. $0 \neq a_{i_0} b_{j_0} = \lambda a'_{i_0} b'_{j_0}$ and so $(a', b') \in U_{i_0} \times V_{j_0}$.

Let $\mu = \frac{a_{i_0}}{a'_{i_0}}$ and $\nu = \frac{b_{j_0}}{b'_{j_0}}$. $\lambda = \mu \cdot \nu$. From

$$a_i b_{j_0} = \lambda a'_i b'_{j_0} \implies a_i = \mu a'_i \implies a = a'.$$

Similarly, $b = b'$.

Let $\mathfrak{P} = \ker(k[z_{ij}] \to k[x_0 \cdots x_n, y_0 \cdots y_m])$ where the map takes $z_{ij} \mapsto x_i y_j$. You can check that $\mathfrak{P}$ contains the ideal

$$\mathfrak{P}' = (z_{ij} z_{k\ell} - z_{i\ell} z_{kj})$$

Obviously, $s(\mathbb{P}^n \times \mathbb{P}^m) \subset V(\mathfrak{P}) \subset V(\mathfrak{P}')$. We now prove that the map is onto this closed subset: $s(\mathbb{P}^n \times \mathbb{P}^m) = V(\mathfrak{P}')$. In particular, $V(\mathfrak{P}) = V(\mathfrak{P}')$. So the radical of $\mathfrak{P}'$ is $\mathfrak{P}$. (Actually, $\mathfrak{P}'$ is the same as $\mathfrak{P}$.)

Let $(c_{ij}) \in V(\mathfrak{P}')$; this means that $c_{ij} c_{k\ell} = c_{i\ell} c_{kj}$. This means that there is some $c_{i_0 j_0} \neq 0.$, and hence $(c_{ij}) \in U_{i_0 j_0}$. Let $a_i = \frac{c_{ij_0}}{c_{i_0 j_0}} \in \mathbb{P}^n$ and $b_j = \frac{c_{i_0 j}}{c_{i_0 j_0}} \in \mathbb{P}^m$. So

$$s(a,b) = (a_i b_j) = \frac{c_{ij_0} c_{i_0 j}}{c_{i_0 j_0}^2} = \frac{c_{ij} c_{i_0 j_0}}{=} (c_{ij}) \in \mathbb{P}^N$$

From this argument we have that $s^{-1}(U_{ij}) = U_i \times_k V_j$.

Finally, we show that

$$s : \mathbb{P}^n \times_k \mathbb{P}^m \to V(\mathfrak{P})$$

is an isomorphism iff

$$s^{-1} : V(\mathfrak{P}) \to \mathbb{P}^n \times_k \mathbb{P}^m$$

is a morphism. But we only need to check on affine opens.

$$s^{-1} : V(\mathfrak{P}) \cap U_{i_0 j_0} \to U_{i_0} \times V_{j_0}$$

We need to check this is a morphism. Consider $i_0 = j_0 = 0$. We have

$$\begin{tikzcd} s^{-1} : V(\mathfrak{P}) \cap U_{i_0 j_0} \arrow[r, "s^{-1}"] \arrow[d, hook] & U_{i_0} \times V_{j_0} \\ \mathbb{A}^{mn+m+n} \cong U_{i_0 j_0} \arrow[ur, "pr_{first\ m+n\ coords}"] & \end{tikzcd}$$

The vertical map is a closed embedding, and hence a morphism. The composition is then a morphism. We are done. The reverse direction of the diagonal map is $(a_1 \cdots a_n, b_1 \cdots b_m) \mapsto (a_1 \cdots a_n, b_1 \cdots b_m, b_{ij})$.

$\boxminus$

6.2. An important class of projective varieties.

DEFINITION 6.6. A hyperplane in $\mathbb{P}^n$ is defined by a linear polynomial $V(a_0x_0 + \cdots . + a_nx_n)$ (a hypersurface where the defining polynomial is linear). A *linear variety* in $\mathbb{P}^n$ is the intersection of hyperplanes. A linear variety of dimension r is also called an r-plane. 1-planes are called lines. For example, $H_i = \{x_i = 0\}$.

LEMMA 6.7.

(1) *Let $X \subset \mathbb{P}^n$ be an r-plane. Then $X \cong \mathbb{P}^r$ and $I(X)$ can be generated by $n - r$ linear polynomials. (i.e. the intersection of planes is a plane!)*
(2) *There is a natural bijection between $(r+1)$-dimensional sub-vector-spaces in k^{n+1} and r-planes in $\mathbb{P}^n$. (Just consider $L \mapsto X = \{(a_0 \cdots a_r) \in L = \{0\}\}/\sim$; this turns out to be a bijection.)*

DEFINITION 6.8. The Grassmannian $G(r, n)$ is the set of all r-dimensional subspaces in k^n (equivalently, the $(r-1)$-planes in $\mathbb{P}^{n-1}$). For example, $G(1, n) = \mathbb{P}^{n-1}$.

NOTATIONS 6.9.

- $G(r, n)$ is also denoted as $\mathbb{G}(r-1, n-1)$.
- Let V be a finite-dimensional k-vector space. Then $G(r, V)$ is the set of all r-dimensional subspaces in V. (No need to choose a basis.) In particular, $G(1, V)$ is denoted by $\mathbb{P}(V)$.

THEOREM 6.10. *$G(r, V)$ is naturally a smooth projective variety of dimension $r(n-r)$, where $n = \dim V$.*

PROOF. We could try to embed this in a projective space. Or, we could give it a cover which gives it a variety structure, and then embed into projective space. We will do the latter.

Let $\{e_1 \cdots e_n\}$ be a basis for V. For each $I \subset \{1 \cdots n\}$ of cardinality $n-r$ consider $k^I = span\{e_i, i \in I\} \subset V$. Let

$$V_I = \{L \subset V \,:\, \dim L = r, L \cap k^I = 0\} \subset G(r,n)$$

Notice that

$$G(r,V) = \bigcup_{\substack{I \subset \{1 \cdots n\} \\ |I| = n-r}} V_I.$$

We claim that there is a natural bijection $\varphi_I : V_I \xrightarrow{\sim} \mathbb{A}^{r(n-r)}$. In fact, there is a natural bijection

$$V_I \xrightarrow{\sim} \mathrm{Hom}(V/k^{I^c}, k^I)$$

where the second thing is $\cong M_{(n-r)\times r}(k)$, the set of $(n-r) \times r$ matrices $\cong \mathbb{A}^{(n-r)r}$ (and I^c means the complement of I). The thing on the left is r-dimensional, therefore we can take the graph. The backwards map is $\varphi \mapsto V = k^{I^c} \oplus k^I \supset \Gamma\varphi$. The other direction takes L to the map

$$L \subset V \twoheadrightarrow k^{I^c}, L \subset V \twoheadrightarrow k^I A$$

Explicitly, this means each $L \subset V_I$ determines uniquely

$$(v_1 \cdots v_r) = (e_1 \cdots e_n)$$

Each $L \subset v_i$ is uniquely given by a basis as follows: have the matrix A where the top half is the identity matrix and the bottom is

$$\begin{pmatrix} a_{r+1,1} & \cdots & a_{r+1,r} \\ & \ddots & \\ a_{n,1} & \cdots & a_{n,r} \end{pmatrix}$$

It is not hard to show $\varphi_J \circ \varphi_I^{-1}$ is a morphism. Therefore, there is a variety structure on $G(r,n)$.

$\boxminus$

We want to construct a closed embedding $\psi : G(r,n) \to \mathbb{P}^N$. There is a natural closed embedding; this is called the Plücker embedding. Let $W = \Lambda^r V$. Then W is a vector space with $\dim W = \binom{n}{r}$. For each $L \in G(r,n)$ we can associate a vector, up to scalar. Namely, we choose a basis $v_1 \cdots v_r \in L$ (for example, the standard basis). This gives us a vector $v_1 \wedge \cdots \wedge v_r \in W$. This depends on the basis, but the difference depends on the determinant of the transition matrix. So it is unique up to scalar. We can define a map

$$\psi : G(r,n) \to \mathbb{P}(W) \qquad L \mapsto v_1 \wedge \cdots \wedge v_r = [L]$$

This is called the Plücker embedding. Consider $\psi : V_I \to \mathbb{P}(W)$, where $I = \{r+1 \cdots n\}$. We have standard coordinates (a_J) with $J \subset \{1 \cdots n\}$ and $|J| = r$. Every element in W can be written

$$\sum a_J e_{j_1} \wedge \cdots \wedge e_{j_r}$$

where $J = \{j_1 < \cdots < j_r\}$. We have

$$(v_1 \cdots v_r) = (e_1 \cdots e_n)A$$

where A was the matrix from earlier; the map takes this to the $r \times r$ minors. a_J is the determinant of something, where the rows are chosen according to J. This is a polynomial map.

Next, we show that ψ is injective. We can recover L by its image.

$$L = \{v \in V \,:\, v \wedge [L] = 0 \text{ in } \Lambda^{r+1}V\}$$

In other words, L is the kernel of $\Lambda[L]$ where $\Lambda[L] : \Lambda^{r+1}V$ is the map that takes $v \mapsto v \wedge [L]$.

Now we want to characterize its image: it is an irreducible closed subset in W, but we will not write down the generators of the basis. Let $w \in W$.

QUESTION 6.11. When can w be written as $w = v_1 \wedge \cdots \wedge v_r$? (When this happens, we say it is decomposable.)

Obvious condition: the kernel of $\wedge w$ has dimension $\geq r$, where

$$\wedge w : V \to \Lambda^{r+1}V$$

We have the following easy lemma:

LEMMA 6.12. *Let $w \neq 0 \in W$. Then $\dim\ker(\wedge w) \leq r$, and equality holds iff w is decomposable.*

PROOF. Use linear algebra. ⊟

We have

$$\delta : W \to \mathrm{Hom}(V, \Lambda^{r+1}V) \supset \{\varphi \,:\, rk\varphi \leq n - r\}$$

and $\psi(G(r,n))$ is exactly $\delta^{-1}\{\varphi \,:\, rk\varphi \leq n - r\}$.

7. September 29

We continue to study the Grassmannian, $G(r, V) = \{r - \textit{planes in } V\}$. We constructed a map

$$\psi : G(r,n) \to \mathbb{P}(W)$$

and showed that it was injective. We want to describe the image of ψ. Let $w \in W - \{0\}$. When does $w = v_1 \wedge \cdots \wedge v_r \iff w \in \psi(G(r,n))$? (If w has this property we say that it is *decomposable*.) Then if $\wedge w : V \to \Lambda^{r+1}V$, then

$$w \in Im\psi \iff \dim\ker(\wedge w) \geq r$$

Consider

$$\delta : W \to \mathrm{Hom}(V, \Lambda^{r+1}V) \quad w \mapsto \wedge w$$

A linear map induces a map between projectivizations of vector spaces. So $\wedge$ induces a map

$$\delta : \mathbb{P}(W) \to \mathbb{P}\operatorname{Hom}(V, \Lambda^{r+1}V)$$

Let $Z_r(V, \Lambda^{r+1}V) \subset \mathbb{P}\operatorname{Hom}(V, \Lambda^{r+1}V)$ be the set

$$\{\varphi : V \to \Lambda^{r+1}V \ : \ rk\varphi \leq n-r\}$$

Then $\psi(G(r,n)) = \delta^{-1}(Z_{n-t}(V, \Lambda^{r+1}V))$.

Digression. In general, let V_1, V_2 be two vector spaces, and let $Z_d(V_1, V_2) \subset \mathbb{P}\operatorname{Hom}(V_1, V_2)$ be defined as $\{\varphi : V_1 \to V_2 \ : \ rk\varphi \leq d\}$. Then $Z_d(V_1, V_2)$ is an algebraic set in $\mathbb{P}\operatorname{Hom}(V_1, V_2)$. In fact, $Z_d(V_1, V_2)$ is a projective variety called the determinantal variety. (It is the variety generated by the $(d+1)\times(d+1)$ minors.) This shows that $\psi(G(r,n))$ is closed, and we have characterized the image as $\psi(G(r,n)) = \delta^{-1}(Z_{n-r}(V, \Lambda^{r+1}V))$. (When $d = 1$ Z_d is smooth; otherwise it is singular.)

REMARK 7.1. Recall we have defined the embedding $s(\mathbb{P}^n \times \mathbb{P}^m)$. This is exactly the determinantal variety $Z_1(k^{n+1}, k^{n+1})$. (We wrote down quadratic polynomials that generated the variety...)

determinant

We want to show that $\psi : G(r,n) \to \psi(G(r,n))$ is an isomorphism of varieties. First we need the target to be a variety. (Why is it irreducible?) First we need it to be a homeomorphism, and you get the variety structure from there. In fact, it is an isomorphism of projective varieties.

It is enough to cover this with open affines, and show that each piece works out. So take the restriction

$$\psi : V_I \to U_I = \mathbb{P}(W) - \{a_I = 0\}$$

Elements in W can be expressed as $\sum a_J e_J$, where $e_J = e_{j_1} \wedge \cdots \wedge e_{j_r}$ for $j_1 < \cdots < j_r$ and $J = \{J_1 \cdots J_r\}$. Also, $U \subset \{1, 2, \cdots, n\}$ and $|I| = n-r$. Recall that elements can be expressed as a matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ a_{r+1,1} & \ddots & a_{r+1,n} \\ & \ddots & \\ a^{n,r} & & a_{n,r} \end{pmatrix}$$

Take the $r\times r$ minors of this matrix; take one row upstairs and one row downstairs (uhhh?). Let $a_J =$ the set of $r \times r$ minors for J, where $J = \{1, 2, \cdots \widehat{i}, \cdots, r\} \cup j$. $a_J = a_{ji}$. We have

$$\begin{tikzcd} V_I \arrow[r, shift left] & U_I \cap \psi G(r,n) \arrow[l, shift left, "\psi^{-1}"] \arrow[dl] \\ U_I \arrow[u] & \end{tikzcd}$$

Somehow this shows that ψ^{-1} is continuous and in fact a morphism.

Now we can say that the Grassmannian is a projective variety.

REMARK 7.2. The ideal of

$$\psi G(r,n)$$

can be generated by quadratic polynomials, called the Plücker relations.

DEFINITION 7.3. A *correspondence* from varieties X to Y is a relation given by a closed subset $Z \subset X \times_k Y$. (Since X and Y are varieties, now we know that the product exists, and that it is a closed subset.) Assume X, Y quasi-projective. Z is said to be a rational map if Z is irreducible and there is an open subset$X_0 \subset X$ such that every point $x \in X_0$ is related to exactly one point in Y.

Z is said to be birational if $Z \subset X \times_k Y$ and $Z^{-1} = \{(y,x) \in Y \times_k X \,:\, (x,y) \in Z\} \subset Y \times_k X$ are both rational.

REMARK 7.4. From now on we will always write $X \times Y$ to mean $X \times_k Y$.

NOTATION 7.5. A rational map from X to Y is also denoted by $f : X \dashrightarrow Y$.

EXAMPLE 7.6. Let X be projective, and $f \in k(X)$ be a rational function. Then f gives rise to a rational map $f : X \dashrightarrow \mathbb{P}^1$. The rational functions are:

$$k(\mathbb{P}^1) = k(t) \to k(X) \quad x \mapsto f$$

$k(U)$ is the fractional field of $\mathcal{O}(U)$. $k(X) - \{0\}$ is the same as k-algebra homomorphisms $k(t) \to k(X)$. Because X is projective, f can be written as a ratio of two homogeneous polynomials of the same degree: $f = \frac{g}{h}$ for $g, h \in k[x_0 \cdots x_n]$ (t must be invertible; it cannot map to zero.) Define

$$X_0 = X - (\{g = 0\} \cap \{h = 0\})$$

Now define

$$F : X_0 \to \mathbb{P}^1 \quad p \mapsto (g(p), h(p))$$

This is well-defined because we were assuming $\deg g = \deg h = d$. Now let y_0, y_1 be the homogeneous coordinates on $\mathbb{P}^1$. Then $V(y_0 h - y_1 g) \subset \mathbb{P}^n \times \mathbb{P}^1$ where the things inside V are bi-homogeneous of degree $(d, 1)$. Let

$$Z = V(y_0 h - y_1 g) \cap X.$$

Over X_0, the intersection

$$pr^{-1}(x_0) \cap Z \text{ inside } X \times \mathbb{P}^1$$

is just the graph of F, i.e.

$$pr^{-1}(x_0) \cap Z = \{(x, F(x)) \,:\, x \in X_0\}$$

In fact, $pr : Z \cap pr^{-1}(x_0) \to x_0$ is an isomorphism. There is a map backwards by the universal property of the product.

Now let Z^* be the closure of $Z \cap pr^{-1}(x_0)$ in $X \times \mathbb{P}^1$. When you restrict to X_0, you have a graph; but it could be more complicated (not irreducible) outside X_0. But inside X_0, $Z \cap pr^{-1}(X_0)$ is irreducible. Taking the closure Z^* is irreducible and $Z^* \cap pr^{-1}(X_0) = Z \cap pr^{-1}(X_0)$. Therefore, Z^* is a rational map. Every point x over X_0 is related to $F(x) \in \mathbb{P}^1$ by Z^*.

If $k(Y) \subset k(X)$ then we get a rational map $X \dashrightarrow Y$.

EXAMPLE 7.7 (Projection from a point, blow up). Let $O = (0, 0, \cdots, 1) \in \mathbb{P}^n$. Let $p : \mathbb{P}^n - \{0\} \to \mathbb{P}^{n-1}$ take $(a_0, \cdots, a_n) \to (a_0, \cdots, a_{n-1})$ be the projection. p is a morphism. $p^{-1}(p(x))$ is the line joining O and x, without the point O. You can show that there is a unique line through two points in projective space.

Consider $Z = V(x_i y_j - y_i x_j)_{0 \le i,j \le n-1} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. $Z = \Gamma p \cup pr^{-1}(O) = \Gamma p \cup \mathbb{P}^{n-1}$. The whole fiber is always inside Z. Call the embedded $\mathbb{P}^{n-1} = E$. By definition, Z is closed.

CLAIM 7.8. *Z is irreducible.*

PROOF. $Z \cap pr^{-1}(\mathbb{P}^n - \{0\})$. This is nothing but the graph $\Gamma p = \{(x, p(x)) \in (\mathbb{P}^n - \{0\}) \times \mathbb{P}^{n-1}\}$. Then Γp is irreducible. To prove the claim, it is enough to show that $Z = \overline{\Gamma p}$. (The closure of an irreducible topological space is irreducible.) Let $q \in \mathbb{P}^{n-1}$ let $\ell_q = p^{-1}(q) \cup \{0\}$. ℓ^q is a line in $\mathbb{P}^n$. $\ell_q \subset \mathbb{P}^n \times \{q\} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ and $(\ell_q - \{0\}) \times \{\} \subset \Gamma p \subset Z$. In particular, $(O, q)\ell_q \times \{q\}$ lands in the closure $\overline{\Gamma p}$. (The point is in the closure of the graph.) $E = \{O\} \times \mathbb{P}^{n-1} \subset \overline{\Gamma p}$ implies $Z \subset \overline{\Gamma p}$. ⊟

So Z is irreducible, and is a projective variety called the blowup of $\mathbb{P}^n$ at O. When you restrict $pr : Z \to \mathbb{P}^n$, then

$$pr^{-1}(O) \cong P^{n-1} = E$$

where E is called an exceptional $\mathbb{P}^{n-1}$ divisor, and $pr : Z - E \to \mathbb{P}^n$ is 1-1. (You replace the origin by $\mathbb{P}^{n-1}$.) (Imagine turning a pair of intersecting curves into a helix; the crossbars are the $\ell_q \times \{q\}$.)

DEFINITION 7.9. Let $X \subset \mathbb{P}^{n-1}$ be quasi-projective. Assume $O \in X$. The blowup of X at O is $\overline{pr^{-1}(X - O)} \subset Z$.

$$\begin{tikzcd} X - \{0\} \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r, hook] \arrow[d, "pr"] & Z \arrow[d, "pr"] \\ X - \{0\} \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

where $\widetilde{X} = pr^{-1}(X - O) \subset Z$. The Z in the top right is the blow up of $\mathbb{P}^n$ at O.

Now, for arbitrary quasi-projective variety containing this point, we can define the blow up as the preimage of $X - \{0\}$ in Z?

EXAMPLE 7.10. $y^2 = x^3 + x^2 \subset \mathbb{A}^2 \subset \mathbb{P}^2$ contains the point $O = (0,0,1) \in X$. Let $\widetilde{X}$ be the blow up of X at O. (Remember, this has a loop at the origin.) Let (u,v) be coordinates on $\mathbb{P}^1$. Then the blow-up is given by

$$Z = V(xu - yv) \subset \mathbb{P}^2 \times \mathbb{P}^1$$

Now take the blowup of $\widetilde{X}$. Let

$$pr^{-1}(X) \subset \mathbb{P}^2 \times \mathbb{P}^1$$

The whole preimage is given by

$$V(y^2z - x^3 - x^2z)$$

Look at $pr^{-1}(X) \cap Z \supset \widetilde{X}$. To get the intersection, we want to solve

$$\begin{cases} xu - yv = 0 \\ y^2z - x^3 - x^2z = 0 \end{cases} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

Just cover $\mathbb{P}^1$ by two affines: $u \neq 0$ and later $v \neq 0$. That is if we have we are solving the equations

$$\begin{cases} xu - y = 0 \\ y^2 - x^3 - x^2 = 0 \end{cases} \implies \begin{cases} y = xu \\ x^2(u^2 - x - 1) \end{cases}$$

This is not irreducible; it is given by two components

$$V(y - x, x^2(u^2 - x - 1)) = V(y - xu, u^2 - x - 1) \cup V(u - xu, x^2)$$

(We're trying to calculate $pr^{-1}(X) \cap Z$.) The second thing is the exceptional locus $E \cap (\mathbb{A}^2 \times \mathbb{A}^1)$. The first thing is $\widetilde{X}$, because when you map to X it is an isomorphism away from zero. (The second one is totally contained in E.)

$E \cup \widetilde{X}$ is the preimage of X. We can remove E because it isn't in the closure of ...something? We will see that $\widetilde{X}$ is smooth. $pr^{-1}(O)$ is given by $x = 0, y = 0; u = 1, v = 1; x = 0, y = 0; u = -1, y = 1$.

8. October 4

We will show that the class of algebraic varieties contains all the quasi-projective varieties. Here is a corrected definition:

DEFINITION 8.1. A rational map is a correspondence $Z \subset X \times Y$ irreducible, such that there is some $X_0 \subset X$ open and $pr_X(Z \cap pr_X^{-1}(X_0)) \to X_0$ is an isomorphism.

(The previous definition works only when the characteristic is zero. Example: consider the Frobenius map $F : \mathbb{A}^1 \to \mathbb{A}^1$, where $x \mapsto x^p$. The graph $\Gamma_F \subset \mathbb{A}^1 \times \mathbb{A}^1$; the projections are bijections. However, this is not a birational map; we will see that birational maps induce an isomorphism of the function field. This does not: $k(X) \to k(X)$ is an injective map of fields, but not an isomorphism. The graph is always isomorphic to the domain, but the projection to Y is not an isomorphism.)

Let $O = (0, \cdots, 0, 1) \in \mathbb{P}^n$. We have a projection: $p : \mathbb{P}^n - \{0\} \to \mathbb{P}^{n-1}$. We get the blowup $Bl_O(\mathbb{P}^n) = Z \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ that has projections to each factor. If we have a subvariety $X \ni O$, then take $pr^{-1}(X - \{0\})$ and take the Zariski closure; this is $Bl_O X$. The preimage of O is called E, which was called an exceptional divisor; it is isomorphic to $\mathbb{P}^{n-1}$. (For example, the blowup of a nodal curve was the union of the exceptional divisor, a copy of $\mathbb{P}^1$, and the curve whose singular point had been separated into another dimension. The original curve was not smooth, but after blowing up it is smooth.)

8.1. Resolution of singularity problem. Given an algebraic variety X, find $Y \to X$ a birational surjective morphism with Y smooth.

Answer: Yes, if the characteristic is zero (Hironaka). Unknown, if $char(k) > 0$, unless the dimension is 1 or 2. A common technique is to blow up the singular locus many times, and we hope that we get something smooth.

Example 8.2 (Incidence correspondence). Let $C \subset \mathbb{G}(r, n) \times \mathbb{P}^n$. Define

$$C = \{(L, x) \,:\, x \in L\}$$

We claim that C is a correspondence. In fact, C is a smooth projective variety of dimension $(r + 1)(n - r)r$.

Proof. First, why is it closed? $\mathbb{P}^n = \mathbb{P}(V)$, $\dim V = n + 1$. Let $e_0 \cdots e_n$ be a basis of V. Let $W = \Lambda^{r+1}V$. We had constructed the Plücker embedding:

$$\mathbb{G}(r, n) \times \mathbb{P}^n \overset{\psi\times ID}{\to} \mathbb{P}(W) \times \mathbb{P}^n \quad \text{where } (L, x) \mapsto ([L], x)$$

$\sum a_I e_I \in W$, $I \subset \{0, \cdots, n\}$ with cardinality $r + 1$. $x \in [L]$ iff $[L] \wedge x = 0$, a.k.a. $\sum a_I e_I \wedge \sum x_i e_i = 0$; this is a bihomogeneous polynomial of degree (1,1). This gives an equation $F(a_I, x) = 0$. So we can write

$$C = V(F) \cap (\psi(\mathbb{G}(v, n)) \times \mathbb{P}^n)$$

We need to show that it is irreducible and smooth. To do this, we claim that for each $I \subset \{0, \cdots, n\}$ with cardinality $n - r$ we have

$$pr^{-1}(V_I) \cap C \cong V_I \times \mathbb{P}^r$$

(It has an open cover such that the open subset is smooth. Check that the RHS is irreducible smooth of dimension $(r + 1)(n - r) + r$; use the homework that $\dim X \times Y = \dim X + \dim Y$.) I parametrizes coordinates in an $n - r$ dimensional vector space; we get a map $I \to k^I$.

You can get a $(n - r - 1)$-plane $k^I/Image := M \subset \mathbb{P}^n$ so $M = \{(0, \cdots, 0, a_1 \cdots a_{n-r-1})\}$ We get a rational map

$$p_M : \mathbb{P}^n \to \mathbb{P}^r \quad \text{where } (a_0 \cdots a_n) \mapsto (a_0 \cdots a_r)$$

and $p_M^{-1}(y)$ is an $(n - r)$-plane. I want to construct a map

$$C \cap pr^{-1}(V_I) \to V_I \times \mathbb{P}^r \quad (L, x) \mapsto (L, p_M(x))$$

Remember $L \cap k^I = \{0\}$ so $x \in L \implies x \notin M$. ($M = \mathbb{P}(k^I)$, and you get an isomorphism between $\mathbb{P}^r$ and $\mathbb{P}(L)$.) $\boxminus$

DEFINITION 8.3. A morphism $f : X \to Y$ is called *dominant* if $f(X)$ contains an (open) dense subset. (Equivalently (?) require that the image is dense). A rational map $Z \subset X \times Y$ is called *dominant* if $pr_Y : Z \to Y$ is dominant.

In the homework we have another notion of rational maps: if $U \subset X$ is open, and $f : U \to Y$ is a morphism. We call this

$$Maps(X,Y)^{rat'} = \{(U,f) \,:\, U \subset X, f : U \to Y\}$$

There is an isomorphism from this to $Maps(X,Y)^{rat}$ given by taking the closure of the graph of f. In the homework, we see that this is injective. But using the new definition, we see that it is bijective. If there is some $X_0 \subset X$ such that the restriction $pr^{-1}(X_0) \cap Z \to X_0$ is an isomorphism, construct $U = X_0$ and f to be the composition of this map with the map to Y. The dominant condition just means: Z is dominant iff $f : U \to Y$ is dominant.

If $f : X \dashrightarrow Y$ is rational, then f^* is the induced map $k(Y) \to k(Y)$. If $g \in k(Y)$ then g is a regular function of some open $V \subset Y$: $g \in \mathcal{O}(V)$. Then $g \circ f \in \mathcal{O}(f^{-1}(V)) \subset k(X)$.

PROPOSITION 8.4. *Consider the functor*

F : (*alg.var*)+(*dominant rational maps*) $\to$ (*f.gen. field extn of* k)+(k*-alg. homomorphisms*)

Then F *is an equivalence of categories.*

PROOF. We have a natural map

$$Maps(X,Y)^{rat,dom} \to Hom_{k.alg}(X,Y)$$

We need first to show that this is bijective; we do this by constructing an inverse. Let $\theta : k(Y) \to k(X)$. The function field of a variety is the same as the function field of any of its open subsets. So we can let $Y_0 \subset Y$, $X_0 \subset X$ be affine opens, and $Y_0 \subset \mathbb{A}^n$, etc. There are some coordinate functions $(y_1 \cdots y_n)$. So $\theta(y_i)$ is a rational function on X. $\theta(y_i)$ is a rational function, so $\theta(y_i) = \frac{f_i}{f_0}$ for $f_i \in \mathcal{O}(X_0)$ and $f_0 \neq 0$. Let $U = X_0 - \{f_0 = 0\} \neq \emptyset$. $\theta(y_i) \in \mathcal{O}(U)$. A morphism $U \to Y$ that is affine is equivalent to a map $\mathcal{O}(Y_0) \to \mathcal{O}(U)$. This gives a map in the other direction.

(We claim that this is dominant: image dense $\iff$ image contains an open dense subset. If $U \to Y_0$ factors through $Z \subset Y_0$ then choose $f \in I(Z) \subset \mathcal{O}(Y_0)$. $Z \subset V(f)$. If it vanishes on a set, then it vanishes on the pullback.) $\boxminus$

REMARK 8.5. The birational maps correspond to isomorphism; they are the invertible morphisms in these categories.

Let $K = k(x_1 \cdots x_r)$ be a finitely generated field over k. Take $A = k[x_1 \cdots x_n] \subset K$, a finitely generated ring. Under the equivalence of categories, denote the image of A as $Spec(A)$; $k(X) = K$.

COROLLARY 8.6. *Every algebraic variety* X *is birational to a hypersurface in* $\mathbb{P}^n$.

PROOF. Let $K = k(X)$ be finitely generated over k. From commutative algebra, k is algebraically closed so it is perfect. Therefore, K/k is separably generated. That is, there are some $x_1 \cdots x_r$ transcendental over k such that $K/(k(x_1 \cdots x_r))$ is finite and separable (and hence generated by one element). So there is some $y \in K$ such that

$$K = k(x_1 \cdots x_n)(y).$$

y has some minimal polynomial $y^n + a_1(x_1 \cdots x_r)y^{n-1} + \cdots + a_n(x_1 \cdots x_r) = 0$; each a_i is an element of the field $k(x_1 \cdots x_n)$ of rational functions, and so they can be written as the ratio of two polynomials. So we can write $f(x_1 \cdots x_r, y) = 0$ for $f \in k[x_1 \cdots x_r, y]$.

We have $V(f) \subset \mathbb{A}^{r+1}$ an affine variety of dimension r and the function field $k(V(f)) = K$. Therefore, X is birational to $V(f)$, and therefore to $\overline{V(r)} \subset \mathbb{P}^{r+1}$. (Birationally, every variety is the same as a hypersurface.) ⊟

THEOREM 8.7. *The projection $\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{p_2} \mathbb{P}^m$ is a closed map. That is, if every $Z \subset \mathbb{P}^n$ is closed, then $p_2(Z)$ is closed.*

We will prove this next time. But this is not the same as for affine varieties.

REMARK 8.8. We claim that $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1$ is not closed. For example, consider $xy = 1$: $V(xy - 1) \mapsto \mathbb{A}^1 - \{0\}$.

COROLLARY 8.9. *Let X, Y be projective varieties. Let $Z \subset X \times Y$ be a correspondence. Then $pr_Y(Z)$ is closed.*

COROLLARY 8.10. *Let $Z \subset \mathbb{G}(r, n)$ be a closed subvariety. Then the union*

$$\bigcup_{L \in Z} L \subset \mathbb{P}^n$$

is a projective variety.

PROOF. C has projections to $Z \subset \mathbb{G}(r, n)$ and to $\mathbb{P}^n$. Since Z is irreducible, $pr_1^{-1}(Z)$ is irreducible. $\cup L \subset \mathbb{P}^n = p_2 p_1^{-1}(Z)$. ⊟

COROLLARY 8.11. *Let $X \subset \mathbb{P}^n - \{0\}$. Then $p_0(X)$ is a projective variety in $\mathbb{P}^{n-1}$. We have $Z = Bl_0\mathbb{P}^n$ with maps to $\mathbb{P}^n$ and $X \subset \mathbb{P}^n$. its image $p_1^{-1}(X)$ is still closed. The projection is exactly the projection $p_0(X)$; hence closed.*

9. October 6

THEOREM 9.1. *$p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.*

DEFINITION 9.2. An algebraic variety X is called *complete* if for any algebraic variety Y, $X \times Y \to Y$ is closed.

The theorem implies: projective varieties are complete. (But it's not obvious.) Let X, Y, Z be projective.

COROLLARY 9.3. *If $Z \subset X \times Y$ is a correspondence, then $pr_Y(Z)$ is closed.*

COROLLARY 9.4. *Let $A \subset X \times Y$, $B \subset Y \times Z$ be correspondences. Then*
$$C =: B \circ A = \{(x, z) \in X \times Z \ : \ \exists y \in Y, (x, y) \in A, (y, z) \in B\}$$
is a correspondence. (So you can compose correspondences.)

(If A and B are graphs of morphisms, then C is the graph of the composition.)

PROOF. Consider $X \times Z \overset{p_{12},p_{23},p_{13}}{\to} X \times Y, X \times Z, Y \times Z$. Then $C = p_{13}(p_{12}^{-1}(A) \cap p_{23}^{-1}(B))$ ⊟

COROLLARY 9.5. *Let $X \subset \mathbb{P}^n$ be projective, with $O \notin X$. Then $p_0(X)$ is projective and $p_0 : X \to p_0(X) = X'$ has finite fibers. In addition, $\dim X = \dim P_0(X)$ and $S(X)S^n/I(X)$ is a finite module over $S(X') = S^{n-1}/I(X')$.*

PROOF. For any point $y \in \mathbb{P}^{n-1}$, $p_0^{-1}(Y) \cong \mathbb{A}^1$ and therefore $p_0^{-1}(y) \cap X$ is closed in $\mathbb{A}^1$. It can't be the whole $\mathbb{A}^1$; X does not contain O. So $p_0^{-1}(y) \cap X$ is finite. (What are the closed things in $\mathbb{A}^1$!)

Say $O = (0, \cdots, 0, 1)$. Because $O \notin X$, there is some $f \in I(X)$ such that $f(0) \neq 0$ and $f = x_n^d + a_1(x_0 \cdots x_{n-1})x_n^{d-1} + \cdots + a_d(x_0 \cdots x_{n-1})$. So $S(X)$ is generated over $S(X')$ by $1, x_n, \cdots, x_n^{d-1}$. (It's a finite module.) So $S(X)$ is finite over $S(X')$, which implies
$$tr.d._k S(X) = tr.d._k S(X')$$
Transcendental degree over an integral domain means the transcendental degree of the fraction field. In the homework we defined the cone as a projective variety. But if we remove one hyperplane, we can consider this as the affine cone; the coordinate ring is exactly the affine coordinate ring. So the previous formula is
$$tr.d._k S(X) \overset{why}{=} \dim C(X) = \dim C(X') = tr.d._k S(X') \implies \dim X = \dim X'$$

⊟

COROLLARY 9.6 (Noether's normalization lemma). *Let $X^r \subset \mathbb{P}^n$ be a projective variety. (The superscript X^r means its dimension is r.) Then there is some $(n-r-1)$-plane L such that $X \cap L = \emptyset$ and $p_L : X^r \to \mathbb{P}^r$ has finite fibers, and $S(X^r)$ is finite over $S^r = S(\mathbb{P}^r)$ (homogeneous polynomial ring of $r+1$ variables.)*

If $L = (0, \cdots, 0, a_{r+1} \cdots, a_n)$ then the projection P_L means you project to the first r coordinates.

PROOF OF COROLLARY. By induction. Choose one point, and use induction on $n - r$. If $n - r = 1$ then there is nothing to do: choose a point, and do the projection $p_0 : X \to$

$P_0(X) \subset \mathbb{P}^{n-1}$. The target space is closed, and has the same dimension of X; therefore, they coincide. In general, choose a point $0 \notin X$, and do the projection

$$P_0 : X \to P_0(X) = X' \subset \mathbb{P}^{n-1}$$

Choose some $(n-1)-(r-1)$ plane $L' \subset \mathbb{P}^{n-1}$. Then

$$P_{L'} : X' \to \mathbb{P}^r, P_{L'} \circ P_0 = P_{span(L',0)}$$

But L is the unique linear variety containing L' and the point O. ⊟

PROOF OF THEOREM. Let $Z - V(f_1 \cdots f_r)$ be an algebraic set of $\mathbb{P}^n \times \mathbb{A}^m$. Closed subsets in this product are given by polynomials which are homogeneous in the first coordinates, and not in the second half of the coordinates. Let $f_i = f_i(x_1 \cdots x_n, y_1 \cdots y_n)$ be homogeneous of degree d_i in X. $p_2(Z) \not\ni b \iff \{f_i(x,b)\}$ do not have common zeroes in $\mathbb{P}^n$. Equivalently, there is some N such that $(x_0 \cdots x_n)^N \subset (f_1(x,b) \cdots f_r(x,b))$ (the radical). Let

$$U_N = \{b \in \mathbb{A}^m : (x_0 \cdots x_n)^N \subset (f_1(x,b) \cdots f_r(x,b))\}$$

$A^m - p_2(Z) = \cap_{N \geq 1} U_N$. It is enough to show that U_N is open. Let $S = k[x_0..x_n] = \bigoplus_{d \geq 0} S_d$. For $b \in \mathbb{A}^m$ define a map $T_b^{(N)} : S_{N-d_1} \oplus \cdots \oplus S_{N-d_r} \to S_N$ that sends

$$(g_1, \cdots, g_r) \mapsto \sum g_i f_i(x,b)$$

This is a k-linear map between two k-vector spaces. This is given by some matrix; the entries of this can be expressed as polynomials of b. Therefore, we get a map

$$T^{(n)} : \mathbb{A}^m \to \mathrm{Hom}_k(S_{N-d_1} \oplus \cdots \oplus S_{N-d_r}, S_N)$$

and $b \in U_N \iff T_b^{(N)}$ is surjective. In terms of b, $T^{(N)}$ is a morphism of affine varieties. Originally, we defined the determinantal variety as $Z_r \subset \mathbb{P}(\mathrm{Hom}(V,W))$; now we have to consider the affine version of this. Here, the cone is the affine cone.

Observations:

(1) $T^{(N)}$ is a morphism of affine varieties.
(2) Let $D = \dim S_N - 1$. Then consider Z_D the determinantal variety, which consists of matrices of rank $\leq D$. This is a closed projective variety; it might not be irreducible. Consider the affine cone

$$C(Z_D) \subset \mathrm{Hom}(S_{N-d_1} \oplus \cdots \oplus S_{N-d_r}, S_N)$$

is closed
(3) Therefore, $\mathbb{A}^m - U_N = (T^{(N)})^{-1}(C(Z_D))$

⊟

We saw an example that $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ might not be closed. (?)

PROPOSITION 9.7. *Let $\varphi : X^r \to Y^s$ be a morphism of (quasi-projective) varieties, $r \geq s$. Let $y \in \varphi(X^r)$, and let $W \subset \varphi^{-1}(y)$ be an irreducible component. (The fiber is closed, but not necessarily irreducible; choose any irreducible component.) Then* $\dim W \geq r - s$.

PROOF. By Noether normalization $Y^s \subset \overline{Y} \subset \mathbb{P}^m \to \mathbb{P}^s$

$$X^r \to Y^s \to \mathbb{P}^s$$

It is enough to assume $Y^s = \mathbb{P}^s \supset \mathbb{A}^s$, because $Y^s \to \mathbb{P}^s$ has finite fibers. We can assume that $Y = \mathbb{A}^s$, and $y = 0$. Let $f_i = y_i \circ \varphi$ be the pullback of a coordinate function. Then

$$\varphi^{-1}(0) = V(f_1 \cdots f_s)$$

It is enough to show: if $X \subset \mathbb{A}^n$ (some arbitrarily large space) is affine then every irreducible component of $X^r \cap V(f)$ has dimension $\geq r - 1$. (When we're all done with the coordinate function y_i, use induction). Assume $S \not\subset V(f)$. We want to show that each irreducible component of $X \cap V(f)$ has dimension $r - 1$. (It can't have dimension 1, because we proved earlier that any subvariety has dimension $<$ that of the original.) Replace $X, V(f)$ by their closures in $\mathbb{P}^n \supset \mathbb{A}^n$ (we're reducing to the projective situation.) Then $\overline{V(f)} = V(F)$, where F is some homogeneous polynomial of some degree d.

Now consider the d-uple embedding; we can assume $\deg F = 1$. (The hypersurface becomes a hyperplane.) So $V(F) = H$ is a hyperplane. (F is the homogenization of f, $f^{\deg f} f(\frac{x_1}{x_0} \cdots \frac{x_n}{x_0})$). We want to say that each projective variety reduces the dimension by one. We know this is true if the projective variety is projective space. Let $O \in H$ with $O \notin X \cap H$. Consider the projection $P_0 : \mathbb{P}^n \to \mathbb{P}^{n-1}$ (since we've used the d-uple embedding, n might be different).

$$P_0 : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

$$(H \cong \mathbb{P}^{n-1} \dashrightarrow P_0(H) \cong \mathbb{P}^{n-1}) \quad \text{where } X \dashrightarrow P_0(X)$$

Observation: $P_0(X \cap H) = P_0(X) \cap P_0(H)$. $\subset$ is obvious. To get the other direction, let $W \subset X \cap H$ be an irreducible component. $P_0(W)$ is irreducible in $P_0(X) \cap P_0(H)$. ($P_0(W)$ is an irreducible component; need to check this.) By induction on $n - r$, $\dim P_0(W) = \dim P_0(X) - 1 = \dim X - 1$. But $\dim P_0(W) = \dim W$.

Next consider if $r = s$. From the proposition we know that the fiber has dimension to ≥ 0; in this case we claim that the fiber has dimension zero. We know how many points are in the fiber.

DEFINITION 9.8. Let $\varphi : X \to Y$ be a dominant morphism. We say that φ is separable if $k(X)$ is separably generated over $\varphi^*(k(y))$: it is a composition of a finite separable extension and a purely transcendental extension. (φ induces an inclusion from $k(Y) \to k(X)$; regard this as a subfield.) This is not automatic: the field is not a perfect field.

EXAMPLE 9.9. Let $F : X \to X$ be the Frobenius morphism. Then F is dominant but not separable. For example, $X = \mathbb{A}^1$; if you look at $k(X) \to k(X)$ the power map $x \mapsto x^p$, this is not a separable field extension.

REMARK 9.10. From now on, assume every variety is algebraic. You can reduce the general case to the affine case in the following.

PROPOSITION 9.11. *Let $\varphi : X^r \to Y^r$ be a separable morphism of varieties of the same dimension. Then there exists $Y_0 \subset Y$ open such that $\varphi^{-1}(y)$ is finite, and*

$$\#\varphi^{-1}(y) = [k(X) : k(Y)]$$

for $y \in Y_0$. Each fiber has one point; but the extension is not degree 1.

Let's start with a special case: $k(Y) = k(X)$. We want a Zariski open subset in Y such that the fiber has just one point. We can assume X, Y are affine (exercise). Let $Y \subset \mathbb{A}^m$, $X \subset \mathbb{A}^n$. Let $x_1 \cdots x_m$ be coordinate functions on $\mathbb{A}^m$. φ^* induces an isomorphism.

$$\varphi^{*-1}(x_i) = \frac{f_i}{f_0} \text{ for } f_i \in \mathcal{O}(Y)$$

Consider $Y_0 = Y - V(f_0)$. The preimage of Y_0 contains only one point. Then $\varphi^{-1}(y) = (\frac{f_1(y)}{f_0(y)} \cdots \frac{f_n(y)}{f_0(y)})$, for $y \in Y_0$ (there is no other choice of point).

10. October 11

PROPOSITION 10.1. *If $f : X^r \to Y^s$ is a morphism (i.e. an r-dimensional variety to an s-dimensional one), then for any $y \in Y^s$, and component $W \subset \varphi^{-1}(y)$, $\dim W \geq r - s$.*

We reduced to proving this for affine varieties. Moreover, we can reduce to proving the following claim. (Assume X, Y affine; replace Y by an affine space by Noether normalization. A map $x \to Y^s$ is given by regular functions. Given one regular function, use the d-uple embedding to make it into a hyperplane.)

CLAIM 10.2. *If $X \subset \mathbb{P}^n$ projective, $\dim X = r$, H a hyperplane. If $X \not\subset H$ then every component of $X \cap H$ has dimension $r - 1$.*

Use induction. $n = r$ is OK, because X is all of $\mathbb{P}^n$ and $X \cap H$ *is* a hyperplane. In the case $r = n - 1$, $X = V(f)$ is a hypersurface. So $X \cap H$ is a hypersurface defined by $f(x_0, \cdots, x_{n-1}, 0)$ (this polynomial might not be irreducible). Each component has dimension $n - 2 = r - 1$.

Now suppose $n - r \geq 2$. Let $O \in H - X \cap H$ and consider the projection $P_0 : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n=1}$. Then last time we said

$$P_0(X \cap H) = P_0(X) \cap P_0(H)$$

Then $P_0(H - 0)$ is a hyperplane in $\mathbb{P}^{n-1}$. Let $W \subset X$ be a component. If $P_0(W)$ is a component, of $P_0(X) \cap P_0(H - 0)$ then we are done by induction. (By induction, $\dim W = \dim P_0(W) = r - 1$.) The claim is that if we choose the projection in a good way, we can always choose $P_0(W)$ to be a component.

CLAIM 10.3. *Fix $W \subset X \cap H$ to be a component. We can find $O \in H \backslash X \cap H$ such that $P_0(W)$ is a component of $P_0(X) \cap P_0(H - 0)$.*

Say $X \cap H = W \cup W^*$ where $W^* = W_1 \cup \cdots \cup W_m$ is the union of other irreducible components. Then it's enough to show that there is some point O such that $P_0(W^*) \not\subset P_0(W^*)$. (We already know that $P_0(X \cap H) = P_0(W) \cup P_0(W_1) \cup \cdots \cup P_0(W_m)$.) Pick

$x \in W\backslash W^*$. Consider $\overline{P_x^{-1}P_x(W^*)} = C(P_x(W^*))$. What is the dimension of W^*?

$$\dim W_i \leq \dim X - 1 = r - 1$$

Therefore, $\dim C(P_x(W_i)) \leq r < n-1 = \dim H$. The cone is inside H, because x and W^* are. So we can find a point

$$O \in H - C(P_x(W^*)) - W$$

It's not inside the cone, nor inside W. Now do the projection $P_O : \mathbb{P}^n \dashrightarrow P^{n-1}$ and we find that

$$P_0(X) \notin P_0(W^*)$$

otherwise $P_0(x) \in P_0(W^*)$ would imply that O is in the line joining x and a point in W^*, which would imply that O is in the cone $C(P_x(W^*))$. This is a contradiction, so we are done.

What if X and Y have the same dimension?

PROPOSITION 10.4. *Let $\varphi : X^r \to Y^r$ be separable. Then there is some Zariski open $Y_0 \subset Y$ such that $\#\varphi^{-1}(y) = [k(X^r) : k(Y^r)]$ for all $y \in Y_0$.*

REMARK 10.5. This is not true if the separability assumption is removed. For example, the Frobenius morphism from last time. Actually, the cardinality of the preimage is *always* equal to the separable degree, and if the field extension is separable, then separable degree = degree.

THEOREM 10.6 (Primitive element theorem). *Let L/K be a finite separable extension. Then L is generated over K by one element. In addition, if $\beta_1 \cdots \beta_r$ form a set of generators, then the generator can be taken to be*

$$\alpha = \sum \lambda_i \beta_i \quad \lambda_i \in K$$

PROOF. First we can assumeX, Y affine. Now, we can assume

$$k(X) = k(Y)p[f_1, f \in \mathcal{O}(X)$$

$\varphi^* : k(Y) \hookrightarrow k(X)$. Consider

$$X \xrightarrow{\varphi} Y \times \mathbb{A}^1 \xrightarrow{p^1} Y$$

where $x \mapsto (\psi(x), f(x)) \mapsto \varphi(x)$ and $Z = \overline{\psi(X)}$. Then $k(Y) \subset k(Z) \xrightarrow{\sim} k(X)$, where the second map is an isomorphism because $k(Z)$ contains the preimage of the generator f of $k(X)$. Apply the case from earlier: there is some $Z_0 \subset Z$ open such that $\#\psi^{-1}(z) = 1$ for $z \in Z_0$. Therefore, we can replace X by Z and assume $X \subset Y \times \mathbb{A}^1$. Let $P(t)$ be the minimal polynomial of f.

$$P(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

for $a_i \in k(Y)$. The a_i are rational functions, but if we replace Y by the open subset Y_0, we can assume that they are regular functions. $P(f) = 0$; because it is separable (?) $P(t)$ and $P'(t)$ do not have common zeroes. Let $W \subset Y \times \mathbb{A}^1$ be defined by

$$W = \{(y,t) \, : \, \sum a_i(y) t^{n-i} = 0\}$$

$X \subset W$. But what if there are other things in W? Call these X^*, so $W = X \cup X^*$.

We claim that $\overline{\varphi(X^*)} \subsetneq Y$. (So the image of the other components X^* doesn't take over the image.) That is, there is some $Y_1 \subset Y$ open such that $W \cap \varphi^{-1}(Y) = X \cap \varphi^{-1}(Y_1)$. The picture is

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\varphi"] & Y \times \mathbb{A}^1 \arrow[d, "p_1"] \\ & Y \end{tikzcd}$$

In fact, let $f_i \in k[y,t]$ be the generators of $I(X)$

$$\begin{tikzcd} I(X) \subset k[y,t] \arrow[r] & \mathcal{O}(Y)[t] \arrow[r] \arrow[d] & \mathcal{O}(X) \arrow[d, hook] \\ & k(Y)[t]/p(t) \arrow[r, "\sim"] & k(X) \end{tikzcd}$$

There is some $b_0 i(y) \in \mathcal{O}(y)$ such that $b_i(y)f_i(y,t) \subset \sum c_i(y,t)p(t) + I(Y)$.

$$Y_1 = Y - \{\prod b_i(y) = 0\}$$

If $(y,t) \in W \iff \sum a_i(y,t)t^{n-i} = 0$, then $b_i(y)f_i(y,t) = 0 \implies (y,t) \in X$, because $b_i \neq 0$ (so f has to be zero there). Therefore, the image must land inside the closed subset defined by $\prod b_i(y)$.

Note that we can replace X and Y by open subsets, because the complement would have lower dimension.

Summary: We know that $k(X) = k(Y)[t]/p(t)$. This is basically saying that there is some open subset $Y_1 \subset Y$ such that this works on the level of coordinate rings:

$$\mathcal{O}(\varphi^{-1}(Y_1) \cap X) = \mathcal{O}(Y_1)[t]/p(t)$$

Something (?) on the right is the same as $\mathbb{P}(W \cap \varphi^{-1}(Y_1))$.

Now replace Y by Y_1. We can assume

$$\mathcal{O}(X) = \mathcal{O}(Y)[t]/p(t)$$

In other words,

$$X = \{(y,t) \,:\, Y \times \mathbb{A}^1 \,:\, \sum a_i(y)t^{n-i} = 0\}$$

For fixed y, this is a degree n polynomial; it has n solutions. But all the roots might not be distinct. $\#\varphi^{-1}(y)$ is just the number of roots of the polynomial $\sum a_i(y)t^{n-1} = 0$. $p(t)$ and $p'(t)$ do not have common zeroes in $k(y)[t]$. This implies that there exist $a(t)$, $b(t)$ in $k(y)[t]$ such that $a(t)p(t) + b(t)p'(t) = 1$. Eliminate the denominators, so we can assume

$$\widetilde{a}(t)p(t) + \widetilde{b}(t)p'(t) = c(y)$$

where $\widetilde{a}$ and $\widetilde{b}$ are in $\mathcal{O}(y)[t]$ and $0 \neq c \in \mathcal{O}(y)$. Over

$$Y_0 = \{y \in Y \,:\, c(y) \neq 0\}$$

$p(t)$ and $p'(t)$ do not have common zeroes. So for $y \in Y_0$, the roots of $\sum a_i t^{n-i}$ are distinct. This implies $\#\varphi^{-1}(y) = n$, where $n = [k(X) : k(Y)]$.

$\boxminus$

10.1. Separable morphisms. Let $\varphi : X \to Y$ be a morphism, $\varphi(X) = Y$. This induces a map of local rings

$$\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

This pulls back a function vanishing on $\mathfrak{m}_{Y,y}$ to one vanishing at $\mathfrak{m}_{X,x}$. In particular, φ induces a map $d\varphi_x$ of tangent spaces:

$$\begin{tikzcd} T_xX \arrow[r] \arrow[d, "="] & T_yY \arrow[d, "\sim"] \\ (\mathfrak{m}_x/\mathfrak{m}_x^2)^* \arrow[r] & (\mathfrak{m}_y/\mathfrak{m}_y^2)^* \end{tikzcd}$$

DEFINITION 10.7. The morphism φ is *smooth* at x if

(1) x (respectively $\varphi(x)$) is smooth in X (respectively in Y)
(2) $d\varphi_x : T_xX \to T_yY$ is surjective

PROPOSITION 10.8. *Let $\varphi : X^r \to Y^s$ be a dominant morphism .Then the following are equivalent:*

(1) *φ is separable*
(2) *there is some $X_0 \subset X$ open such that φ is smooth at $x \in X_0$*
(3) *there is some point $x \in X$ such that φ is smooth at x*

If φ is the Frobenius morphism, then there is no point at which this is smooth.

PROOF. We can assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine.

$$\begin{tikzcd} X \arrow[dr, "\varphi"'] & \Gamma\varphi \arrow[l, "p_1"'] \arrow[r, hook] \arrow[d, "p_2"] & \mathbb{A}^n \times \mathbb{A}^m \arrow[d] \\ & Y \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

so we can just assume

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\varphi"] & \mathbb{A}^n \times \mathbb{A}^m \arrow[d] \\ Y \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

Let $g_1 \cdots g_p \in I(Y)$ be generators. $\varphi^*(g_1) \cdots \varphi^*(g_p), f_1, \cdots , f_q$ are generators of $I(X)$. For $x \in X$ and $y \in Y$

$$T_xX = \{(\xi_1 \cdots \xi_{n+m}) \in k^{n+m} \,:\, \sum \frac{\partial g_i}{\partial y_j}\xi_{j+n} = 0, \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\xi_j + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j}\xi_{j+n} = 0\}$$

This is the definition we gave at the beginning of this course. There is a projection of this to

$$T_yY = \{(0, \cdots, 0, \xi_{n+1} \cdots \xi_{n+m}) \,:\, \sum \frac{\partial g_i}{\partial y_j}\xi_{j+n} = 0\}$$

The kernel of this projection $d\varphi_x$ is

$$\{(\xi_1 \cdots \xi_n) \,:\, \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\xi_j = 0\}$$

This is just the kernel of the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$. Now φ is smooth at x iff $\dim\ker d\varphi_x = r - s$. Assume X and Y are smooth to prove the proposition. The dimension of the kernel is always $\geq r - s$. (A linear map from r-dimensional space to s-dimensional space has kernel with dimension $\geq r - s$.) This is equivalent to having $rk\frac{\partial f_j}{\partial x_i} \geq n - (r - s)$. This determines an open subset of X. If this is nonempty then the second condition is satisfied. This shows (2) $\iff$ (3). Next time: (1) $\iff$ (2).

$\boxminus$

11. October 13

From last time, we were trying to prove the following proposition:

Proposition 11.1. *Let $\varphi : X \to Y$ be a dominant morphism. Then TFAE:*

1. *φ is separable*
2. *there exists $X_0 \subset X$ open such that φ is smooth for every point $x \in X_0$*
3. *there is some $x \in X$ such that φ is smooth at x*

Assume X, Y affine. We have already shown (2) $\iff$ (3). By replacing X by its graph, we can assume $X \subset \mathbb{A}^n \times \mathbb{A}^m$. Also assume that X and Y are smooth. Then the map φ is just

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\varphi"] & \mathbb{A}^n \times \mathbb{A}^m \arrow[d, "p_2"] \\ Y \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

$(g_1 \cdots g_p) = I(Y)$. We can choose f_i such that $(f_1 \cdots f_q, \varphi^* g_1 \cdots \varphi^* g_p) = I(X)$. Then the tangent space is

$$T_xX = \{(\xi_1 \cdots \xi_n, \eta_1 \cdots \eta_m) \,:\, \begin{pmatrix} 0 & g_y \\ f_x & f_y \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0\}$$

Via $d\varphi_x$ this maps to $T_yY = \{(\eta_1 \cdots \eta_m) \,:\, (\frac{\partial g}{\partial y})\eta = 0\}$ So the kernel of $d\varphi_x$ is $\{(\xi_1 \cdots \xi_n) \,:\, (\frac{\delta f}{\delta x})\xi = 0\}$ φ is smooth at x iff $\dim\ker(d\varphi_x) \leq r - s$. Equivalently, $rk(\frac{\partial f}{\partial x}) \geq n - r + s$.

$$\{p \in X \,:\, rk(\frac{\partial f}{\partial x})(x) \geq n - r + s\} \subset X$$

is open. (We know that the set of smooth points is open.) It is nonempty, so there is an open subset where this is true. This shows (2) $\iff$ (3).

By the same trick used to get $Der_k(k(X), k(X))$:

$$\begin{aligned} Der_{k(Y)}(k(X), k(X)) &\xrightarrow{\sim} Der_{k(Y)}(k(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y), k(X)) \\ &\cong \{D \in Der_{k(Y)}(k(Y)[x_1 \cdots x_n], k(X)) \,:\, Df_i = 0\} \\ &= \{(\xi_1 \cdots \xi_n) \in k(X)^n \,:\, \sum \xi_i \frac{\delta f_i}{\partial x_j}\} \end{aligned}$$

We used:

$$k(Y) \otimes \mathcal{O}(X) \cong k(Y)[x_1 \cdots x_n]/(f_1 \cdots f_q)$$

where $I(Y)$ is generated by the f_i. So

$$\dim Der_{k(Y)}(k(X), k(X)) = n - rk_{k(X)}(\frac{\partial f}{\partial x})$$

By the same argument as used in the smoothness proof, there is some $X_0 \subset X$ open such that

$$rk_{k(X)} \frac{\partial f}{\partial x} = rk_k(\frac{\partial f}{\partial x}(x))$$

for all $x \in X_0$. We can multiply this by a matrix to eliminate denominators.

We want to show (1) $\implies$ (2). Assume $k(X)/k(Y)$ is separable. By a fact of commutative algebra, this is equivalent to

$$\dim_{k(X)} Der_{k(Y)}(k(X), k(X)) = r - s$$

where $r = \dim X$ and $s = \dim Y$. If L/K is a finitely generated field extension, then L/K is separably generated iff $\dim_L Der_k(L, L) = tr.d._{L/K}$. We used this earlier with $K = k$ and $L = k(X)$. Now we use $K = k(Y)$ and $L = k(X)$. So there's an open subset where the rank of the matrix is $n - r + s$. So the points here are smooth.

REMARK 11.2. If $\varphi : X \to Y$ is smooth over $X_0 \subset X$, then φ is dominant. (So the assumption of dominant-ness in the proposition is not necessary.)

$$\varphi : X \xrightarrow{\psi} \overline{\varphi(X)} \xhookrightarrow{i} Y$$

$$d\varphi_x - di_{\psi(X)} \circ d\psi_x$$

$\psi(X)$ contains an open smooth subset Z of $\varphi(X)$. This is possible because the smooth points are an open sets, and the image contains a smooth subset. (Open sets are dense.) $Z = \varphi(X)$. *I think we're using the homework: the image of a constructible set contains an open set.* We get a factorization

$$d\varphi_x : T_xX \to T_zZ \hookrightarrow T_zY$$

The second map can't be an isomorphism, because the dimension is greater on the right. $d\varphi_x$ is not surjective; contradiction.

COROLLARY 11.3 (Sard's lemma for varieties / Generic smoothness). *Let $char(k) = 0$, $\varphi : X \to Y$ dominant. Then there is some nonempty Zariski-open subset $Y_0 \subset Y$ such* *that*

$$\varphi|_{\varphi^{-1}(Y_0)}\varphi^{-1}(Y_0) - Sing(X) \to Y$$

is smooth. (We have to exclude the singular points of X.) (This is wrong for the Frobenius morphism in characteristic p; it is dominant but nowhere smooth.)

PROOF. Assume X and Y are smooth. (We have already excluded the singular points explicitly, so assume it was this way to start with.) Let $X_0 \subset X$ be the open subset such that φ is smooth at $x \in X_0$. X_0 is always nonempty, because we are in characteristic zero: any dominant map is separable, since any field extension is separable. Let $Z = X \backslash X_0$. We just need to show that $\psi = \varphi|_Z : Z \to Y$ is not dominant. The image of Z lies in a proper closed subset; the complement of this subset is going to be Y_0. If not, $\psi : Z \to Y$ is dominant, and there exists some $Z_0 \subset Z$ open such that $\psi|_{Z_0} : Z_0 \to Y$ is smooth. (Again, dominant morphisms are separable.)

$$Z \overset{i}{\to} X \overset{\varphi}{\to} Y$$

where the composition is ψ. Choose $z \in Z$;

$$d\psi_z : T_z Z \twoheadrightarrow T_{\psi(Z)}(Y)$$

is surjective. This factors through $T_z X$ via $d\psi_z : T_z X \to T_{\psi(Z)} Y$; but $d\psi$ should not be surjective. By definition, this is the point where the morphism is not smooth. ⊟

PROPOSITION 11.4. *Let $\varphi : X \to Y$ be smooth at $x \in X$. Then there exists a unique component Z of $\varphi^{-1}(\varphi(X))$ passing through x. In addition, Z is smooth at x and* $\dim Z = \dim X - \dim Y$.

For example, consider $X = (xy = t) \subset \mathbb{A}^3$; this maps to $\mathbb{A}^1$ by sending $(x, y, t) \mapsto x$. Away from the axis, the fibers are hyperbolas; at the origin, the image is the union of two coordinate axes, and is not smooth.

PROOF. Again, assume X and Y are two smooth varieties that are affine. Assume $X \subset \mathbb{A}^n \times \mathbb{A}^m$ and $Y \subset \mathbb{A}^m$. (This is a statement about local things.) Choose $g_1 \cdots g_{m-s} \in I(Y)$ with independent linear terms. (X and Y have dimensions r and s, respectively.) Let $f_1, \cdots, f_{n-r+s} \in I(X)$ such that $f_1 \cdots f_{n-r+s}, g_1, \cdots, g_{m-s}$ have independent linear terms. Because it's smooth at the point, $\dim \ker d\varphi_0 = r - s$ so

$$rk \frac{\partial f}{\partial x}(0,0) = n - r + s$$

and so $f_1(x, 0), \cdots, f_{n-r+s}(x - 0)$ have independent linear terms. Consider the algebraic set $V(f_1 \cdots f_{n-r+s}, g_1, \cdots, g_{m-s}) = X \cup X'$ where $0 \notin X'$ (this was from... lecture 3-ish...) Also consider

$$V(f_1 \cdots f_{n0r_s} g_1 \cdots g_{m-s}, y_1 \cdots y_m) - p_2^{-1}(0) \cap (X \cup X') = \varphi^{-1}(0) \cup (p_2^{-1}(0) \cap X') \not\ni 0$$

But the set on the left is

$$\mathbb{A}^n \supset V(f_1(x, 0) \cdots, f_{n-r+s}(x, 0))$$

Think of this $\mathbb{A}^n$ as $\mathbb{A}^n \times \{0\}$; this set is exactly $Z \cup Z'$ and $0 \notin Z'$. Comparing the two sides, $Z \subset \varphi^{-1}(0)$ is the unique component passing through zero. Z is smooth at zero, and so the dimension is the number of equations necessary to define it, which is $n - (n - r + s) = r - s$.

$\boxminus$

THEOREM 11.5 (Zariski's main theorem, smooth case). *Let $\varphi : X \to Y$ be a birational morphism of quasi-projective varieties. Assume Y is smooth. Let $Y_0 = \{y \in Y \;:\; \#\varphi^{-1}(y) = 1\}$ (We know that there is an open subset of Y inside Y_0.)*

(1) *$\varphi|_{\varphi^{-1}(y_0)} : \varphi^{-1}(y_0) \to Y_0$ is an isomorphism*
(2) *For any $y \in Y \backslash Y_0$ and a point $x \in \varphi^{-1}(Y)$ there exists a subvariety $E \subset X$ of dimension $r - 1$ passing through x, and $\dim \overline{\varphi(E)} \leq r - 2$. In particular, for any $y \in Y \backslash Y_0$, there exists a component in $\varphi^{-1}(Y)$ of positive dimension.*
Such E is called the exceptional divisor. (This is a generalization of the exceptional divisor associated with the blowup.)

(AKA: If the preimage is not one point, then the preimage is a component of positive dimension.) The assumption that Y is smooth is necessary: consider the map of $\mathbb{A}^1$ to the cuspidal curve $y^2 - x_3$. Every point has a unique preimage, but it is not an isomorphism.

PROOF. Let $x \in X$ and $y \in Y$. φ induces a morphism of local rings

$$\varphi^* : \mathcal{O}_{Y,y} \hookrightarrow \mathcal{O}_{X,x}$$

that reduces to a morphism $k(Y) \to k(X)$. There are two cases. Let

$$Y_1 = \{y \in Y \;:\; \exists x \in \varphi^{-1}(y), \varphi^* : \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}\}$$

$$Y_2 = \{y \in Y \;:\; \exists x \in \varphi^{-1}(y), \varphi^* : \mathcal{O}_{Y,y} \subsetneq \mathcal{O}_{X,x} \;\forall x \in \varphi^{-1}(Y)\}$$

CLAIM 11.6. *$Y_1 \subset Y_0$ and $Y_2 \subset Y \backslash Y_0$. (All of this implies $Y_1 = Y_0$, $Y_2 = Y \backslash Y_0$.)*

Let $y \in Y_1$. We have $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$ for some point $x \in X$. If $x' \in \varphi^{-1}(Y)$ is another point, then

$$\mathcal{O}_{x,x} \cong \mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x'}$$

The claim follows from the following lemma:

LEMMA 11.7. *Let X be quasi-projective, x and x' two points such that $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x'} \subset k(X)$. Then $x = x'$.*

By some old homework, if there is an isomorphism between local rings, then there is an isomorphism on open subsets.

PROOF OF LEMMA. Assume $X \subset \mathbb{P}^n$ is projective: take its closure. Choose some appropriate hyperplane that does not pass through x and x'; we can assume that X is affine. (For quasi-projective varieties, every two points are contained in one affine subvariety.) Now there is a 1-1 correspondence between points of X and maximal ideals of $\mathcal{O}(X)$. If one is contained in the other, then there is an inclusion of maximal ideals; but since they're *maximal* ideals, then they are the same. That is,

$$\mathcal{O}_{X,x} \subset \mathcal{O}_{X,'x} \implies \mathfrak{m}_x \supset \mathfrak{m}_{x'}$$

$A_{\mathfrak{m}_x} \subset k(A) \supset A_{m'_x}$. ⊟

We want to show that $Y_2 \subset Y \backslash Y_0$. Take $y \in Y_2$, and consider the preimage. Let $f \in \mathcal{O}_{X,x}$ be such that f is in $\mathcal{O}_{Y,y}$: that is, $f = \varphi^*(\frac{a}{b})$ for $a, b \in \mathcal{O}_{Y,y}$. (We're using the fact it's birational.) $b(y) = 0$. We need the following theorem:

THEOREM 11.8. *If $y \in Y$ is smooth, $\mathcal{O}_{Y,y}$ is a unique factorization domain. (*Using smoothness!*)*

Proof: commutative algebra.

Now, because $\mathcal{O}_{Y,y}$ is a UFD, we can assume that a, b are coprime in $\mathcal{O}_{Y,y}$. Let $b = \beta b'$ where β is prime. Now $\mathbb{P} = \mathcal{O}(Y) \cap \beta\mathcal{O}_{Y,y}$ is a prime ideal. Let E be a component of $V(\varphi^*\beta) \cap X$. The set of zeroes of this contain the point x. β is prime, so it will vanish at y. So $\varphi^*\beta$ will vanish at x.

$\dim E = r - 1$ (we proved before that a component is cut out by one equation, as it has dimension $r - 1$). $\varphi^*(a) = f\varphi^((b) = f\varphi^*(\beta)\varphi^*(b_1)$ which implies that $\varphi^*(a)$ vanishes on E. This implies a, β vanish on $\overline{\varphi(E)}$. But a, b coprime implies $a \notin \mathfrak{P}$. Therefore, $\overline{\varphi(E)} \subsetneq V(\mathfrak{P}) \subsetneq Y$ and so $\dim \overline{\varphi(E)} \leq r - 2$.

$y \in Y_2$ imlies $\dim \varphi^{-1}(y) \geq 0$ implies $y \in Y \backslash Y_0$.

⊟

12. October 18

THEOREM 12.1 (Zariski main theorem, smooth case). *Let $\varphi : X \to Y$ be birational, with Y smooth. There is some $Y_0 \subset Y$ open and $\varphi : \varphi^{-1}(Y_0) \xrightarrow{\sim} Y_0$ and for every $y \in \overline{Y \backslash Y_0}$, there exists a subvariety E of dimension $r - 1$ through $x \in \varphi^{-1}(y)$ such that $\dim \overline{\varphi(E)} \leq r - 2$*

$$\varphi : E \to \overline{\varphi(E)} \implies \dim \varphi^{-1}(y) \geq 1$$

Recall the blowup of the point in the plane: $V(xu - yv) = Z \subset \mathbb{A}^2 \times \mathbb{P}^1$. E is the exceptional divisor. Choose $O = (0, 0)$ and $P = (0, 0, 0, 1) \in E$. (There is a map of local rings $\mathcal{O}_{Z,p} \hookleftarrow \mathcal{O}_{\mathbb{A}^1,O} : \varphi^*$ that is not an isomorphism.) Setting $v = 0$, we get the subset $V(xu - y)$. $u \in \mathcal{O}_{Z,p}$ and you can write $u = \varphi^*(\frac{y}{x})$ so in the proof x is an irreducible element in the local ring. Choose $E = V(\varphi^*x) \subset Z - \{v = 0\}$ defined by $x = 0$. E is defined by the pair $xu - y = 0$ and $x = 0$ so $x = y = 0$; this is exactly the exceptional divisor. x determines a variety of codimension 1 on the base copy of $\mathbb{A}^2$: this is what we denoted $V(\mathfrak{P})$. Since y does not vanish, the image of E is not the whole y axis: it is just one point. The image of $Z - \{v = 0\}$ is $\mathbb{A}^2 - \{x = 0\} \cup \{(0,0)\}$ (this is constructible: not open or closed, or locally closed).

In the proof, we used the result that the local ring at a smooth point is a UFD. (Not proven here; final project?)

12.1. Divisors. Let X be a smooth algebraic variety.

LEMMA 12.2. *Let X be affine, and $x \in X$. Let $f \in \mathcal{O}_{X,x}$ be an irreducible element (a prime element, and in particular $f\mathcal{O}_{X,x}$ is a prime ideal). $f\mathcal{O}_{X,x} \cap \mathcal{O}(X) = \mathfrak{P}$ is prime, and $X' = V(\mathfrak{P})$ is a subvariety of dimension $r - 1$ containing x.*

Conversely, if $X' \subset X$ is a subvariety of dimension $r - 1$ and $x \in X'$ then there is some irreducible element $f \in \mathcal{O}_{X,x}$ such that $X' = V(\mathfrak{P})$ and $\mathfrak{P} = f\mathcal{O}_{X,x} \cap \mathcal{O}(X)$. In other words, $I(X') \cdot \mathcal{O}(X, x) = f \cdot \mathcal{O}(X, x)$.

PROOF. By shrinking the affine variety a little bit, I can assume that $f \in \mathcal{O}(X)$. There exists some open $U \subset X$ such that $V(f) \cap U = V(\mathfrak{P}) \cap U$. (Like in lecture 3, f generates this locally (but maybe not globally).) Each irreducible component of this has dimension $r - 1$ (it's given by one function).

Conversely, choose $f \in I(X')$. Then f is not a unit in $\mathcal{O}_{X,x}$ (because it vanishes at a point). Because it is a UFD, you can factor f into prime divisors, and then assume that f is really a prime element. $X' \subset V(f)$ but they have the same dimension, so it is exactly one component of $V(f)$. So $X' = V(\mathfrak{P})$. $\boxminus$

DEFINITION 12.3. Let $X' \subset X^r$ of dimension $r - 1$. Let $x \in X$;. An element $f \in \mathcal{O}_{X,x}$ is called a local equation of X' at x if $f \cdot \mathcal{O}_{X,x} = I(X') \cdot \mathcal{O}_{X,x}$.

We showed that every smooth variety of dimension $r - 1$ has a local equation. Let X be a smooth algebraic variety. The divisor group $Div(X)$ is the free abelian group generated by the codimension 1 subvarieties. (The codimension 1 subvarieties form a set; consider the free abelian group over this set.) For example, elements in $Div(X)$ can be written as a formal sum

$$\sum_{\substack{Z_i \subset X \\ Z_i \text{ codim. } 1}} n_i Z_i$$

where all but finitely many of the n_i are zero.

DEFINITION-LEMMA 12.4. There exists a natural group homomorphism $k(X)^x \to Div(X)$ taking $f \mapsto (f)$.

CONSTRUCTION. Write $(f) = \sum n_Z Z$ and we need to define $n_Z \in \mathbb{Z}$. Choose any point $x \in Z$. In $\mathcal{O}_{X,x}$ choose a local equation f_Z. Then we can write

$$f = \frac{g}{h} f_Z^r \quad g, h \in \mathcal{O}_{X,x} \text{ and } gh, f_Z \text{ are coprime}$$

Define $ord_{Z,x}(f) = r$ for $r \in \mathbb{Z}$. (This is independent of the choice of local equation, because these differ by a unit. Why? If f_1 and f_2 are two, then they divide each other, because look at where they vanish. Up to multiplication by a constant, primes are in 1-1 correspondence with codimension-1 subvarieties.) Therefore, this is well-defined. We claim that this does not depend on the choice of x: more precisely, $x \mapsto ord_{Z,x}(f)$ is locally constant on Z and is therefore independent of the choice of x. (Z is irreducible, hence connected, so we can drop the "locally" in the "locally constant.")

In a neighborhood of x in X, $U \cap Z = V(f_Z) \cap U$ so f_Z is a local equation for every point $y \in Z$. Because $f = \frac{g}{h} f_Z^r$ (assume both g and h are regular functions on some open set U), assume we have moved x to any point $y \in U$. Also $(gh, f_Z) = 1$ because if $f_Z | gh$ then this would vanish on Z. $ord_{Z,y}(f) = r = ord_{Z,x}(f)$ for all $y \in U \cap Z$.

CLAIM 12.5.

- $ord_Z(fg) = ord_Z(f) + ord_Z(g)$
- $ord_Z(f+g) \geq \min\{ord_Z(f), ord_Z(g)\}$ *if* $f + g \neq 0$

PROOF. Exercise. ⊟

CLAIM 12.6. *Fix* $f \in k(X)^x$ *(nonzero rational function); then for all but finitely many* Z*,* $ord_Z(f) = 0$.

PROOF. We can assume that X is affine: there are only finitely many closed subvarieties of codimension 1 in the complement; so we can ignore them. Almost all varieties of codimension 1 intersect this open affine. $f = \frac{g}{h}$ for $g, h \in \mathcal{O}(X)$. $ord_Z(f) \neq 0 \implies Z \subset V(gh)$ but the latter has only finitely many irreducible components. ⊟

$ord_Z(f)$ measures the vanishing order of f on this variety. If it's a rational function, it measures the order of the pole on the variety.

Finally, we can define $n_Z = ord_Z(f)$. So define

$$(f) = \sum ord_Z(f) \cdot Z$$

⊟

If it's a function over $\mathbb{C}$ there is a similar situation. Around the origin, you can always write $g = z^r$; this is the same r as defined above.

DEFINITION 12.7.

$$Div^0(X) = Im(k(X)^\times \to Div(X))$$

is called the group of principal divisors.

$$Pic(X) = Cl(X) := Div(X)/Div^0(X)$$

is called the Picard group, or divisor class group, of X.

EXAMPLE 12.8. If $X = \mathbb{A}^n$, then $Pic(X) = 0$. This is implied by the following fact: every subvariety of codimension 1 is a hypersurface (we proved this a while ago). So each Z is a principal divisor, and every divisor is in the image of this map.

If X is affine, then $\mathcal{O}(X)$ is a UFD iff $Pic(X) = 0$. Equivalently, a ring of integers is a UFD iff its class group is zero. Assume that $\mathcal{O}(X)$ is a UFD. We need to show that every subvariety of codimension 1 is defined by one equation. Use the trick from before: let $Z \subset X$ be codimension 1. $f \in I(Z)$ is not a unit, because it vanishes somewhere. Factor f into prime divisors; we can assume that f is irreducible. We know $Z \subset V(f)$ but because f is irreducible, f is prime and (f) is a prime ideal. $V(f)$ is an irreducible subset, and it has the same dimension of Z; therefore, $Z = V(f)$ (it's a component, and $V(f)$ is connected since it's irreducible).

Conversely, assume that $Pic(X) = 0$; we want to show that $\mathcal{O}(X)$ is a UFD. So every subvariety is a principal divisor: $Z = (f)$. $f \in k(X)$. This implies that $ord_{Z'} f = 0$ if $Z' \neq Z$ and 1 if $Z' = Z$. But in fact, $f \in \mathcal{O}(X)$: it is enough to show that $f \in \bigcap_{x \in X} \mathcal{O}_{X,x} = \mathcal{O}(X)$. Every f can be written $\frac{g}{h}$; factor g and h into primes. The order of f is ≥ 0 for any Z, the denominator does not contain a prime factor. This shows that h will not contain a prime element.

We are trying to show that $\mathcal{O}(X)$ is a UFD. If another function $g \in \mathcal{O}(X)$ is irreducible, then $V(g)$ is irreducible. So $V(g)$ is irreducible. Why? If $Z \subset V(g)$ then we know that $Z = V(f)$ for some f, which implies that $f|g$. But g is irreducible, so f and g differ by a unit. Therefore, this is a UFD. If $g|fh$ then $V(g) \subset V(fh) = V(f) \cap V(h)$. Because $V(g)$ is irreducible, it is contained in one of the components. This implies that $g|f$ or $g|h$.

EXAMPLE 12.9. Say $Z = \mathbb{P}^n$. We have the degree map $deg : Div(X) \to \mathbb{Z}$, where $Z = V(f) \mapsto deg(f) := deg(Z)$. The map is surjective, because you can take a hyperplane, that maps to 1. The group $Div^0(X)$ of principal divisors is in the kernel of the degree map: these contain ratios $\frac{f}{g}$ where f, g have the same degree (these are homogeneous polynomials). If $\frac{f_1^{n_1} \cdots f_r^{n_r}}{g_1^{m_1} \cdots g_s^{m_s}}$ where f_i and g_i are irreducible. The divisor associated to this rational function is exactly $\sum n_i V(f_i) - \sum m_i V(g_i)$. These define a subvariety of codimension 1. This goes to zero, because the degrees of the numerator and denominator are the same. We know that

$$k(X)^\times \to Div(X) \stackrel{deg}{\to} \mathbb{Z} \to 0$$

is exact. The kernel is exactly the image of the principal divisors. If we have a constant function, the associated divisor is zero. Let's complete the exact sequence:

$$1 \to k^\times \to k(X)^\times \to Div(X) \stackrel{deg}{\to} \mathbb{Z} \to 0$$

If I have a divisor that maps to zero in $\mathbb{Z}$ I want to show that it is principal. Let $D = \sum n_i Z_i$ such that $\deg D = 0$. Each Z_i can be written as some $V(f_i)$. We can rewrite

$$\prod f_i^{n_i} = \frac{\prod_{n_i>0} f_i^{n_i}}{\prod_{n_j<0} f_j^{-n_j}} = \frac{f}{g}$$

where $D = (\frac{f}{g})$. All the things that were not really explained are an exercise.

PROPOSITION 12.10.

$$\deg : Pic(\mathbb{P}^n) \tilde{\to} \mathbb{Z}$$

and $H \mapsto 1$ where H is the hyperplane.

13. October 20

DEFINITION 13.1. A prime divisor is $D = Z$ where Z is a subvariety of codimension one. (i.e. a generator.)

An effective divisor has the form $D = \sum n_i Z_i$ with all coefficients positive.

PROPOSITION 13.2. *Let $f \in k(X)^{\times}$. Then (f) is effective iff $f \in \mathcal{O}(X)$.*

We proved this last time. Assume X is affine. It's in the local ring if it's in every coordinate ting. Then $ord_Z f \geq 0$ implies $f \in \mathcal{O}_{x,Y}$ for all $x \in X$.

COROLLARY 13.3. *Let X be smooth (actually, you can just assume normal), $F \subset X$ closed of codimension ≥ 2. Then $\mathcal{O}(X) = \mathcal{O}(X - F)$.*

If I remove a subvariety of codimension 2, the divisor group will not change. Any function $f \in \mathcal{O}(X - F)$ defines a rational function on X, but the associated divisor is still effective, so $f \in \mathcal{O}(X)$.

REMARK 13.4. If $U \subset X$ is open, then there is a natural restriction $Div(X) \to Div(U)$. If $X = U \cup V$, then

$$0 \to Div(X) \to Div(U) \oplus Div(V) \to Div(U \cap V)$$

is exact. In order to define the divisor on X, you just have to specify the divisors on U and V that coincide on their intersection.

NOTATION 13.5. If a divisor D is effective we write $D \geq 0$.

DEFINITION 13.6. Assume X is projective. Let $D \in Div(X)$. Define

$$\mathcal{L}(D) = \{f \in k(X) \,:\, f = 0 \text{ or } (f) + D \text{ is effective }\}$$

$\mathcal{L}(D)$ is a k-vector space. Last time we said that

$$ord_Z(f + g) \geq \min\{ord_Z(f), ord_Z(g)\}$$

Also define

$$|D| = \{\text{effective divisors of the form } (f) + D\}$$

This gives a natural identification $|D| \cong \mathbb{P}(\mathcal{L}(D))$. (By definition, this is the set of 1-dimensional subspaces of $\mathcal{L}(D)$.) If $(f) + D = (g) + D$ then $(f) = (g)$, or equivalently $(\frac{f}{g}) = 0$. By the remark, this means $\frac{f}{g} \in \mathcal{O}(X)^\times$. But this is a projective variety, so $\mathcal{O}(X)^\times = k^\times$. We will see that $\mathcal{L}(D)$ is finite-dimensional. So $\mathfrak{P}(\mathcal{L}(D))$ has a natural structure as an algebraic variety.

DEFINITIONS 13.7. A linear subvariety of $|D|$ is called a *linear system*. A linear system is called *complete* if it is of the form $|D|$, for some D.

If $D = \sum n_i Z_i$ then $Supp(D) = \bigcap_{n_i \neq 0} Z_i$.

Let L be a linear system on X. The base points of L are defined to be

$$\bigcap_{D \in L} Supp(D)$$

REMARK 13.8. One of the most important questions in additive function theory is to calculate the dimension of a complete linear system.

EXAMPLE 13.9 (Some linear systems on a projective variety). Let $X \subset \mathbb{P}^n$ and $S(X)$ be the homogeneous coordinate ring. Define a linear system on X as follows. Write $S(X) = \oplus S(X)_d$. Let $f \in S(X)_d$. The elements here are not really functions (they're only defined up to constant multiples because we are in projective space), but they do make well-defined divisors. We want a divisor on $X \cap (\mathbb{P}^n - H_i)$ where $H_i = V(x_i)$. The associated divisor $\left(\frac{f}{x_i^d}\right)$ is an effective divisor on $X \cap (\mathbb{P}^n - H_i)$. When you restrict to an intersection $X \cap (\mathbb{P}^n - H_i - H_j)$ of open subsets, the difference is given by the principal divisor $\frac{x_i^d}{x_j^d}$ but this is invertible on the intersection, and its divisor is zero. Therefore, there exists an effective divisor $(f) =: X \cdot V(f)$ on X whose restriction to $X \cap (\mathbb{P}^n - H_i)$ is $\left(\frac{f}{x_i^d}\right)$.

Another definition of (f): consider $X \cap V(f) = Z_i$. (The intersection doesn't depend on the lifting.) Each of its irreducible components has codimension 1, so Z_i has codimension 1, and $X \cdot V(f) = \sum n_i Z_i$. Let $x_{k(i)}$ be coordinates such that $Z_i \not\subset H_{k(i)}$. Then $n_i = ord_{Z_i} \frac{f}{x_{k(i)}^d}$. Top and bottom are homogeneous functions of degree d. You can check that n_i has nothing to do with the choice of $k(i)$, and that the two definitions are the same.

In this way, we defined a map from $S(X)_d \to Div(X)$. The claim is that the image is a linear system. Why? It is easy to check: $(f) = (g) + \left(\frac{f}{g}\right)$; this is well-defined because f and g have the same degree. So (f) and (g) are in a linear system.

If you fix g, then every other element (f) is of the form $(f) = (g) + \left(\frac{f}{g}\right)$. You can add the rational functions: if

$$(f_1) = (g) + \left(\frac{f_1}{g}\right)$$

$$(f_2) = (g) + \left(\frac{f_2}{g}\right)$$

then $\left(\frac{f_1+f_2}{g}\right)+(g)=(f_1+f_2)$. So the image has a linear structure.

To summarize, this linear system, which Mumford denotes by $L_X(d)$, is isomorphic to $\mathbb{P}(S(X)_d)$.

REMARK 13.10. If two divisors are linearly equivalent – that is, if $D_1 - D_2 = (f)$ – then they define the same linear systems: $|D_1| = |D_2|$. If $|D_1| = (g) + D_1$ then this can also be written $(gf) + D_2$.

THEOREM 13.11. *For $d >> 0$, $L_x(d)$ is complete.*

COROLLARY 13.12. *Every complete linear system is finite dimensional. That is, $\ell(D) < \infty$.*

PROOF. If $|D| \neq \emptyset$ then we can assume that D is effective. Therefore we can write $D = \sum n_i Z_i$ with $n_i \geq 0$. Choose a homogeneous polynomial $F_i \in S(X)_{d_i}$ of degree d_i such that $Z_i \subset X \cap V(F_i)$, but $X \not\subset V(F_i)$. (Z_i is cut out by homogeneous polynomials; choose one that does not vanish on all of X.) Let $F = \prod F_i^{n_i}$ of degree $e = \sum n_i d_i$. F does not vanish identically on X, because each F_i does not. $(F) = G \in L_x(e)$ can be written as $D + D'$ where D' is effective. (Each F_i is Z_i + some effective divisor.) $(F) = \sum n_i(F_i) = D+$ effective thing.

Therefore, from the definition, if an effective divisor is the sum of two effective divisors, then $\mathcal{L}(D) \subset \mathcal{L}(G)$ which implies $\ell(D) \leq \ell(G)$. Replacing G by some multiple mG if necessary, we can assume that e is very large, so that $L_X(e)$ is complete. Therefore, $\ell(G) = \dim L_X(e)$. But the latter thing is finite-dimensional, as it is the projectivization of a homogeneous complement. ⊟

Let $\varphi : X \dashrightarrow \mathbb{P}^n$ be a rational map such that $\varphi(X)$ is not contained in any hyperplane. We will construct a linear system L_φ on X as follows. First, the rational map is given by some correspondence $Z = \overline{Z_\varphi}$ with projections p_1 to X and p_2 to $\mathbb{P}^n$. $Z \to X$ is birational. According to the Zariski main theorem, let $F \subset X$ so that p_1 is not an isomorphism; then $codim F \geq 2$. If p_1 is not an isomorphism, then the fiber is positive-dimensional (always assume X is smooth). Therefore, $\dim p_1^{-1}(F) \leq r - 1$ if X has dimension r. (On a birational map, there is an isomorphism on an open set, and the fiber has positive dimension.)

$\dim p_1^{-1}(F) \leq r-1$ but because the fiber has positive dimension, by the Zariski main theorem we have $\dim F \leq r - 2$.

Let $H \subset \mathbb{P}^n$ be a hyperplane. We define a divisor $\varphi^* H$ on X is follows. (The divisors on X are the same as the divisors on $X - F$. But on $X - F$ the rational map is really a morphism.) So $\varphi^* H$ on $X - F - \varphi^{-1}(H_i)$ is $(\varphi^* \left(\frac{\ell}{x_i}\right))$. Pull back on

$$X - F - \varphi^{-1}(H_i) \xrightarrow{\varphi} \mathbb{P}^n - H_i$$

to get an effective divisor on $X - F - \varphi^{-1}(H_i)$. But
$$\varphi^*\left(\frac{\ell}{x_0}\right)|_{X-F-\varphi^{-1}}(H_i \cup H_j) = (\varphi^*\left(\frac{\ell}{x_j}\right))|_{X-F-\varphi^{-1}}(H_i \cup H_j)$$
Therefore there exists $\varphi^* H$ on $X - F$ (and hence on X).

It is easy to see that if $H = V(\ell)$ and $H' = V(\ell')$ then $\varphi^* H = \varphi^* H' + (\varphi^*(\frac{\ell}{\ell'}))$. This implies that the $\varphi^* H$ form a linear system denoted by $L_\varphi \overset{\cong}{\to} G(n-1, \mathbb{P}^n) =: \mathbb{P}^n$.

PROPOSITION 13.13. *The base points of L_φ are just F.*

PROOF. If $x \notin F$ then $\varphi(x) \in \mathbb{P}^n$ there is some hyperplane H such that $\varphi(x) \notin H$. Therefore, $x \notin \varphi^{-1}(H) = Supp\varphi^* H|_{X-F}$. SO any base point of the linear system is contained in F. Now we want to show that, in fact, every point in F is in the support of this linear system. Let $H \subset \mathbb{P}^n$ be a hyperplane. We need to show that $F \subset Supp\varphi^* H$. Without loss of generality, if I take a linear change of coordinates I can assume that $H = H_0$ (the hyperplane given by the first coordinate plane). Then observe that $\frac{x_i}{x_0}$ is a rational function whose pullback is a rational function on X, and the associated divisor is exactly $\varphi^* H_i - \varphi^* H_0$ (as before). Therefore, on $X - Supp\varphi^* H$, the principal divisor $\left(\varphi^* \frac{x_i}{x_0}\right)$ is effective. This means that $\varphi^* \frac{x_i}{x_0}$ is a regular function on $X - Supp\varphi^* H_0$. Therefore, $\varphi|_{X - Supp\varphi^* H_0}$ is a morphism given by
$$p \mapsto (\varphi^*\left(\frac{x_i}{x_0}\right)(p), \cdots, \varphi^*\left(\frac{x_1}{x_0}\right)(p))$$
The rational map is not single-valued in F. So $F \subset Supp\varphi^* H_0$. ⊟

THEOREM 13.14. *There is a 1-1 correspondence between:*

(1) *a linear system on X, with base point of codimension ≥ 2, and an isomorphism $\psi : L \overset{\sim}{\to} \widehat{P}^n$;*
(2) *a rational map $\varphi : X \dashrightarrow \mathbb{P}^n$, such that the image $\varphi(X)$ is not contained in any hyperplane.*

In addition, this correspondence is PGL_{n+1}-equivariant: namely, on the projective space there is a natural action of PGL_{n+1} by linear change of coordinates. If there is an action of PGL_{n+1} on the projective space, it also acts on the dual projective space. So there is an action on the data (a).

In other words, given a linear system L on X with base points of codimension ≥ 2, there exists a rational map $\varphi : X \dashrightarrow \widehat{L} = \mathbb{G}(n-1, L)$ not contained in any hyperplane.

14. October 25

Recall that there is a natural isomorphism from $V \to V^{\vee\vee}$. But there is a canonical isomorphism $\mathbb{P}(V^\vee) = (\mathbb{P}(V))^\vee$, and there is a canonical isomorphism $\mathbb{P}(V) \cong (\mathbb{P}(V))^{\vee\vee}$ Let X be smooth and projective, and L a linear system on X, with base points of codimension at least 2. We want to construct a rational map $\varphi : X \dashrightarrow \breve{L}$.

SKETCH OF PROOF. Fix $\psi : L \xrightarrow{\sim} (\mathbb{P}^m)^\vee$. Let $D_i = \psi^{-1}(H_i)$, where $H_i = V(x_i)$. We want to use this divisor as coordinates of our variety. (Rational functions are principal divisors; you can regard divisors as functions, up to constant. So you can think of D_i as a function.) Write $D_i - D_0 = (f_i)$ where $f_i \in k(X)$. $f_i \in \mathcal{O}(X - Supp(D_0))$: on $X - Supp(D_0)$ it is effective, and is hence given by a regular function. There is a map

$$X - Supp(D_0) \to \mathbb{P}^m \quad x \mapsto (1, f_1(x), \cdots, f_m(x))$$

...but this doesn't appear to be well-defined, since f_i are only defined up to constant. Oops? $\boxminus$

EXAMPLE 14.1. Consider $X = \mathbb{P}^n$, and consider $L \subset L_X(1)$, which is the linear system of hyperplanes $(\mathbb{P}^n)^\vee$. Given a linear system, the rational map $\varphi_L : X \to L^\vee$, which we claim is just the projection $P_M \to L^\vee$, where $M = \bigcap_{H \in L} H$. If $\dim L = r$ then $\dim M = n - r - 1$.

Conversely, given any linear subvariety $M \subset \mathbb{P}^n$ of dimension $n-r-1$, you can construct

$$L = \{H \in (\mathbb{P}^n)^\vee \,:\, H \supset M\}$$

This is a linear subvariety, hence gives a linear system. So $\varphi_L = P_M$. More generally, if $L_1 \subset L_2 \subset (\mathbb{P}^n)^\vee$, then there are maps

$$\begin{tikzcd}
 & L_2^\vee \arrow[dd, dashed, "p_M"] \\
X = \mathbb{P}^n \arrow[ur, dashed, "\varphi_{L_2}"] \arrow[dr, dashed, "\varphi_{L_1}"'] & \\
 & L_1
\end{tikzcd}$$

EXAMPLE 14.2. If $X = \mathbb{P}^n$ and $L = L_X(d)$ then the rational map $\varphi_L : X \dashrightarrow L^\vee$ is just the d-uple embedding. This linear system has no base point, so the rational map is a regular map.

THEOREM 14.3. *For large enough d, the linear system $L_X(d)$ is complete.*

PROOF. *Wrong in Mumford!* Let $D - dH_i \in L_X(d)$. There are elements (X_i^d), where the corresponding divisor is $d \cdot H_i$. Then $D - dH_i$ is a principal divisor. So $f_i \in k(X)$. You can consider the divisor

$$\left(\frac{f_i}{f_j}\right) = (D = dH_i) - (D - dH_j) = dH_j - dH_i = \left(\frac{x_j^d}{x_i^d}\right)$$

Therefore, in the fractional field L of $S(X)$, $x_i^d f_i = x_j^d f_j$ (up to constant). Because the choice of f_i is determined up to constant, you can choose the constant to make $x_i^d f_i = x_j^d f_j$ an actual equality, for all i and j. So we obtain a well-defined element $F \in L$, given by $F = x_i^d f_i$ for any i. The divisor $(f_i)|_{\mathbb{P}^n - H_i}$ is effective: away from H_i this is the same as D which is effective. This $f_i \in \mathcal{O}(X - H_i \cap X)$. For each i, you can find some N large enough such that $x_i^N f_i \in S(X)$. Therefore, $x_i^{N-d} F = x_i^{N-d} x_i^d f_i \in S(X)_N$.

Define $S'_d \subset L$ by $S'_d = \{\{x \in L \ : \ \exists N, x_i^N s \in S(X)_{N+d} \ \ \forall i\}\}$ Let $S' = \bigoplus_{d \geq 0} S'_d$. Then this is a graded ring containing $S(X)$. Now $F \in S'$. Saying $D \in L_X(d)$ is equivalent to saying $F \in S(X)$. (Remember $F = x_i^d f_i$ and $D = (f_i) + dH_i$.) To show that the linear system is complete for d large enough, it suffices to show that $S'_d = S(X)_d$ for d large enough.

To do this, fix $s \in S'_d$ with $d \geq 0$. Because $x_i^N s \in S(X)_{N+d}$, if we take $\ell_0 = N(n+1)$ then $S(X)_\ell s \subset S(X)_{\ell+d}$ for $\ell \geq \ell_0$ (since $\ell \geq N(n+1)$,at least one power in the monomial goes to N). Likewise, $S(X)_\ell \cdot s^q \subset S(X)$ for all $q \geq 0$. For example, consider $x_0^{\ell_0} \in S(X)$ for $q \geq 0$. $s^q \in \frac{1}{x_0^{\ell_0}} S(X)$ for all $q \geq 0$. (This is like proving all the regular functions on a projective variety are constant.)

Consider $M = \frac{1}{x_0^{\ell_0}} S(X) \subset L$, where M is a finitely generated $S(X)$-module. By some commutative algebra fact, this implies that there exists some polynomial relations

$$s^q + a_1 s^{q+1} + \cdots + a_q = 0$$

where $a_i \in S(X)$. We say that s is integral over $S(X)$. So S' is integral over $S(X)$.

THEOREM 14.4. *Let A be a finitely-generated integral k-algebra, and $K = Frac(A)$. Let L/K be a finite field extension. Then the integral closure A' of A in L is finite over A and A' is also a finitely generated k-algebra.*

PROOF. Commutative algebra. ⊟

So S' is finite over $S(X)$. We can choose a set of homogeneous generators. Let $s_1, \cdots, s_m$ be a set of homogeneous generators of degree d_i. There exists ℓ_0 such that $S(X)_\ell s_j \in S(X)_{\ell+d_i}$ for any $\ell \geq \ell_0$. Therefore, $S'_d = S(X)_d$ if $d \geq \max\{d_i\} + \ell_0$. Elements in S'_d can be written as combinations $\sum S(X)_{d-d_0} s_i$.

($F = x_i^d f_i$. If $F \in S(X)_d$ then $L_X(d) \ni (F) = (x_i^d) + (f_i) = dH_i + (f_i) = D$.) ⊟

14.1. Canonic divisors. Right now, it's hard to construct linear systems on a variety. Of course, there's $0 \in Div(X)$. If we know there's a map $X \dashrightarrow \mathbb{P}^n$, then we can make a linear system. But we would like to do the reverse: start with a linear system, and try to get an embedding of X into projective space.

DEFINITION 14.5. Let $A \to B$ be a (commutative) ring homomorphism. The Kähler differential $(\Omega_{B/A}, d)$ is a pair, where $\Omega_{B/A}$ is a B-module, and d is a derivation in

$\mathrm{Der}_A(B, \Omega_{B/A})$, that satisfies the following universal property: for any B-module M, the natural map $\mathrm{Hom}_B(\Omega_{B/A}, M) \mapsto \mathrm{Der}_A(B, M)$ given by $\varphi \mapsto \varphi \circ d$ is an isomorphism. (Recall from before $\mathrm{Der}_A(B, M)$ is really a B-module.)

LEMMA 14.6. $(\Omega_{B/A}, d)$ *exits and is unique up to a unique isomorphism.*

PROOF. Uniqueness is clear from the universal property. We need to show existence. Define $\Omega_{B/A}$ be the free B-module $\{db : b \in B\}/\sim$ where the relations $\sim$ are $da = 0$, $d(b_1 + b_2) - db_1 - db_2 = 0$, $d(b_1 b_2) - b_1 db_2 - b_2 db_1$ and $d : B \to \Omega_{B/A}$ by $b \mapsto db$. These relations are created by making it a derivation. It is easy to check that this satisfies the universal property. Starting with $D : B \to M$, then $\varphi : \Omega_{B/A} \to M$ by $\varphi(db) = D(b)$. Therefore, $\mathrm{Hom}(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$. ⊟

EXAMPLE 14.7. Let $A = k$, $B = k[x_1, \cdots, x_n]$. Then $\Omega_{B/A} = Bdx_1 \oplus \cdots \oplus Bdx_n$.

COROLLARY 14.8. *If B/A is finitely generated as an A-algebra, with generators $b_1, \cdots, b_r$. Then $\Omega_{B/A}$ is finitely generated over B with generators $db_1 \cdots db_r$.*

REMARK 14.9. "A is finite over B" means A is a finite module over B; "finitely generated" refers to algebras.

LEMMA 14.10. *Let $S \subset B$ be a multiplicative set. Then $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$.*

PROOF. There is a natural map by restriction. $\mathrm{Der}(S^{-1}B, M) = \mathrm{Der}_A(B.M)$ for any $S^{-1}B$-module of M. ⊟

COROLLARY 14.11. *Let X/k be an algebraic variety. Then $\Omega_{\mathcal{O}_{X,x}/k}$ is a finitely generated $\mathcal{O}_{X,x}$-module.*

NOTATION 14.12. If X/k is affine, denote

$$\Omega_{\mathcal{O}(X)/k} = \Omega_X \text{ and } \Omega_{\mathcal{O}_{X,x}/k} = \Omega_{X,x}$$

REMARK 14.13. If X is not affine, then we denote Ω_X by the functor

$$\{\text{Affine opens on } x\} \to \{Ab : U \to \Omega_U\}$$

PROPOSITION 14.14. *Let X/k be an algebraic variety. Then X is smooth at x iff $\Omega_{X,x}$ is a free module over $\mathcal{O}_{X,x}$ of $rk = \dim X = r$.*

PROOF. $\Omega_{X,x} \otimes_{\mathcal{O}_X} k$ is a k-vector space: the cotangent space $T_x^* X = (T_x X)^\vee$. Why? We want $\mathrm{Hom}_k(\Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} k, k)$ to be the tangent space. This is $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X,x}, k)$; by the universal property, this is just $\mathrm{Der}_k(\mathcal{O}_{X,x}, k) = T_x X$.

Now, if $\Omega_{X,x}$ is free of rank r then $\dim_k \Omega_{X,x} \otimes k = r$. This is the same as $\dim T_x X = r = \dim X$, which implies that X is smooth at x. Conversely, suppose X is smooth at x. Then $\dim_k \Omega_{X,x} \otimes k = r$. On the other hand, $\dim_K \Omega_{X,x} \otimes K = r$ (where $K = k(X)$). This is because

$$\mathrm{Hom}(\Omega_{X,x} \otimes K, K) = \mathrm{Hom}(\Omega_{K/k}, K) = \mathrm{Der}_k(K, K)$$

(Taking the differential commutes with localization. But we know the dimension of $\mathrm{Der}_k(K,K)$ is r, because this is how we *defined* dimension.)

Now we are in the following situation: we want to show the following lemma.

LEMMA 14.15. *Let $(A,\mathfrak{m},k=A/\mathfrak{m})$ be a Noetherian local domain with fractional field K, and M be a finite A-module. If $\dim_k M\otimes k = r = \dim_K M$ then M is a free A-module of rank r.*

LEMMA 14.16 (Nakayama lemma). *Let $(A,\mathfrak{m},k)$ be a Noetherian local ring, and M be a finite A-module. If $M\otimes k=0$, then $M=0$.*

PROOF. Commutative algebra. ⊟

PROOF OF 14.15. Let $m_1,\cdots,m_r\in M$ s.t. $m_1\cdots m_r$ mod $\mathfrak{m}$ form a basis of $M\otimes k$. Then consider the map of A-modules

$$A^r\to M \text{ where } e_i\to m_i$$

We claim that this is an isomorphism. If not, then it has a cokernel, N

$$A^r\to M\to N\to 0$$

Tensor with k:

$$A^r\otimes k\to M\otimes k\to N\otimes k\to 0$$

But $M\otimes k$ is generated by m_i, so $N\otimes k=0$, which implies $N=0$ by Nakayama's lemma. Therefore, $A^r\to M\to 0$ is surjective. Consider the kernel

$$0\to N\to A^r\to M\to 0$$

If we tensor with the fractional field we get

$$0\to N\otimes K\to A^r\otimes K\to M\otimes K\to 0$$

But this is surjective, and $N\otimes K=0$. This is a contradiction, because N is a submodule of a free module. $N\otimes K=0$ means: for all $n\in N$ there exists some $a\in A$ such that $an=0$. ($N\otimes K$ is basically the localization.) This is impossible, because n is in a free module A^r, in which no element is torsion.

(We're using the fact that localization preserves short exact sequences.) ⊟

15. October 27

PROPOSITION 15.1. *Let V and W be two (finite-dimensional) vector spaces. Suppose there is an isomorphism*

$$\psi:\mathbb{P}(V)\to\mathbb{P}(W)$$

Then there is a lifting

$$\widetilde{\psi} : V \overset{\cong}{\to} W$$

of vector spaces. In addition, $\widetilde{\psi}$ *is unique up to a non-zero multiple.*

Let L be a linear system on X. Suppose we choose an isomorphism $\psi : L \xrightarrow{\sim} (\mathbb{P}^m)^\vee$. $L = \mathbb{P}(V)$; V is a subspace of a complete linear system $\mathcal{L}(D)$ for $D \in L$. Then

$$\widetilde{\varphi} : V \to k\{x_0, \cdots, x_m\}$$

(Recall $(\mathbb{P}^m)^\vee$ is the space of all hyperplanes in $\mathbb{P}$, and hence the projectivization $\mathbb{P}(k\{x_0, \cdots, x_m\}) = \mathbb{P}(S_1)$.) We can define $f_i = \widetilde{\psi}^{-1}(x_i) \in V \subset \mathcal{L}(D) \subset k(X)^\times$. So we get a map

$$X \dashrightarrow \mathbb{P}^m \text{ where } x \mapsto (f_0(x), \cdots, f_m(x))$$

that is well-defined because $\widetilde{\psi}$ is unique up to constant multiple. The codimension 2 base points are not used.

PROPOSITION 15.2. $x \in X$ *is smooth iff* $\Omega_{X,x}$ *is free of rank* r *over* $\mathcal{O}_{X,x}$.

In the proof, we showed that $\Omega_{X,x} \otimes k \xrightarrow{\sim} T_x^* X = \mathfrak{m}/\mathfrak{m}^2$. More explicitly, we have

$$\begin{array}{ccccc} \mathfrak{m} & \hookrightarrow & \mathcal{O}_{X,x} & \longrightarrow & \Omega_{X,x} \\ \downarrow & & & & \downarrow \\ \mathfrak{m}_x/\mathfrak{m}_x^2 & & \xrightarrow{\sim} & & \Omega_{X,x} \otimes k \end{array}$$

In particular, if $\{x_1, \cdots, x_r\} \subset \mathfrak{m}$ such that $x_1, \cdots, x_r$ mod $\mathfrak{m}^2$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$, then $dx_1, \cdots, dx_r$ form a basis of $\Omega_{X,x}$ over $\mathcal{O}_{X,x}$. (If they form a basis mod $\mathfrak{m}^2$, then the $dx_i \otimes k$ form a basis of $\Omega_{X,x} \otimes k$. Using Nakayama's lemma, if there are elements $dx_1, \cdots, dx_r$ that form a basis mod k, then they form a basis of $\Omega_{X,x}$.) Such $x_1, \cdots, x_r$ are called *local parameters* of X at x.

Then for every $f \in \mathcal{O}_{X,x}$,

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$\frac{\partial}{\partial x_i} \in \mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X,x}, \mathcal{O}_{X,x}) = Der_{\mathcal{O}_{x,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$$

Observation: For X affine, every element in Ω_X can be regarded as a map from $X \to \sqcup T_x^* X$. Namely, let $\omega \in \Omega_X$. Since the map commutes with localization:

$$\Omega_x \to \Omega_{X,x} \to \Omega_{X,x} \otimes k \cong T_x^* X$$

$$\omega \mapsto \omega_x \mapsto \omega(x)$$

There are many such maps; we claim that this one is algebraic. If $f \in \mathcal{O}(X)$ then $df \in \Omega_X$. Regard df as a map like above:

$$df : X \to \sqcup T_x^* X \text{ where } x \mapsto f - f(x) mod \mathfrak{m}^2$$

$\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space; $f - f(x)$ is map that vanishes at x. This does not require the definition of Ω_X. In the spirit of defining regular functions on X,

$$\Omega_{\mathcal{O}(X)/k} = \Omega_X = \{\omega : X \to \sqcup_{x\in X} T_x^* X \ : \ \forall p \in X\ \exists V \in p \text{ open},\ f_i, g_i \in \mathcal{O}(V) s.t. \omega = \sum f_i dg_i\}$$

We know that Ω_X is generated over $\mathcal{O}$ by those relations; this automatically gets an inclusion $\subset$. It is an exercise to show that this is actually an equality. Unlike our other definition, this makes sense for all varieties, not just affine ones.

DEFINITION 15.3. Let X be a smooth variety of dimension r.

$$\omega_{X,x} = \wedge^r_{\mathcal{O}_{X,x}} \Omega_{X,x}$$

is a free $\mathcal{O}_{X,x}$-module of rank one, with a basis given by

$$dx_1 \wedge \cdots \wedge dx_r$$

If $\{x_1', .., x_r'\}$ is another set of local parameters then $dx_1', \cdots, dx_r'$ is another basis, and the difference is given by the determinant $\det(\frac{\delta x_i'}{\delta x_j}) dx_1 \wedge \cdots \wedge dx_r$. Remember $\det(\frac{\delta x_i'}{\delta x_j}) \in \mathcal{O}_{X,x}$ that is invertible, because both the x_i's and x_i''s are bases.

DEFINITION 15.4.

$$\omega_{K/k} = \wedge^r \Omega_{K/k}$$

is a 1-dimensional K-vector space. Elements in $\omega_{K/k}$ are called rational r-forms.

DEFINITION-CONSTRUCTION 15.5. Let $\omega \in \omega_{K/k} - 0$.

$$\omega_{K/k} \cong \omega_{X,x} \otimes_{\mathcal{O}_{X,x}} K \implies \omega = f(dx_1 \wedge \cdots \wedge dx_r)$$

Define $(\omega) = \sum n_i Z_i$ where $n_i = ord_{Z_i} f$. For each i, the definition depends on $x \in Z_i$. You can show that, in fact, everything is well-defined: the dx_i are defined on open neighborhoods.

EXERCISE 15.6. n_i does not depend on the choice of $x \in Z_i$ and local parameters $\{x_1, \cdots, x_r\} \in \mathcal{O}_{X,x}$. (This is similar to the proof of the definition of principal divisors.)

If $f \in K$ then $(f\omega) = (f) + (\omega)$, by definition. This implies that

$$\{(\omega) \ : \ \omega \in \omega_{K/k} - 0\}$$

is a well-defined divisor class called the *canonical divisor [class]*; this is usually denoted by K_X.

Likewise, we define the n-fold canonical class as follows. Let $\omega \in \omega_{K/k} \otimes n$ (recall $\omega_{K/k}$ is a one-dimensional vector space over K, but is infinite-dimensional over k). Locally, $\omega = f(dx_1 \wedge \cdots \wedge dx_r)^n$, where $(\omega) = \sum n_i Z_i$ and $n_i = ord_{Z_i}(f)$ for $x \in Z_i$.

Obvious fact: If $\omega_1 \in \omega_{K/k}^{n_1}$ and $\omega_2 \in \omega_{K/k}^{n_2}$ then $(\omega_1 \otimes \omega_2) = (\omega_1) + (\omega_2)$. This immediately implies that if $\omega \in \omega_{K/k}^n$ then $(\omega) \sim nK_X$.

Now assume X is projective. If $\omega \in \omega_{K/k}^{\otimes n}$ then consider the complete linear system

$$\begin{aligned}\mathcal{L}((\omega)) &= \{f \in K \,:\, f = 0 \text{ or } (f) + (\omega) \geq 0\}\\ &= \{\omega \in \omega_{K/k}^{n} \,:\, (\omega) \geq 0\}\\ &=: \mathcal{L}(nK_X)\end{aligned}$$

DEFINITION 15.7.

$$P_n(X) = \ell(nK_X)$$

is called the n^{th} *plurigenera* of X. In particular, $P_1(X) = P_g(X)$ is called the geometric genus of X.

$\bigoplus_{n\geq 0}\mathcal{L}(nK_X)$ has a k-algebra structure, where multiplication is given by the tensor product, which preserves effectiveness. If $\omega_1 \in \mathcal{L}(n_1K_X)$ and $\omega_2 \in \mathcal{L}(n_2K_X)$ then $\omega_1\otimes\omega_2 \in \mathcal{L}((n_1+n_2)K_X)$. This is called the *canonical ring* of X. Here is a quite recent theorem:

THEOREM 15.8 (Siu, Hacon-McKernan,…). *The canonical ring is a finitely generated k-algebra.*

EXAMPLE 15.9. $X = \mathbb{P}^n$. (The divisor class group is isomorphic to $\mathbb{Z}$; the divisors are multiples of hyperplanes.)

CLAIM 15.10. $K_X = -(n+1)H$ *where H is a hyperplane. This is not effective;* $P_g(\mathbb{P}^n) = 0$.

Let $x_0, \cdots, x_n$ be homogeneous coordinates $y_i = \frac{x_i}{x_0}$, and $y_1, \cdots, y_n$ are coordinates of $\mathbb{P}^n - H_0 \cong \mathbb{A}^n$.

$$\omega = dy_1 \wedge \cdots \wedge dy_n$$

These form a basis at every point. $(\omega)|_{\mathbb{P}^n - H_0} = 0$. Therefore, there is some multiple m such that $(\omega) = mH_0$. We need to show that $m = -(n+1)$. To do this, represent ω in a different coordinate system. Define $z_i = \frac{x_i}{x_n}$ for $i = 0, \cdots, n-1$. $y_0 = \frac{1}{z_0}$ and $y_i = \frac{z_i}{z_0}$ for $1 \leq i \leq n-1$.

$$\begin{aligned}\omega &= d(\frac{z_1}{z_0}) \wedge \cdots \wedge d(\frac{z_{n-1}}{z_0} \wedge d(\frac{1}{z_0}))\\ &= \pm\frac{1}{z_0^{n+1}} dz_0 \wedge \cdots \wedge dz_{n-1}\end{aligned}$$

The dx_i are independent; this is determined by the coefficients $\frac{1}{z_0^{n+1}}$. So $(\omega) = -(n+1)H_0$.

EXAMPLE 15.11. $X = V(y^2z = x^3 + xz^2) \subset \mathbb{P}^2$ where $char(k) \neq 2$. I claim that $K_X = 0$, and so the genus is $P_g(X) = 1$.

$(\mathbb{P}^2 - H_Z) \cap X = \overset{o}{X} = V(y^2 = x^3 + x)$.

$$\Omega_{K/k} = Kdx + Kdy/(2ydy - (3x^2+1)dx)$$

For $(a, b) \in \overset{o}{X}$, $\Omega_{X,(a,b)} \otimes k = kdx + kdy/(2bdy - (3a^2+1)dx)$. (The tangent space is given by the thing you're modding out by; this is how you get the cotangent space.) Homework…

$dx \neq 0$ in $\Omega_{X,(a,b)} \otimes k$ if $b \neq 0$ implies that x is a local parameter at (a,b) is $b \neq 0$.

Let $\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x^2+x}}$. This is a rational differential. The restriction $(\omega)|_{\overset{o}{X}-(y=0)}$ is zero, because it is invertible away from $y = 0$. This is almost the whole curve, just missing the points where $y = 0$, and the point at infinity. In $\Omega_{K/k}$, $2ydy = (3x^2+1)dx$ so $\frac{dx}{y} = \frac{2dy}{3x^2+1}$. At (a,b) if $3a^2+1 \neq 0$ then $dy \neq 0$ in $\Omega_{X,(a,b)} \otimes k$ and this implies that y is a local parameter at (a,b) if $3a^2+1 \neq 0$, i.e.

$$(\omega)|_{\overset{o}{X}-(3x^2+1=0)}$$

Two closed subsets do not have an intersection on the curve; that is, these two open affines cover the curve:

$$\overset{o}{X} = (X - \{y = 0\}) \cup (X - \{3x^2+1 = 0\})$$

so $(\omega) = 0$ on X. Now $X - \overset{o}{X} = X \cap H_Z = p(0,1,0)$. So there is one more point to deal with.

First get local parameters at p. Using the affine coordinates $\widetilde{X} = X - (X \cap H_y) = V(z = x^3 + xz^2)$. Now $\Omega_{X,p} \otimes k = kdx + kdz/(dz = 3x^2dx + z^2dx + 2xzdz)$. At this point, $x = 0$ and $z = 0$, so you only have to mod out by dz (but the relation $dz = 0$ only holds at the point p, and not necessarily in a neighborhood; below, you will need to make the whole expression well-defined there before plugging in the relation). Therefore, x is a local parameter at p. Write $\omega = fdx$ around p. $\omega = \frac{dx}{y}$; changing coordinates $(x,y,1) \mapsto (x,1,z)$ means you're doing $x \mapsto \frac{x}{z}$ and $y \mapsto \frac{y}{z}$. So this differential is

$$\omega = \frac{d(\frac{x}{z})}{\frac{y}{z}} = \frac{zdx - xdz}{z} \in \Omega_{K/k}$$

In $\Omega_{K/k}$ we have $dz = 3x^2dx + z^dx + 2xzdz$. So we can solve for dz:

$$dz = \frac{3x^2+z^2}{1-2xz}dx$$

We get

$$\omega = dx - \frac{x}{z} \cdot \frac{3x^2+z^2}{1-2xz}dx$$

Remember $z = x^3 + xz^2$. $\frac{x^3}{z} = 1 - xz$ so we can rewrite

$$\begin{aligned}\omega &= dx - \frac{\frac{3x^3}{z} + xz}{1-2xz}dx \\ &= dx - \frac{3 - 3xz + xz}{1-2xz}dx \\ &= dx - \frac{3-2xz}{1-2xz}dx \\ &= \frac{-2}{1-2xz}dx\end{aligned}$$

But this is invertible in the local ring (it takes the value 1 at the point), so the order is zero. That is, $(\omega) = 0$ on X.

PROPOSITION 15.12. *If X, Y are birational then $P_n(X) = P_n(Y)$.*

PROOF. If X and Y are birational, then there are neighborhoods such that $X \supset U \xrightarrow{\cong} V \subset Y$. By Zariski's main theorem, $X \backslash U$ and $Y \backslash V$ have codimensions ≥ 2. Removing a codimension 2 variety doesn't affect divisors. Birationality means $k(X) \cong k(Y)$. $\omega_{k(X)/k} \cong \omega_{k(Y)/k}$. All of this implies that there is a canonical isomorphism

$$\mathcal{L}(nK_X) \cong \mathcal{L}(nK_Y)$$

$\boxminus$

COROLLARY 15.13. *$X = V(y^2 = x^3 + x)$ is not birational to $\mathbb{P}^1$.*

(This was proved differently in the homework.)

16. November 1

If $\varphi : X \to Y$ is a morphism, define the pullback $\varphi^* : \Omega_Y \to \Omega_X$. If X, Y are affine, we get a pullback $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Suppose $A \to B \xrightarrow{\varphi^*} C$ is a chain of maps of commutative rings. (B represents $\mathcal{O}(Y)$, and C represents $\mathcal{O}(X)$.) We claim that there is a natural map $\varphi^* : \Omega_{B/A} \otimes_B C \to \Omega_{C/A}$. For any C-module M (which is necessarily also a B-module), $\mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M) \xleftarrow{restriction} \mathrm{Der}_A(C, M) = \mathrm{Hom}_C(\Omega_{C/A}, M)$. Take $M = \Omega_{C/A}$ and take the identity map in $\mathrm{Hom}_C(\Omega_{C/A}, M)$. Then we get

$$\varphi^* = restr(Id) : \Omega_{B/A} \otimes_B C \to \Omega_{C/A}$$

REMARK 16.1.

(1) This map commutes with the derivation d associated with Ω: $\varphi^*(df) = d(\varphi^* f)$.
(2) The kernel is $\mathrm{Der}_B(C, M)$ so there is an exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

which gives

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

Geometrically, if $\omega \in \Omega_Y$ then ω is a map $Y \to \sqcup_{y \in Y} T_y^* Y$ and $\varphi^* \omega$ is a map $X \to \sqcup_{x \in X} T_x^* X$ which takes $x \mapsto (d\varphi)^\vee(\omega(\varphi(x))) \in T_x^* X$ where

$$x \overset{\varphi}{\mapsto} \varphi(x) \xrightarrow{\omega} T^*Y \xrightarrow{(d\varphi)^\vee} T_x^* X$$

It's easy to see that if $\omega = df$ then $\varphi^*(df) = d(\varphi^* f) = d(f \circ \varphi)$ Therefore if locally on Y $\omega = \sum f_i dg_i$ then locally on X

$$\varphi^* \omega = \sum (f_i \circ \varphi) d(g_i \circ \varphi)$$

Now consider the case $A = k$, $B = k(Y)$, $C = k(X)$. Then $\varphi^* : k(Y) \to k(X)$ is induced by a dominant morphism $\varphi : X \to Y$.

$$\Omega_{k(Y)/k} \otimes k(X) \to \Omega_{k(X)/k} \to \Omega_{k(X)/k(Y)} \to 0$$

If $\varphi : X \to Y$ is separable (the dimension is the same as the degree of the transcendental field extension, and $\mathrm{Der}_k(K, k) = \mathrm{Hom}_K(\Omega_{K/k}, K)$ is the dual space)

$$\dim_{k(X)} \Omega_{k(X)/k(Y)} = \dim(X) - \dim(Y)$$

So the right exact sequence is also left exact, by dimension reasons. In particular, if $\dim(X) = \dim(Y)$, then $\Omega_{k(Y)/k} \otimes_{k(Y)} k(X) \cong \Omega_{k(X)/k}$.

Last time we had define $\omega_{k(Y)/k} = \wedge^{\dim(Y)}\Omega_{k(Y)/k}$ then

$$\varphi^* : \omega_{k(Y)/k} \otimes_{k(Y)} k(X) \cong \omega_{k(X)/k}$$

Let $\omega \in \omega_{k(Y)/k}$ (a rational top form on Y). We could associate two divisors on X. We could consider $\varphi^*(\omega \otimes 1)$ (which we will write $\varphi^*(\omega)$). On the other hand, we get a divisor ω on Y, and we can get a divisor on X $\varphi^{-1}(\omega)$. These are not the same, but it turns out that the difference is an effective divisor.

Lemma 16.2. *Let $\varphi : X^r \to Y^r$ be separable. $B := (\varphi^*(\omega \otimes 1)) - \varphi^{-1}(\omega)$ is an effective divisor, called the branched divisor.*

Recall $\varphi^{-1} : Div(Y) \to Div(X)$ which takes a divisor $D = \sum n_i Y_i \mapsto \varphi^{-1} D = \sum n_i \varphi^{-1}(Y_i)$. Write $\varphi^{-1}(Y_i) = \sum m_j Z_j$. Let $x \in X_j$ and a local equation for Y_i at $\varphi(x)$. Locally, Y_i is just (f) around $\varphi(x)$. $m_j = \mathrm{ord}_{x_j} \varphi^*(f)$ is shown to be well-defined in the homework.

Proof. Write $\omega = f dy_1 \wedge \cdots \wedge dy_r$ where $(y_1, \cdots, y_r)$ are local parameters at $y = \varphi(x)$ (we can always represent a rational differential this way). Then $\varphi^*\omega = (f \circ \varphi) d(y_1 \circ \varphi) \wedge \cdots \wedge d(y_r \circ \varphi)$. If $x_1, \cdots, x_r$ are local parameters at x, then this is

$$(f \circ \varphi) \det\left(\frac{\partial(y_i \circ \varphi)}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_r$$

This determinant is in $\mathcal{O}_{X,x}$.

$$\begin{aligned}(\varphi^*\omega) - \varphi^{-1}(\omega) &= (f \circ \varphi) \det(\frac{\partial(y_i \circ \varphi)}{\partial x_j}) - (f \circ \varphi) \\ &= (\det \frac{\partial(y_i \circ \varphi)}{\partial x_j})\end{aligned}$$

In other words,

$$Supp(B) = \{x \in X \,:\, (\det \frac{\partial(y_i \circ \varphi)}{\partial x_j}) \notin \mathcal{O}^*_{X,x}\}$$

If this element is a unit, then the associated divisor is zero. The condition that the determinant is not in $\mathcal{O}^*_{X,x}$ is the same as it being in $\mathfrak{m}_x$. So

$$Supp(B) = \{x \in X \,:\, (d\varphi_x)^\vee : T^*_{\varphi(x)}Y \to T^*_x X \text{ not an isomorphism}\}$$

If these are two vector spaces of the same dimension, this is equivalent to requiring that $d\varphi_x : T_x X \to T_{\varphi(x)Y}$ is not surjective. (It has to be injective, maybe because the variety is smooth(?)) *Supp*(B) is sometimes called the *branched locus of* φ. (Geometrically, think about the places where the tangent is vertical: the size of the fiber is anomalous).

$\boxminus$

EXAMPLE 16.3. Let $X = (y^2z = x^3 + x^2)$ in characteristic $\neq 2$. There is a projection $P_{(0,1,0)} : (x, y, z) \mapsto (x, z)$. There is a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$; but since this is a smooth curve, it extends to a morphism.

Let (x, y, z) be homogeneous coordinates on $\mathbb{P}^2$ and (u, v) be homogeneous coordinates on $\mathbb{P}^1$. Then $\varphi : (x, y, z) \mapsto (x, z)$ is a map $\varphi : \overset{o}{X} \to \mathbb{A}^1 = \{v \neq 0\}$ (where $\overset{o}{X}$ is the affine piece). Then $\varphi^* : k[u] \to k[x, y]/y^2 - x^3 - x$ is $u \mapsto x$ the projection to the first coordinate. Let $du \in \Omega_{k(\mathbb{P}^1)/k}$. By last time, u is a local parameter on the entire affine line, so $\varphi^{-1}(du) = 0$ but $\varphi^*du = dx$. x is a local parameter on $\overset{o}{X} - \{y = 0\}$ is a local parameter. Around $\{y = 0\} = \{(1, 0), (0, 0), (-1, 0)\}$, dx is a local parameter (no zeroes or three anomalous points. But at all of these three points, $3x^2 + 1 \neq 0$; and y is a local parameter so the vanishing order is 1. Therefore, (dx) on $\overset{o}{X} = [1] + [0] + [-1]$.

There is a point at infinity. Consider the blowup $\widetilde{X} \subset \mathbb{P}^1 \times \mathbb{P}^2$. There is an isomorphism $\widetilde{X} \to X$, and a projection to $\mathbb{P}^1$.

$$\begin{cases} y^2z = x^3 + xz^2 \\ xv = zu \end{cases}$$

These two equations cut out an algebraic set with two components, one of which is the blowup $\widetilde{X}$, and the other is the exceptional divisor. In the case $y \neq 0$ and $v \neq 0$ (we're in $\mathbb{A}^2 \times \mathbb{A}^1$) this turns into

$$\begin{cases} z = x^3 + xz^2 \\ xv = z \end{cases}$$

$\widetilde{X} \cap (\mathbb{A}^2 \times \mathbb{A}^1)$ is given by $k[x, z, v]/xv = z$. Combine the equations above as $xv = x^3 + xz^2$. The exceptional. divisor corresponds to $x = 0$ in the first of the equations above. Outside of the exceptional divisor, we can divide this by x.

Remember that the rational differential ω is du. This is given by $\omega = -\frac{dv}{v^2}$. If $p \in X \cap \{y \neq 0\}$, then there is a $\widetilde{p} \in \widetilde{X} \cap (\mathbb{A}^2 \times \mathbb{A}^1)$. x is a local parameter around $\widetilde{p}$. Therefore, B around p or $(\widetilde{p})$ is given by the local parameter $\frac{\partial(v \circ \varphi)}{\partial x_j}$. Last time, we showed that x is a local parameter around p. $X \cap \mathbb{P}^2$ is given by $z = x^3 + xz^2$; x is a local parameter because the x-derivative is nonzero. So B around p or $\widetilde{p}$ is given by $3d(v \circ \varphi)/dx = \det(dy \circ \varphi/\partial x_j)$.

$v = x^2 + y^2v^2$ and $dv = 2xdx + 2xv^2dx + 2x^2vdv$ which gives $\frac{dv}{dx} = \frac{2x+2xv^2}{1-2x^2r} = x \cdot \frac{2+2v^2}{1-2x^2v}$. So this function has vanishing order exactly one at p. So $B = [1] + [0] + [-1] + [\infty]$.

Alternatively, write $z = \frac{x^3}{y^2=xz}$ and map $(x, y, z) \mapsto (1, \frac{x^2}{y^2-xz})$, and do some calculations.

The point is that an elliptic (cubic) curve can be written as a double cover of $\mathbb{P}^1$ with branching at four points.

16.1. Hilbert polynomial.

Theorem 16.4 (Hilbert). *Let M be a finitely generated graded module over the graded polynomial ring $S = k[x_0, \cdots, x_n]$. Then there exists a polynomial $p_M(t) \in \mathbb{Q}[t]$ of degree at most n such that $p_M(d) = \dim_k M_d$ for d sufficiently large.*

Proof. By induction. If $n = -1$ this is trivial, because $S = k$ and the module M is a finite-dimensional k-vector space. So $M_d = 0$ for d sufficiently large; this implies $p_M = 0$.

Now, suppose the theorem holds for $< n$. Let

$$N' = \{m \in M \,:\, x_n m = 0\}$$
$$N'' = M/x_n M$$

Then N' and N'' are finitely generated graded $k[x_0, \cdots, x_{n-1}]$-modules. By induction, there is a $P_{N'}(t) \in \mathbb{Q}[t]$ and $P_{N''}(t) \in \mathbb{Q}[t]$ such that $P_{N'}(d) \dim_k N'_d$ and $P_{N''}(d) = \dim_k N''_d$. We have the following short exact sequence:

$$0 \to N'_d \to M_d \xrightarrow{x_n} M_{d+1} \to N''_{d+1} \to 0$$

This implies that

$$\dim M_{d+1} - \dim M_d = \dim N''_{d+1} - \dim N'_d = P_{N''}(d+1) - P_{N'}(d)$$

Observe that $R(t) := P_{N''}(t+1) - P_{N'}(t) \in \mathbb{Q}[t]$ is of degree $\leq n-1$ by induction.

Lemma 16.5. *Let $f(t) \in \mathbb{Q}[t]$ be of degree m. There exists $g(t) \in \mathbb{Q}[t]$ of degree 1 such that $g(t+1) - g(t) = f(t)$. Therefore, there exists $p(t) \in \mathbb{Q}[t]]$ such that $p(t+1) - p(t) = p_{N''}(t+1) - p_{N'}(t) = \dim M_{d+1} - \operatorname{im} M_d$ for d large enough. This implies that $\dim M_{d+1} - p(d+1) = \dim M_d - p(d)$. Then $\dim M_d - p(d) = constant$ and $\dim M_d = \underbrace{p(d) + constant}_{p_M(d)}$ for d large enough.*

∎

17. November 3

Theorem 17.1. *If M is a finitely generated graded $S = k[x_0, \cdots, x_n]$-module, then there exists a polynomial $P_M \in \mathbb{Q}[t]$ of degree $\leq n$ such that $P_M(d) = \dim_k M_d$ for d large enough. In particular, let $M = S/I(X)$ where X is an algebraic set in $\mathbb{P}^n$. Then $P_X = P_{S/I(X)}$ is the Hilbert polynomial of X.*

Example 17.2.

(1) Let $X = \mathbb{P}^n$. Then $M = S$, S_d is the set of homogeneous polynomials of degree d, and the dimension $\dim_k S_d = \binom{n+d}{n}$. Therefore, the Hilbert polynomial for $\mathbb{P}^n$ is
$$P_{\mathbb{P}^n}(t) = \binom{t+n}{n} = \frac{(t+n)(t+n-1)\cdots(t+1)}{n!}$$
This is $\frac{1}{n!}t^n +$ lower terms.

(2) Now suppose $X = V(f) \subset \mathbb{P}^n$ is a hypersurface. $S(X) = S/(f)$. Suppose $\deg f = d$. Then there is a short exact sequence
$$S_m \xrightarrow{f} S_{m+d} \to S(X)_{m+1} \to 0$$
So $\dim_k S(X)_{m+d} = \dim_k S_{m+d} - \dim_k S_m = \binom{m+d+n}{n} - \binom{m+n}{n}$ for $m \geq d$. So the Hilbert polynomial is
$$P_X(t) = \binom{t+n}{n} - \binom{t-d+n}{n}$$
This is $\frac{d}{(n-1)!}t^{n-1}+$ lower terms.

(3) $X = \{p_1, \cdots, p_m\}$. We can assume that $p_i \notin H_0 = V(X_0)$ (there is a hyperplane that does not intersect any of the finite set of points). We get a short exact sequence
$$0 \to I(X)_d \to S_d \xrightarrow{\Phi} k^m$$
where
$$f \mapsto \left(\frac{f}{x_0^d}(p_1)\cdots\frac{f}{x_0^d}(p_m)\right)$$
A priori this is just left exact (the RHS might not be surjective). But for d large enough, there exist f_i where $\deg f_i = d$, and $f_i(p_j) \neq 0$ iff $i = j$. Specifically, $f_i = \prod_{j\neq i}(a_j x_0 - x_1)$. So for d large enough, Φ is surjective. So for large enough d, $S(X) \cong k^m$ and $P_X(t) = m$.

REMARK 17.3.

(1) If X is smooth an projective, $L_X(d) = \mathbb{P}(S(X))$ and $\dim L_X(d) = P_X(d) - 1$ for d large enough. But we also showed that this is complete, for d large enough. So we can write this as $\dim |dH| = P_X(d) - 1$, and $P_X(d) = \ell(dH)$.

(2) Let X be smooth and projective. Then consider the graded ring $\bigoplus_{d\geq 0}\mathcal{L}(dK_X)$. You can show that there exists some polynomial $P(t)$ such that $P_t(d) = \ell(dK_X)$ for d large enough. (This does not follow from our theorem.) The degree of this polynomial is called the Kodaira dimension of X, and is in the range $[-1, \dim X]$. (-1 is for the zero polynomial.)

DEFINITION 17.4. Let $X^r \subset \mathbb{P}^n$ be projective. The arithmetic genus of X is
$$P_a(X) = (-1)^r(P_X(0) - 1)$$

THEOREM 17.5.

(1) *If $X \cong Y$ as varieties, then $P_a(X) = P_a(Y)$.*

(2) *If X, Y are smooth and birational, then $P_a(X) = P_a(Y)$.*

This is hard; no proof.

EXAMPLE 17.6. (1) $P_a(\mathbb{P}^n) = 0$, because $P_{\mathbb{P}^n}(0) = 1$

(2) $P_a(V(f))$.

$$P_X(t) = \binom{t+n}{n} - P\binom{t+n-d}{n}$$

$$P_X(0) = 1 - \binom{n-d}{n}$$

$$P_a(X) = (-1)^n \binom{n-d}{n} = \begin{cases} 0 & d \ll n \\ \binom{d-1}{n} & d > n \end{cases}$$

(3) $P_a(y^2z = x^3) = P_a(y^2z = x^3 + zx^2) = P_a(y^2z = x^3 + z^2x) = 1$ because these are all smooth projective rational curves, and hence isomorphic to $\mathbb{P}^1$

THEOREM 17.7 (Serre duality). *If X is a smooth projective curve, then $P_a(X) = P_g(X)$.*

Note that geometric genus was only defined for smooth varieties, but algebraic genus is defined for any variety.

PROPOSITION 17.8. *Let $X^r \subset \mathbb{P}^n$ be projective. Then the Hilbert polynomial $P_X(t) = \frac{d}{r!}t^r +$ lower terms, where $d \in \mathbb{Z}_{>0}$.*

DEFINITION 17.9. d as used in the previous proposition is called the degree of X. (This coincides with the degree of a hypersurface.)

PROOF. We know that this is true if X is a hypersurface. Now let $(n-r-2)$-plane. Then

$$P_M : X \to X' := P_M(X) \subset \mathbb{P}^{r+1}$$

is birational (homework). We know that

(1) $S(X)$ is a finite $S(X')$ module (proven before)

(2) $k(X) \overset{\cong}{\leftarrow} k(X')$ because they're birational. But this implies $Frac(S(X)) \cong Frac(S(X'))$, because $Frac(S(X)) = k(X)(x_0)$, and $Frac(S(X')) = k(X')(x_0)$ can be written as transcendental extensions. Why? Elements are the ratio of homogeneous polynomials of the same degree. But $\frac{f}{g} \in Frac(S(X))$ can be written as

$$\sum \frac{\lambda_\alpha x^\alpha}{g} = \sum \frac{\lambda_\alpha}{\frac{g}{x^\alpha}} = \sum \lambda_\alpha / x_0^{\alpha \cdot \beta} \sum \lambda_\beta \frac{x^\beta}{\beta x^\alpha} x_0^{\alpha \cdot \beta}$$

Let $f_1, \cdots, f_s$ be the generators of $S(X)$ over $S(X')$. Because they have the same fractional field we can write $f_i = \frac{g_i}{h_i}$ for $g_i, h_i \in S(X')$. If $h = \prod h_i$ then $S(X) \subset \frac{1}{h}S(X')$. This is equivalent to saying that $hS(X) \subset S(X')$. Decompose h into homogeneous components: $h = \sum_d$. Each component is in $S(X')$. So we can assume that there is some $h \in S(X')$ homogeneous such that $h \cdot S(X) \subset S(X')$ homogeneous of degree d_0. Therefore,

$$\dim S(X)_d \leq \dim S(X)_d \leq \dim S(X)_{d+d_0}$$

Therefore,

$$P_{X'}(d) \le P_X(d) \le P_{X'}(d+d_0)$$

for d large enough. So $P_X - P_{X'}$ is a polynomial of degree $< \deg P_{X'}$. But we know that $P_{X'}(t) = \frac{D}{r'}t^r + \cdots$. So $P_X = \frac{D}{r!}t^r +$ lower terms.

$\boxminus$

17.1. Introduction to intersection theory.

PROPOSITION 17.10. *Let Z^n be an affine variety and X^r, Y^s be closed subvarieties. Let $x \in X^r \cap Y^s$. Assume that X is smooth in Z^n and write $X^r \cap Y^s = W_1 \cup \cdots \cup W_m \cup W^*$ where W_i are irreducible components and W^* is some algebraic set that does not contain x. Then* $\dim W_i \ge r+s-n$.

COROLLARY 17.11. *Let $X^r, Y^s \subset \mathbb{P}^n$ be projective and $r+s \ge n$. Then $X^r \cap Y^s \neq \emptyset$.*

PROOF OF COROLLARY. Take the affine cone $C(X^r), C(Y^s) \subset \mathbb{A}^{n+1}$.

$$C(X^r) \cap C(Y^s) \ni 0$$

where $\dim C(X^r) = r+1$ and $\dim C(Y^s) = s+1$. The proposition implies that $\dim C(X^r) \cap C(Y^s) \ge (r+1)+(s+1)-(n+1) \ge 1$. So $X^r \cap Y^s \neq \emptyset$. $\boxminus$

PROOF OF PROPOSITION. $X \cap Y = (X \times Y) \cap \Delta$ where $\Delta : Z \to Z \times Z$ is the diagonal embedding. $X \times Y$ has dimension $r+s$. Once the intersection is cut out by n equations, each component has dimension $\ge r+s-n$. It is enough to show that around (x,x), Δ is cut out by $f_1, \cdots, f_n \in \mathcal{O}(Z \times Z)$. Each time you cut out a variety by one equation, the dimension drops by at most one.

Let $x_1, \cdots, x_n$ be local parameters at x (we're assuming that x is a smooth point). By shrinking Z, we can assume that x_i is actually in $\mathcal{O}(Z)$. Consider $f_i = x_i \otimes 1 - 1 \otimes x_i \in \mathcal{O}(Z \times Z)$. $f_i(p,p') = x_i(p) - x_i(p')$ for $(p,p') \in Z \times Z$. Clearly, $f_i|_\Delta = 0$. But I claim that $df_1, \cdots, df_n$ are linearly independent in the cotangent space $T^*_{(x,x)}(Z \times Z) = T^*_x Z \oplus T^*_x Z$. This is because

$$\mathfrak{m}_{(x,x)} = \mathfrak{m}_x \otimes \mathcal{O}(Z) + \mathcal{O}(Z) \otimes \mathfrak{m}_x$$

implies that $x_1 \otimes 1, \cdots, x_n \otimes 1, 1 \otimes x_1, \cdots, 1 \otimes x_n$ form local parameters at (x,x). $df_i = d(x_i \otimes 1) - d(1 \otimes x_i)$ are linearly independent. Consider the regular map

$$\varphi = (f_1, \cdots, f_n) : Z \times Z \to \mathbb{A}^n(y_1, \cdots, y_n)$$

which is smooth at (x,x). $\varphi^*(dy_i) = df_i$. We can write $f^{-1}(0) = Z^* \cup Z^{**}$, where $Z^* \ni (x,x)$ and $Z^{**} \not\ni (x,x)$. $\dim Z^* = n$, $\mathbb{Z}^*$ is smooth at (x,x). $\Delta \subset f^{-1}(0)$ implies $Z^* = \Delta$ i.e. $\Delta = V(f_1, \cdots, f_n)$ around (x,x). $\boxminus$

THEOREM 17.12. *Let $X^r \subset \mathbb{P}^n$ be of degree d (i.e. degree defined by the Hilbert polynomial). Then there is an open subset $U \subset \mathbb{G}(n-r, \mathbb{P}^n)$ such that for any $L \in U$, $\#L \cap X^r = d$.*

PROOF. Recall the incidence correspondence

$$\begin{tikzcd} & C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{G}(r,n) & & \mathbb{P}^n \end{tikzcd}$$

$C \xrightarrow{\pi_2} \mathbb{P}^n$ is smooth with fibers isomorphic to $\mathbb{G}(n-r-1, n-1)$. Why? We proved that π_1 is a fiber bundle with fiber = projective space. In this case, if we consider $\pi_2 : \pi_2^{-1}(U_i) \to U_i = \mathbb{P}^n - H_i$ then

$$\pi_2^{-1}(U_i) \cong U_i \times \mathbb{G}(n-r-1, n-1)$$

in which $(x, L) \in \pi_2^{-1}(U_i)$ maps to $(x, L \cap H_i)$. You can check that this map induces an isomorphism. Therefore,

$$\begin{tikzcd} & \pi_2^{-1}(X) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{G}(n-r,n) & & X \end{tikzcd}$$

where $\pi_2^{-1}(X)$ is irreducible of dimension $r+r(n-r) = r(n-r+1)$, since the Grassmannian has dimension $r(n-r+1)$. The claim is that π_1 is separable. If this is separable, then there is an open subset of the Grassmannian such that the fiber is the transcendental degree; identify this with d. $\boxminus$

18. November 8

THEOREM 18.1. *Let $X^r \subset \mathbb{P}^n$ be projective of degree d. Then there exists an open $U \subset \mathbb{G}(n-r, \mathbb{P}^n)$ nonempty such that for any $L \in U$*

- $X^r \cap L = \{p_1, \cdots, p_k\}$
- *For any $p_i \in X^r \cap L$, X^r, L are smooth at p_i and $T_{p_i}X^r + T_{p_1}L = T_{p_i}\mathbb{P}^n$. Equivalently (by dimension reasons) $T_{p_i}X^r \cap T_{p_i}L = 0$*

In this case, $k = d$.

(So degree is the number of intersection points of a "generic" hyperplane.)

PROOF.

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{G}(n-r,n) & & \mathbb{P}^n \end{tikzcd}$$

p_1 is smooth with fibers isomorphic to $\mathbb{P}^{n-r}$, and p_2 is smooth with fibers isomorphic to $\mathbb{G}(n-r-1, n-r)$. Consider the preimage $p_2^{-1}(X) \subset C$ where $X \subset \mathbb{P}^n$. Then $p_2^{-1}(X)$

is irreducible of dimension $(n-r+1)r$. For any L, $p_1^{-1}(L) \cap p_2^{-1}(X) \cong X \cap L$ which is nonempty, so p_1 is surjective, and in particular it is dominant.

Smooth at a point $\iff$ separable. We claim that $p_1 : p_2^{-1}(X) \to \mathbb{G}(n-r,n)$ is separable. Equivalently, there exists $(x,L) \in p_2^{-1}(X)$ such that $p_1 : p_2^{-1}(X) \to \mathbb{G}(r,n)$ is smooth at (x,L). Equivalently there exists $(x,L) \in p_2^{-1}(X)$ such that $p_2^{-1}(X)$ is smooth at (x,L) and $dp_1 : T_{(x,L)}p_2^{-1}(X) \to T_L\mathbb{G}(n-r,n)$ is surjective. Locally on $\mathbb{P}^n$ $C \to \mathbb{P}^n$ is a fibration, with fiber $\mathbb{G}(n-r-1,H)$. If $x \in X \cap (\mathbb{P}^n - H)$ has preimage (x,L) then

$$T_{(x,L)}p_2^{-1}(X) \cong T_xX \oplus T_L\mathbb{G}(n-1-r,H) = dp_2^{-1}(T_xX)$$

So $p_2^{-1}(X)$ is smooth at (x,L) iff X is smooth at x.

(Note: $\ker dp_{1(x,L)} = T_{(x,L)}p_1^{-1}$)

What is the kernel of $dp_1 : T_{(x,L)}C \to T_L\mathbb{G}(r,n)$?

$$dp_2(\ker(dp_1)) = T_xL$$

(The kernel are the vectors tangent to L.) $dp_2 : \ker(d[_1) \cap T_{(x,L)}p_2^{-1}(X) \xrightarrow{\cong} T_xL \cap T_xX$ Injectivity means this kernel is zero: i.e. $T_xL \cap T_xX = 0$ is equivalent to $dp_1 : T_{(x,L)}p_2^{-1}(X) \twoheadrightarrow T_L\mathbb{G}(n-r,n)$. Now choose $x \in X$ such that X is smooth at x. Choose $x \in L^{n-r} \subset \mathbb{P}^n$ such that $T_xX \cap T_xL = 0$. Then $dp_1 : p_2^{-1}(X) \to \mathbb{G}(n-r,n)$ is smooth at (x,L). But we proved that once there is one point where the map is smooth, there is an open set where it's smooth. But these spaces have the same dimension, and we don't need $char = 0$ to say: there exists $U \subset \mathbb{G}(n-r,n)$ non-empty and open such that $p_1 : p_1^{-1}(U) \cap p_2^{-1}(X) \to U$ is smooth.

For all $L \in U$, $L \cap X = \{p_1, \cdots, p_k\}$ (it's smooth, and the relative dimension is zero).

$$T_xL \cap T_xX = 0 \iff T_xL + T_xX = T_x\mathbb{P}^n$$

We need to show that $k = d$. Reduce to the case where X is a hypersurface. Let $L \in U$. Let $L \in U$. Let $x \in L$ such that $x \notin \bigcup \overline{p_ip_j}$ (the union of the lines through the p_i). $p_x : \mathbb{P}^n \to \mathbb{P}^{n-1}$ induces a birational morphism $X \to p_2(X)$. $\deg X = \deg P_x(X)$ (last time we proved that birational projection does not change the leading term of the Hilbert polynomial.) Also, $X \cap L \xrightarrow{\cong} p_x(X) \cap p_x(L)$

Therefore, we reduce to the case X is a hypersurface in $\mathbb{P}^n$ of degree d. Set $X = V(f)$. L is a line $\overline{pq}$; $X \cap L$ contains the zeroes of $f(\lambda p + \mu q) = 0$, and this is homogeneous of degree d. The condition $T_xX \cap T_xL = 0$ is equivalent to f and f' not having common zeroes; i.e. f has distinct roots. Since this is an equation of degree d there are exactly d solutions.

$\boxminus$

DEFINITION 18.2. Let $X^r, Y^s \subset Z^n$ where Z is smooth. Let $X^r \cap Y^s = \bigcup W_i$ where W_i are irreducible components. We say that X and Y intersect at W_i properly if $\dim W_i = r+s-n$ (the minimum). We say that X, Y intersect transversely if there exists $x \in W_i$ such that

X, Y are smooth at x and $T_xX + T_xY = T_xZ$ (so the intersection of these vector spaces has dimension $r + s - n$).

LEMMA 18.3. *If X, Y intersect transversely at W and $x \in W$ is a point as in the definition then W is smooth at x.*

PROOF. $X \cap Y = X \times Y \cap \Delta$; since Z is smooth, the diagonal is locally cut out by n equations. So $X \cap Y = \varphi^{-1}(0)$ where $\varphi : X \times Y \to \mathbb{A}^n$ is smooth at (x, x). ⊟

THEOREM 18.4 (Bézout's Theorem). *Let $X^r, Y^s \subset \mathbb{P}^n$ are projective varieties. Assume that $X \cap Y$ transversely. Then*

$$\deg X \cdot \deg Y = \sum \deg \underset{\textit{irred. components}}{{}_{W_i \in X \cap Y}} W_i$$

PROOF.

NOTATION 18.5. For W_i let $W_i' \subset W_i$ be the open subset consisting of x such that X, Y are smooth at x and $T_xX + T_xY = T_x\mathbb{P}^n$.

Case 1: $Y^s = L^s$ is an s-plane. $X^r \cap L^s = \bigcup W_i$ where $\dim W_i = r + s - n$. Choose a plane M^{2n-r-s} such that $M \cap W_i$ transversely for any i.

$$\#X \cap (L \cap M) = \#(X \cap L) \cap M = \sum \#W_i \cap M = \sum \deg W_i$$

Since $L \cap M$ is an $(n - r)$-plane, the expression on the left is $\deg X$.

Case 2: $r + s = n$. Consider $X \subset \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ where the map $\mathbb{P}^n \to \mathbb{P}^{2n+1}$ has coordinates $(a_0, \cdots, a_n) \mapsto (a_0, \cdots, a_n, 0, \cdots, 0)$. $Y \subset \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ with the embedding $(b_0, \cdots, b_n) \mapsto (0, \cdots, 0, b_0, \cdots, b_n)$. The Hilbert polynomial does not change in this embedding.

DEFINITION 18.6. $J(X, Y) = \bigcup_{x \in X, y \in Y} \overline{xy}$. In other words,

$$J(X, Y) = \{(a_0, \cdots, a_{2n+1}) : (a_0, \cdots, a_n, 0, \cdots, 0) \in X, (0, \cdots, 0, a_{n+1}, \cdots, a_{2n+1}) \in Y\}$$

In terms of coordinate rings, $S(J(X, Y)) = S(X) \otimes S(Y)$; the homogeneous ideal is $S \otimes I(Y) + I(X) \otimes S$. $\dim S(J(X, Y))_\ell = \sum_{i=0}^{\ell} \dim S(X)_i \dim S(Y)_{\ell-1}$.

$$\dim S(X)_i = \frac{d_x}{r!} i^r + o(i^r) = \binom{i + r}{r} + o(i^r)$$

$$\dim S(Y)_{\ell-i} = \binom{\ell - i + s}{s} + o((\ell - i)^s)$$

$$\begin{aligned}\dim S(J(X,Y))_\ell &= \sum_{i=0}^{\ell}\left(\binom{i+r}{r}+o(i^r)\right)\left(\binom{\ell-i+s}{s}+o((\ell-i)^s)\right)\\ &= d_x d_y \sum_{i=0}^{\ell}\binom{i+r}{r}\binom{\ell-i+s}{s}+o(\ell^{r+s+1})\\ &= d_x d_y\binom{i+r+s+1}{r+s+1}+o(\ell^{r+s+1})\end{aligned}$$

Therefore,

$$P_{J(X,Y)}(t)=\frac{dxdy}{(r+s+1)!}t^{r+s+1}+\text{ lower terms}$$

i.e. $J(X,Y)$ is of dimension $r+s+1$ and $\deg = d_x d_y$.

Take $L=(x_0-x_{n+1},x_1-x_{n+2},\cdots,x_n-x_{2n+1})$ (an n-plane). Then $L\cap J(X,Y)=X\cap Y$. $X\cap Y$ is transverse means that $L\cap J(X,Y)$ is transverse. Then $\#X\cap Y=\#L\cap J(X,Y)=d_x d_y$.

General case. Choose a plane M^{2n-r-s} that intersects each W_i transversely. Then $\#M\cap(X\cap Y)=\#M\cap(\bigcup W_i)=\sum\deg W_i$. On the other hand, $M\cap(X\cap Y)=X\cap(Y\cap M)$. We want $Y\cap M$ to be transverse. Let $x\in Y\cap M$. Then $T_xW_i+T_xM=T_x\mathbb{P}^n$ and this implies that $T_xY=T_xM=T_x\mathbb{P}^n$ which shows that $Y\cap M$ transversely. Therefore, $Y\cap M=\bigcup Y_i$. Finally, $\#X\cap(Y\cap M)=\#X\cap(\bigcup Y_i)=\sum\#(X\cap Y_i)$. But Y_i has dimension $n-r$, by transversality. We reduce to case 2, which gives $\cdots=\sum\deg X\deg Y_i=\deg X(\sum\deg Y_i)$ which is $\deg X\deg Y$ by case 1.

$\boxminus$

19. November 10

THEOREM 19.1 (Bézout's theorem). *If $X^r,Y^s\subset\mathbb{P}^n$ intersect properly and transversely (i.e. X and Y are smooth at a point and their tangent spaces together span the tangent space of $\mathbb{P}^n$), and $X\cap Y=\bigcup W_i$ then*

$$\deg X\cdot\deg Y=\sum\deg W_i$$

In particular, if X and Y are plane curves in $\mathbb{P}^2$ of degree d and e respectively, then $\#X\cap Y=de$.

We introduced a new variety $J(X,Y)\subset\mathbb{P}^{2n+1}$, the union of all the lines joining a point of X with a point on Y. By calculating the Hilbert polynomial we know that $\deg J(X,Y)=\deg X\cdot\deg Y$. $\#X\cap Y=\#J(X,Y)\cap L$ for some plane L. This eventually reduces to the case where Y is a plane. In the general case, we had $X\cap Y=\cup W_i$ where $W_i'\subset W_i$ is open. Look at the points where X and Y intersect transversely. If X ad Y intersect transversely, then there is only one component through that point. We can always find a plane M^{2n-r-s} such that $M \cap W_i = \{p_1, ..., p_{k_1}\}$. Then $\sum \deg W_i = \#(X \cap Y) \cap M = \#X \cap (Y \cap M) = \deg X \cdot \deg Y$.

Now we want to drop the restriction that the varieties intersect transversely; what if they only intersect properly?

CLARIFICATION 19.2. Two varieties intersecting transversely means that for every component there is a point at which they intersect transversely. (If the intersection has multiple components, usually the intersection will not be transverse in the intersection point of the intersection variety.) In this case, transverse intersection implies proper intersection.

Assume $X^r, Y^s \subset Z^n$ are smooth and intersect properly. So each irreducible component has dimension $r+s-n$. Let $X \cap Y = \bigcup W_i$. Then one can define the intersection number $i(X, Y, W)$ of X, Y along W_i. The definition will not be given here. But here is a special case: if $X = V(f)$ and $Y = V(g)$ are plane curves in $\mathbb{A}^2$ or $\mathbb{P}^2$ and $p \in X \cap Y$ then $i(X, Y, W_i)$ is the length of $\mathcal{O}_{\mathbb{A}^2,p}/(f,g)$ as an $\mathcal{O}_{\mathbb{A}^2,p}$-module.

THEOREM 19.3 (Bézout's theorem, general case). *If $X, Y \subset \mathbb{P}^n$ intersect properly then*

$$\deg X \cdot \deg Y = \sum_{W_i \in X \cap Y} i(X, Y, W_i) \deg W_i$$

19.1. Curve theory.

DEFINITION 19.4. An algebraic curve is an algebraic variety of dimension one.

LEMMA 19.5. *Let X be a smooth curve and Y be a projective variety. Then every rational map $\varphi : X \dashrightarrow Y$ is a morphism.*

This is basically Zariski's main theorem, but there we require X to be at least quasiprojective. But here, look at the graph $\Gamma\varphi$ with projections to X and Y. We need to show that $\Gamma\varphi \to X$ is an isomorphism. But $\Gamma\varphi_{|_U} = p^{-1}(U)$, so we can reduce to the affine case. So $\Gamma\varphi_{|_U}$ is quasiprojective and we can use Zariski's main theorem.

COROLLARY 19.6. *Let X, Y be two smooth projective curves. If X, Y are birational, then they are isomorphic.*

So classifying smooth projective curves up to birational isomorphism is the same as classifying them up to actual isomorphism. Note that this is not true for surfaces: take a surface and blow up at a point.

Now suppose we have $X \to k(X)$ where $tr.d.k(X)/k = 1$. Can we find a smooth projective curve with function field $k(X)$? (If there is one it's unique by the corollary.(So classifying smooth projective curves up to birational isomorphism is the same as classifying them up to actual isomorphism.))

DEFINITION 19.7. A function field K/k is a finitely generated field over k of transcendental degree 1.

QUESTION 19.8. Given K can we find a smooth projective curve with $k(X) \cong K$? (You can always find a curve with the right function field, but is it birational to something smooth and projective?) Equivalently, given any curve Y can we find a smooth projective X birational to it?

THEOREM 19.9. *Given K there exists such X.*

DEFINITION 19.10. A variety X is called *normal* if for every $x \in X$, $\mathcal{O}_{X,x}$ is integrally closed in $k(X)$.

FACT 19.11. If X is affine, then X is normal iff $\mathcal{O}(X)$ is integrally closed in $k(X)$.

PROOF. Suppose $\mathcal{O}(X)$ is integrally closed; we want to show each local ring is integrally closed. let $u \in k(X)$ where

$$u_n + a_1 u^{n-1} + \dots + a_n = 0$$

for $a_i \in \mathcal{O}_{X,x}$. Write $a_i = \frac{b_i}{c_i}$ where $b_i, c_i \in \mathcal{O}(X)$ and $c_i \notin \mathfrak{m}_x$. Multiplying out denominators,

$$d_0 u^n + d_1 u^{n-1} + \dots + d_n = 0$$

for $d_i \in \mathcal{O}(X)$ and if $v = d_0 u$, $v^n + e_1 v^{n-1} + \dots + e_n = 0$ for $e_i \in \mathcal{O}(X)$. So $v \in \mathcal{O}(X)$ and $u = \frac{v}{d_0} \in \mathcal{O}_{X,x}$.

Now suppose X is normal. Let $u \in k(X)$.

$$u^n + a_1 u^{n-1} + \dots + a_n = 0$$

for $a_i \in \mathcal{O}(X) \subset \mathcal{O}_{X,x}$. So $u \in \mathcal{O}_{X,x}$. etc. ⊟

Let X be an algebraic variety. Then the normalization of X is $\widetilde{X}$ normal, together with a morphism $\pi : \widetilde{X} \to X$ such that if Y is normal and $f : Y \to X$ is dominant, then there exists a unique $\widetilde{f} : Y \to \widetilde{X}$ making the diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

FACT 19.12. The normalization exists and is unique.

PROOF. If X is affine, then let A be the integral closure of $\mathcal{O}(X)$ in $k(X)$, and $\widetilde{X} = \mathit{Spec}(A)$. By the previous fact, $\widetilde{X}$ is normal, because A is integrally closed.

Let $f : Y \to X$ be dominant. Because X is affine, this corresponds to a map $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$, which produces an injective (because the morphism is dominant) map $k(X) \hookrightarrow k(Y)$. Write $Y = \cup Y_i$ for Y_i an affine open. SO $k(Y) \supset \mathcal{O}(Y_i)$: But now the composition map naturally factors through A:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{O}(Y_i) \arrow[dd, hook] \\
& A \arrow[dl] & \\
k(X) \arrow[rr] & & k(Y)
\end{tikzcd}$$

If $u \in k(X)$, $u \in A$ then

$$u^n + a_1 u^{n-1} + ... + a_n = 0$$

for $a_i \in \mathcal{O}(X)$. So

$$f^*(u)^n + ... + f^*(a_n) = 0$$

which implies $f^*(u) \in \mathcal{O}(Y_i)$. So f factors through its normalization:

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d, hook] & \widetilde{X} \arrow[d] \\
Y \arrow[r, "\widetilde{f}"] & X
\end{tikzcd}$$

In the general case, $X = \bigcup X_i$ where X_i are affine. You can make the maps factor through the intersection, and glue all the $\widetilde{X}_i$'s to get a $\widetilde{X}$:

$$\begin{tikzcd}
& \widetilde{X}_i \arrow[r, "\pi"] \arrow[dl] & X_i \arrow[d, hook] \\
\widetilde{X} & \widetilde{X_i \cap X_j} \arrow[u] \arrow[d] \arrow[r] & X_i \cap X_j \arrow[d, hook] \\
& \widetilde{X}_j \arrow[r] \arrow[ul] & X_j
\end{tikzcd}$$

The intersection of two affine varieties is still affine, because it is $X \times Y \cap \Delta$, and the product of two affine varieties is affine and Δ is closed by assumption.

$\boxminus$

THEOREM 19.13. *If X is smooth, then X is normal.*

But the converse is not true: $xy = z^2$ is normal but not smooth. (Exercise for you: $k[x, y, z]/z^2 - xy$ is integrally closed.)

FACT 19.14. If $\mathcal{O}_{X,x}$ is a UFD then it is integrally closed.

PROOF.

$$u^n + a_1 u^{n-1} + ... + a_n = 0$$

for $a_i \in \mathcal{O}_{X,x}$. $u = \frac{f}{g}$ and $(f,g) = 1$

$$f^n + a_1 g f^{n-1} + ... + a_n g^n = 0$$

Let $\mathfrak{P}$ be a prime divisor of g. Then $\mathfrak{P} \nmid f^n$, which is a contradiction. ⊟

THEOREM 19.15. *If X is normal, then X_{sing} has codimension ≥ 2.*

PROPOSITION 19.16. *Let X be normal and $Y \subset X$ be a closed subvariety of codimension one. Then there exists some open subset $U \subset X$ with $U \cap Y \neq \emptyset$ and $Y = V(f)$ on U.*

PROOF OF THEOREM, GIVEN PROPOSITION. Suppose X_{sing} has codimension 1. That is, there exists a subvariety $Y \subset X_{Sing}$ of codimension one. By shrinking X we can assume that $Y = V(f)$. Let $y \in Y$ be a smooth point (of Y, not X of course!). Let $y_1, ..., y_{n-1}$ be local parameters of Y at y. That is, $y_1, ..., y_{n-1}$ generates the maximal ideal $\mathfrak{m}_{Y,y}$. But the y_i are regular functions on Y that can be lifted to regular functions on X.

$$0 \to (f) = I(Y) \to \mathcal{O}(X) \to \mathcal{O}(Y) \to 0$$

so $y_1, ..., y_{n-1}, f$ generate $\mathfrak{m}_{X,y}$. So $dy_1, ..., dy_{n-1}$ generate $\mathfrak{m}_{X,y}/\mathfrak{m}^2_{X,y}$. This is a contradiction: this vector space has dimension $> n$ because X is singular there. ⊟

PROOF OF PROPOSITION. We will prove the proposition for $\dim X = 1$; the higher dimensional case just requires more definitions. $Y = \{y\} \in X$. If X is normal, then we claim that this is defined by one equation on some open neighborhood of X. We can assume X is affine. Let $f \in \mathcal{O}(X)$ such that $f(y) = 0$. By shrinking X we can assume that the zero locus of f is just $\{y\}$ (if there are other zeroes can just throw them away). By Hilbert's Nullstellensatz, $\sqrt{(f)} = \mathfrak{m}_y$. Therefore, there exists some r such that $\mathfrak{m}_y^r \subset (f) \subset \mathfrak{m}_y$ (the last inclusion is automatic because f vanishes at y). Let r be the smallest integer such that $\mathfrak{m}_y^r = (f)$.

To be continued... ⊟

20. November 15

Presentations: December 1-3, 20 minute presentations. Proofs only if there is time. Maybe examples. Written survey due later: 5-10 pages.

PROPOSITION 20.1. *Let X be a normal variety, $Y \subset X$ of codimension 1. Then locally on X, Y is defined by one equation. That is, there is some open $U \subset X$ such that $U \cap Y \neq \emptyset$ and $I(U \cap Y) \subset \mathcal{O}(U)$ is generated by one element.*

PROOF. First do the case where $\dim X = 1$. Let $Y = \{x\}$. By shrinking X we can find $f \in \mathcal{O}(X)$ such that x is the only zero of f. In $\mathcal{O}_{X,x}$, $\mathfrak{m}_x^r \subset (f) \subset \mathfrak{m}_x$. Choose r such that $\mathfrak{m}_x^{r-1} \not\subset (f)$ and $\mathfrak{m}_x^r \subset (f)$. Then there are some $a_1, \cdots, a_{r-1} \in \mathfrak{m}_x$ such that $g = a_1, \cdots, a_{r-1} \notin (f)$ and $g\mathfrak{m}_x \subset (f)$.

Let $u = \frac{g}{f} \notin \mathcal{O}_{X,x}$ but $u\mathfrak{m}_x \subset \mathcal{O}_{X,x}$. I claim that $u\mathfrak{m}_x \not\subset \mathfrak{m}_x$. Otherwise, $u : \mathfrak{m}_x \to \mathfrak{m}_x$ is an $\mathcal{O}_{X,x}$-module homomorphism. So there exists $F \in \mathcal{O}_{X,x}[t]$ monic such that $F(u) = 0$, which means that u is integral over $\mathcal{O}_{X,x}$. But $\mathcal{O}_{X,x}$ is integrally closed. So $u \in \mathcal{O}_{X,x}$, which is a contradiction. Therefore, $u\mathfrak{m}_x = \mathcal{O}_{X,x}$. This implies that $\mathfrak{m}_x = (\frac{1}{u})$. We are done: we were trying to prove that the ideal is generated by one element.

REMARK 20.2. In fact, what we showed is that if X is a normal curve, $\mathcal{O}_{X,x}$ is in fact a principal domain.

Now do the general case. Recall that if $x \in X$ is a point, then we have defined $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}(U)$. Now if $U \subset X$ is a closed subvariety, we can define $\mathcal{O}_{X,Y} = \varinjlim_{U \cap Y = \emptyset} = \mathcal{O}(U)$. It is easy to see that $\mathcal{O}_{X,Y}$ is a local ring with residue field $k(Y)$ and maximal ideal $\mathfrak{m}_{X,Y} = \varinjlim_{U \cap Y \neq \emptyset} I(U \cap Y)$.

Now the proof is the same if we replace $\mathcal{O}_{X,x}$ in the 1-dimensional case with $\mathcal{O}_{X,Y}$. Namely, (1) there exists some open set $U \subset X$ such that $U \cap Y \neq \emptyset$ and $U \cap Y$ is set-theoretically defined by one equation; and (2) $\mathcal{O}_{X,Y}$ is integrally closed (proof is the same as the fact that in the affine case X normal $\iff$ $\mathcal{O}_{X,x}$ is integrally closed). $\boxminus$

For normal varieties, the singular locus has codimension ≥ 2. So normal curves are smooth.

THEOREM 20.3. *For any finitely-generated function field K/k, there is some smooth variety such that $k(X) = K$.*

PROOF. Let K/k be a function field; i.e. $tr.d.K/k = 1$. Choose any projective curve X such that $k(X) = K$: take any finitely generated k-algebra $B \subset K$ (whose fraction field is K), and take $U = Spec(B)$ affine, embed this in $\mathbb{P}^n$ and take the closure. However, the point is that this might not be smooth.

Now let $\widetilde{X}$ be the normalization of X. Then $\widetilde{X}$ is smooth. Cover X by affine opens U_i such that any two points of X are contained in some U_i. By construction, $\widetilde{X} = \bigcup \widetilde{U}_i$, where $\widetilde{U}_i$ is the normalization of U_i. The $\widetilde{U}_i$ are affine smooth. Let $\overline{U_i}$ be the closure of $\widetilde{U}_i$ in some projective space; each $\overline{U_i}$ is projective. There is a rational map $\widetilde{U}_i \to \overline{U_i}$, and $\widetilde{U}_i \subset \widetilde{X}$, so this extends to a rational map from $\widetilde{X} \stackrel{\varphi}{\to} \overline{U_i}$. Let $\varphi : \widetilde{X} \to \prod \overline{U_i}$, and $\overline{X}$ be the closure of the image of φ. So we have

$$\widetilde{X} \to \overline{X} \stackrel{closed}{\hookrightarrow} \prod \overline{U_i}$$

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi"] & \overline{X} \arrow[r, hook, "closed"'] & \prod \overline{U_i} \arrow[d] \\ \widetilde{U}_i \arrow[u, hook] \arrow[rr, "open"] & & \overline{U_i} \end{tikzcd}$$

Now I claim that φ is an isomorphism (this implies the theorem). It is clear that φ is injective. If $x, y \in \widetilde{X}$, there is some $\widetilde{U_i}$ such that $x, y \in \widetilde{U_i}$. If $\varphi(x) = \varphi(y)$ then $x = y$ by diagram chasing. (Same in $\overline{U_i}$ means same to begin with.)

To show that φ is surjective, we need a lemma:

LEMMA 20.4. *Let $\pi : \widetilde{X} \to X$ be the normalization of a variety. Then π is surjective.*

PROOF. By construction, it is enough to show that this is true if X is affine. (The whole thing is made by gluing together affine pieces.)

$$A = \mathcal{O}(X) \subset B = \mathcal{O}(\widetilde{X}) \subset k(X)$$

Let $x \in X$; then $\mathfrak{m}_x \subset A$. It is enough to show $\mathfrak{m}_x B \neq B$. If B is integral over A then it is finite over A. By Nakayama's lemma, if B is a finite A-module and $\mathfrak{m}_x B = B$ then $B = 0$.

(If $\widetilde{X}$ is smooth, then the closure of its image is smooth and projective.)

$$\begin{tikzcd} & & \overline{\overline{X}} \arrow[d] \\ X & \widetilde{X} \arrow[l] \arrow[r] & \overline{X} \end{tikzcd}$$

Let $\overline{\overline{X}}$ be the normalization of $\overline{X}$. $\widetilde{X}$ is a smooth normal variety that maps to $\overline{X}$ via a dominant morphism. I claim that φ is an open embedding; this will be proved at the end.

$\widetilde{X}$ can be regarded as an open subset of $\overline{\overline{X}}$. We can extend $\widetilde{X} \to X$ to get a morphism $\overline{\overline{X}} \to X$. This is dominant, and factors through a unique morphism $\overline{\overline{X}} \to \widetilde{X}$.

$$\begin{tikzcd} & \widetilde{X} \arrow[r, hook] & \overline{\overline{X}} \arrow[d] \arrow[dl] \arrow[dll, dashed] \\ X & \widetilde{X} \arrow[l] \arrow[r] & \overline{X} \end{tikzcd}$$

The morphism $\widetilde{X} \to X$ extends $\overline{\overline{X}} \to X$ because X is projective and $\overline{\overline{X}}$ is smooth. Since $\widetilde{X}$ is the normalization of X, it factors as

$$\begin{tikzcd} \overline{\overline{X}} \arrow[r, "\psi"] \arrow[dr] & \widetilde{X} \arrow[d] \\ & X \end{tikzcd}$$

Now let $x \in \overline{X}$, and choose $x' \in b\overline{X}$ a lift of x. Then $x = \psi(x') \in \widetilde{X}$ so $\varphi : \widetilde{X} \to \overline{X}$ is surjective.

So it remains to show that φ is an open embedding.

$$\begin{tikzcd} \widetilde{X} \arrow[r] & \overline{X} \subset \prod \overline{U_i} \arrow[d, "dominant"] \\ \widetilde{U}_i \arrow[u, hook] \arrow[r, hook] & \overline{U_i} \end{tikzcd}$$

At the level of local rings we have

$$\begin{tikzcd} \mathcal{O}_{\widetilde{X},x} \arrow[d, "\sim"] & \mathcal{O}_{\overline{X},\varphi(x)} \arrow[l] \\ \mathcal{O}_{\widetilde{U_i},x} & \mathcal{O}_{\overline{U_i},x} \arrow[l, "\sim"'] \arrow[u, hook] \end{tikzcd}$$

The isomorphisms and injections in this picture force $\mathcal{O}_{\widetilde{X},x} \xleftarrow{\varphi^*} \mathcal{O}_{\overline{X},\varphi(x)}$ to be injective.

$\boxminus$

DEFINITION 20.5. Let $\varphi : X \to Y$ be a morphism of algebraic varieties. Then φ is called finite if for any affine open $V \subset Y$, $\varphi^{-1}(V) =: U$ is affine and $\mathcal{O}(U)$ is a finite $\mathcal{O}(V)$-module.

PROPOSITION 20.6. *Let $\varphi : X \to Y$ be a non-constant morphism of smooth projective curves. Then φ is finite.*

PROOF. φ is surjective and therefore dominant: the image is closed, it is not a point, and is connected, so it must be all of Y. Let $V \subset Y$ affine. Consider $A = \mathcal{O}(V) \subset k(V)$

$$\begin{tikzcd} A = \mathcal{O}(V) \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ k(Y) \arrow[r, hook] & k(X) \end{tikzcd}$$

where B is the integral closure of A in $k(X)$. Let $U = Spec(B)$. U is integrally closed, hence normal, hence smooth. In terms of varieties, this is

$$\begin{tikzcd} V \arrow[d] & U \arrow[l, "\varphi_{|U}"] \arrow[d, dashed] \\ Y & X \arrow[l, "\varphi"] \end{tikzcd}$$

Where $U \dashrightarrow X$ is the natural rational map that is induced. But this extends to a morphism because U is smooth and projective.

B is finite over A by general commutative algebra, so we are done if we show $\varphi^{-1}(V) = U$. Suppose this is false. Then there is some $y_0 \in V$ and $x_0 \notin U$ such that $\varphi(x_0) = y_0$.

Sub-claim: there is some $f \in k(X)$ such that f has poles at x_0, and $f \in \mathcal{O}(U)$. I first show that, if true, this leads to a contradiction. $f \in B = \mathcal{O}(U)$ so

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0$$

for $a_i \in \mathcal{O}(V) = A$. So

$$f = -a_1 - \frac{a_2}{f} - \cdots - \frac{a_n}{f^{n-1}}$$

But f having poles at x_0 means $\frac{1}{f} = \mathcal{O}_{X,x_0}$ because $\mathcal{O}_{X,x_0}$. (If you choose a local parameter you can write $f = ux^r$ for $r < 0$, so $\frac{1}{f}$ is a regular element.) This is a contradiction.

Now I prove the sub-claim. Let $\varphi^{-1}(y_0) = \{x_0, \cdots, x_n\}$. Let $\widetilde{f}$ be any function regular on $\{x_1, \cdots, x_n\} \cap U$, but with poles at x_0. This is possible, because there's an affine open such that $x_0, \cdots, x_n$ are in the affine open. So choose some rational function on the affine open that is regular on the other x_i but has a pole on x_0. If $\widetilde{f}$ has no poles in U, we are done (function is regular on U and has a pole on x_0). Otherwise, let $x' \in U$. $\widetilde{f}$ has a pole at x'. $\widetilde{f}$ is regular on x_i so x' is not one of these x_i. So $\varphi(x') = y' \in V$ for $y' \neq y_0$. Choose $h \in \mathcal{O}(V)$, $h(y_0) \neq 0$, where h vanishes at y' with sufficiently high order. (You can do whatever you want, over an open affine.) Take $f = \widetilde{f}\varphi^*(h)$. Then f will have no pole at x'. Continue: if $\widetilde{f}$ has more than one pole, then do this finitely many times.

$\boxminus$

21. November 17

If $\varphi : X \to Y$ is a nonconstant projective morphism, and X and Y are smooth, projective curves, then φ is a finite morphism: i.e. for $V \subset Y$ affine open, $U = \varphi^{-1}(V)$ is affine open and $\mathcal{O}(U)$ is a finite $\mathcal{O}(V)$-module.

Definition 21.1. If X is a smooth projective curve, recall the prime divisors are just points. Define the degree map

$$\deg : X \to \mathbb{Z} \text{ where } D = \sum n_i p_i \mapsto \deg(D) = \sum n_i$$

Theorem 21.2. *If X is a smooth projective curve and $D = (f)$ is principal, then* $\deg D = 0$.

Corollary 21.3. *The degree map descends to a map*

$$\deg : Cl(X) = \mathrm{Pic}(X) = Div(X)/Div^0(X) \to \mathbb{Z}$$

Definition 21.4.

$$\mathrm{Pic}^0(X) = \ker(\deg : \mathrm{Pic}(X) \to \mathbb{Z})$$

Proposition 21.5. *Let $\varphi : X \to Y$ be a dominant (i.e. nonconstant) morphism of smooth projective curves. Let $D \in Div(Y)$. (We defined the pullback $\varphi^* D \in Div(X)$: you just have to define it by prime divisors, which are locally given by one equation. Pull back this equation and take the degree; this defines a divisor.) Then* $\deg \varphi^* D = \deg D \cdot \deg \varphi$ *where* $\deg \varphi = [k(X) : k(Y)]$.

PROOF OF THEOREM. Let $f \in k(X)$; this can be regarded as a rational morphism $\varphi : X \dashrightarrow \mathbb{P}^1$; but we can extend this to a morphism. This is nonconstant if f is nonconstant. Exercise: $(f) = \varphi^*([0] - [\infty])$. Then $\deg(f) = \deg \varphi \cdot \deg([0] - [\infty])$. ⊟

COROLLARY 21.6. *Let X be a smooth projective curve. If there are $x, y \in X$ and $x - y$ is principal, then $X \cong \mathbb{P}^1$.*

PROOF. If $x - y = (f)$, $\varphi : X \to \mathbb{P}^1$ then $x = \varphi^*([0])$. So $\deg \varphi = 1$. So φ is birational; if it's birational, we have proved before that it is an isomorphism. ⊟

PROOF OF PROPOSITION. (Use the natural embedding $k(Y) \subset k(X)$.) We can assume that $D = p$. Write $\varphi^{-1}(p) = \{p_1, \cdots, p_\ell\} \subset X$.Let y be a local parameter at $p \in Y$. In $\mathcal{O}_{X,p_i}$, $y = u_i x_i^{r_i}$ where u_i is a unit of $\mathcal{O}_{X,x_i}$. Then $\varphi^* D = \sum r_i p_i$. We want to show that $\sum r_i = \deg \varphi$. Let $V \ni p$ be an affine open. Define $A = \int(V)$, $U = \varphi^{-1}(V)$ (affine), $B = \mathcal{O}(U)$. Then B/A is a finite A-module. Let $\mathfrak{m}_p \subset A$ be the maximal ideal corresponding to p. By localization, there is a map $\mathcal{O}_{Y,p} \cong A_{\mathfrak{m}_p} \to B_{\mathfrak{m}_p}$, where $B_{\mathfrak{m}_p}$ is a finite $A_{\mathfrak{m}_p}$-module. But $B_{\mathfrak{m}_p}$ is a free. $A_{\mathfrak{m}_p}$ is a principal domain; any module over one of these is free, plus some torsion module. The torsion could only happen in y; but y is not torsion on B. Alternatively, this is an integral domain so it has no torsion. So we can write $B_{\mathfrak{m}_p} \cong A_{\mathfrak{m}_p}^{\oplus n}$, where $n = \deg \varphi$. $B_{\mathfrak{m}_p} \otimes k(Y) = k(X)$. Therefore, $\dim_k B_{\mathfrak{m}_p}/y$ is an n-dimensional vector space. (y generates the maximal ideal $\mathfrak{m}_p B_{\mathfrak{m}_p}$, so we are here just modding out by the maximal ideal.)

Claim: $B_{\mathfrak{m}_p} = \bigcap \mathcal{O}_{X,p_i}$. The inclusion $\subset$ is obvious; we prove the other direction.

$$B_{\mathfrak{m}_p} = \{\frac{f}{g} \,:\, g \in \mathcal{O}(V), g(p) \neq 0, f \in B = \mathcal{O}(U)\}$$

Let $h \in \bigcap \mathcal{O}_{X,p_i}$. h is regular away from $q_2, \cdots, q_m$, $q_i \neq p_j$. $p \in W := V - \varphi(q_1, \cdots, q_m) \subset V$ (the q_i are the places where h is not regular: these are not any p_i). [If the q_i are defined by f, then $\mathcal{O}(W) = \int(V)_f$. $h \in \mathcal{O}(\varphi^{-1}(W)) = \mathcal{O}(W) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$.]

$$h \in \mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) \subset B_{\mathfrak{m}_p}$$

We can write $y - u(x_1^{r_1} \cdots x_n^{r_n})$. We have

$$B_{\mathfrak{m}_p}/y \cong \prod \mathcal{O}_{X,p_i}/x_i^{r_i}$$

This implies that $\dim_k B_{\mathfrak{m}_p}/y = \sum \dim_k \int_{X,p_i}/x_i^{r_i} = \sum r_i$. ⊟

EXAMPLE 21.7 (Calculating $\mathrm{Pic}^0(X)$). $\mathrm{Pic}^0(\mathbb{P}^1) = 0$.

Let X be the elliptic curve defined by $y^2 z = x_3 - xz^2$. Consider the divisor class given by a hyperplane section $L_X(1)$. We can consider the hyperplane $\{z = 0\} =: H_z$. Take $p_0 = (0, 1, 0)$ to be the origin, and note that $H_Z \cap X = p_0$; around this point, the local equation is $z = x^3 - xz^2$. x is a local parameter at p_0. By the non-Archimedean property of ord, $\mathrm{ord}_{p_0} z = \mathrm{ord}_{p_0} x^3 - xz^2 \geq \min\{\mathrm{ord}_{p_0} x^3, \mathrm{ord}_{p_0} xz^2\} \geq 3$. Since $\mathrm{ord}_{p_0}(xz^2) \geq 7$, $\mathrm{ord}_{p_0} z = ord_{p_0}(x^3 - xz^2) = 3$. So $3p_0 \in L_X(1)$. Define a map $AJ : X \to \mathrm{Pic}^0(X)$ sending $p \mapsto p - p_0$. I claim that AJ is injective: if $p - p_0 \sim q - p_0$ then $p \sim q$. But by the corollary, if two points are equivalent, then your curve is $\mathbb{P}^1$. But this curve is not $\mathbb{P}^1$ (for example, they have different genuses). So this is a contradiction.

Now I claim that AJ is surjective. Let $D \in Div(X)$, where $\deg D = 0$. Because D has degree zero, it can be written $\sum n_i p_i$ with $\sum n_i = 0$, we can write this as $D = \sum n_i(p - p_0)$. All the n_i can be assumed to be positive: given $x \in X$, consider the line $\overline{xp_0}$ which intersect X in x, p_0, and somewhere else, say y. Then $L_X(1) \ni x + p_0 + y \sim 3p_0$. Then $x - p_0 \sim p_0 - y$. So if one of the n_i were negative, we could replace x with y, which is linearly equivalent, and switch the sign.

If $\sum n_i = 1$ we are done. Otherwise, write $D = (p_1 - p_0) + (p_2 - p_0) + \cdots$. $\overline{p_1p_2} \cap X = \{p_1, p_2, q\}$.

$$L_X(1) \ni p_1 + o_2 = q \sim 3p_0$$
$$p_1 - p_0 + p_2 - p_0 \sim p_0 - q \sim q' - p_0$$

where q' is the other point on the line through q, p_0. We have reduced $\sum n_i$ by one; after continuing this procedure sufficiently many times, we will eventually get $D \sim p - p_0$.

So here is the Abel-Jacobi map:

$$AJ : X \xrightarrow{\cong} \mathrm{Pic}^0(X)$$

Since the RHS is a group, this gives a group structure on X. More precisely, p_0 is the identity element. Addition is given as follows: $p_1 + p_2 := AJ^{-1}(AJ(p_1) + AJ(p_2)) = AJ^{-1}(p_1 - p_0) + (p_2 - p_0)$. This is just q'. So in other words, take a line through p_1 and p_2, find where else q it hits the curve. Now find the other point q' on the line through q and p_0. q' is the desired sum. To get $-p_1$, take the line through p_1 and p_0, and find where else it intersects the curve.

THEOREM 21.8. *Let X be an elliptic curve. The multiplication map $m : X \times X \to X$ and the inverse $s : X \to X$ are morphisms of algebraic varieties.*

PROOF. Push points around. ⊟

Let $x = (a, b, 1) \in X$. The line $\overline{xp_0}$ is given by $az - x = 0$. Therefore,

$$\overline{xp_0} \cap X = \begin{cases} az - x = 0 \\ y_2 z = x^3 + xz_2 \end{cases}$$

There should be three solutions: $(0, 1, 0), (a, b, 1), (a, -b, 1)$. So $s : (a, b, 1) \mapsto (a, -b, 1)$. This shows that this is a morphism when you restrict to this affine open.

DEFINITION 21.9. A *group variety* is an algebraic variety G together with an element $e \in G$, and regular morphisms $m : G \times G \to G$ and $s : G \to G$, such that (G, e, m, s) form a group in the usual sense.

A group variety is called an *abelian variety* if G is projective. If G is projective, then the group structure is necessarily abelian. Unfortunately, the converse is false.

COROLLARY 21.10. *X defined by $xy^2 = x^3 - xz^2$ is an abelian variety.*

REMARK 21.11. In fact, if $char(k) \neq 2, 3$ then every (geometric) genus one projective curve is isomorphic to a cubic curve in $\mathbb{P}^2$ of the form $y^2z = x_3 + axz^2 + bz^3$, with $4a^3 + 27b^2 \neq 0$. This is called the *Weierstrass normal form*. By the same argument, X is an abelian variety, choosing $(0, 1, 0)$ as the "infinity point."

Suppose $k = \mathbb{C}$. Write $\omega = \frac{dx}{y}$; this is a regular differential. In the elliptic case, this is $\frac{dx}{\sqrt{x^3+ax+b}}$, and was called the elliptic integral. Let's construct a map $X \to \mathbb{C}/L$ where L is a lattice, by taking $p \mapsto \int_{p_0}^{p} \omega$. This does not give a well-defined map $X \to \mathbb{C}$; it's possible to choose a path from $p_0 \mapsto p_1$ (i.e. one containing a non-contractible loop) that makes a different integral. These are called periods. So to make this map well-defined, you have to mod out by the lattice generated by the periods (in this case, there are only two periods, which is good, since that way you get a two-dimensional $\mathbb{R}$-lattice inside of $\mathbb{C}$). This map ends up being an isomorphism, so X turns out to be isomorphic to a torus.

EXAMPLE 21.12 (Linear affine algebraic groups). $GL_n = \{A \in M_{n\times n} : \det A \neq 0\}$. Since the determinant is a polynomial, this cuts out a linear variety.

$PGL_n = \{GL_n/k^\times I_n\}$ has a group structure. You can also show that this is an affine variety.

$\mathbb{G}_m = GL_1 = \mathbb{A}^1 - \{0\}$ is a special case of GL_n. This is a commutative group... but we don't call it an abelian variety.

$\mathbb{G}_a \cong (\mathbb{A}^1, +)$ is $\mathbb{A}^1$ with the natural addition structure.

Given a quadratic form on your space, you can construct SO, the subgroup of GL_n preserving your form. Every linear affine algebraic group can be realized as a subgroup of GL_n.

But the situation is different for projective varieties:

THEOREM 21.13. *Every abelian variety is abelian.*

Elliptic curves are examples here. In general, if X is a smooth projective curves of genus g, $Pic^0(X)$ is an abelian variety of dimension g. (This is just a group, but it has a natural algebraic variety structure.)

In fact, every algebraic group G is an extension

$$0 \to N \to G \to A \to 1$$

where A is an abelian variety and N is a linear algebraic group (any closed subgroup of GL_n).

22. November 22

22.1. Riemann-Roch.

THEOREM 22.1 (Riemann-Roch Theorem). *Let X be a smooth projective curve over k of geometric genus g*[1]*, and D a divisor on X. Then*

$$\ell(D) - \ell(K_x - D) = 1 - g + \deg D$$

Both sides depend only on the divisor class group of D. All the regular functions on X are constant, so $\ell(0) = 1$. Also, $\ell(D) = 0$ if $\deg D < 0$, because $(f) + D \geq 0 \implies \deg D \geq 0$. Substituting $D = K_x$ in Riemann-Roch gives

COROLLARY 22.2.

$$\deg K_x = 2g - 2$$

COROLLARY 22.3. *If $\deg D \geq 2g - 1$ then $\ell(D) = 1 - g + \deg D$.*

PROOF. If $\deg(K_x - D) < 0$ then $\ell(K_x - D) = 0$. ⊟

COROLLARY 22.4. *If $\deg D \geq 2g$ then $\ell(D - p) = \ell(D) - 1$.*

COROLLARY 22.5. *If $\deg D \geq 2g$ then $|D|$ has no base point; in particular, the rational map $X \dashrightarrow |D|^*$ (the dual projective space $(|D|)^\vee$) is a morphism. (The base points are exactly where the rational map cannot be extended to a morphism.)*

PROOF. If p were a base point, then $\ell(D - p) = \ell(D)$, contradicting the previous corollary. ⊟

THEOREM 22.6. *Let D be a divisor on X. If $p, q \in X$ such that $\ell(D - p - q) = \ell(D) - 2$ then $Z_{|D|} : X \to |D|^*$ is a closed embedding.*

COROLLARY 22.7. *If $\deg D \geq 2g + 1$ then $Z_{|D|} : X \to |D|^*$ is a closed embedding.*

REMARK 22.8. $|D|^*$ is noncanonically isomorphic to $\mathbb{P}^{\deg D - g}$.

COROLLARY 22.9. *Let $g \geq 2$. Then $Z_{|3K_x|} : X \to \mathbb{P}^{5g-6}$ is a closed embedding.*

PROOF. $\deg 3K_x = 6g - 6 \geq 2g + 1$ ⊟

If you fix the genus, then every curve of genus g can be embedded into a projective space of dimension $5g - 6$. This is how you construct the moduli space of curves.

PROOF OF THEOREM 22.6. Easy fact: $\ell(D) - \ell(D - p) \leq 1$, which implies $\ell(D - p) = \ell(D) - 1$. So $|D|$ has no base points. Consider $\varphi = Z_{|D|} : X \to |D|^*$. Then φ is injective: if $\varphi(p) = \varphi(q) = x \in |D|^*$. Let H be a hyperplane of $|D|^*$ then $\varphi^* H \in |D|$, and all elements

[1] recall $g - \ell(K_x)$ and $\ell(D) = \dim \mathcal{L}(D)$ where $\mathcal{L}(D) = \{f \in k(X)^* : (f) + D \geq 0\} \cup \{0\}$

in the linear system $|D|$ are obtained in this way. If $x \in H$ then $supp(\varphi^*(H)) \supset \{p, q\}$; otherwise, $supp(\varphi^* H) \not\ni p, \not\ni q$. But $\mathcal{L}(D-p-q) \subsetneq \mathcal{L}(D-p) \subsetneq \mathcal{L}(D)$. So there is $f \in \mathcal{L}(D-p) \backslash \mathcal{L}(D-p-q)$ and $(f)+D \in |D|$ where $p \in supp((f)+D)$ (because $(f)+D \geq p$) but $q \notin supp((f)+D)$. Contradiction.

Let $X' = \varphi(X) \subset |D|^*$. We need to show that $\varphi : X \to X'$ is an isomorphism. (The image is closed because we are mapping from projective varieties.) We know it's injective; so it suffices to show that it induces an isomorphism on the local rings. We have

$$X \xrightarrow{f} X' \hookrightarrow |D|^*$$

The second map is a closed embedding, hence a finite morphism. The first map is also finite, because it is a nonconstant map of curves. Suppose $V \subset |D|^*$ is affine. Then $A = \mathcal{O}(V)$ comes from $B = \mathcal{O}(U)$ where $U = \varphi^{-1}(V)$. Recall $x \in V$ is in $\varphi(X)$ if for any $f \in \mathcal{O}(U)$. X' is defined by

$$V \cap X' = \{x : \varphi^* f = 0 \implies f(x) = 0\}$$

So $\mathcal{O}(V \cap X') = Im(A \to B)$ and $I(V \cap X') = \ker(A \to B)$. (We can always factor a map as a surjection followed by an injection; in this case, we have produced $A \twoheadrightarrow \mathcal{O}(V \cap X') \hookrightarrow B$.)

Let $p \in X$. Then $\mathcal{O}_{X,p}$ is finite over $\mathcal{O}_{X',\varphi(p)}$ (it was finite to begin with, and then we localized).

Claim: φ is injective on tangent spaces. On the level of local rings, this says $\mathfrak{m}_{X',\varphi(p)} \to \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ is surjective.

If the claim is true then we're done. This condition means it's surjective on the cotangent space; dually, it's injective on the tangent space. Let $\mathscr{A} = \mathfrak{m}_{X',\varphi(p)}\mathcal{O}_{X,p}$ so $\mathscr{A} \subset \mathfrak{m}_{X,p}$. From the claim, $\mathscr{A}$ contains a local parameter of X at p. So $\mathscr{A}$ is exactly the maximal ideal $\mathfrak{m}_{X,p}$ (it's a principal ideal generated by any local parameter).

Now consider the maps of $\mathcal{O}_{X',\varphi(p)}$-modules $\mathcal{O}_{X',\varphi(p)} \to \mathcal{O}_{X,p}$. So the map

$$\begin{array}{ccccccc}
\mathcal{O}_{X',\varphi(p)}/\mathfrak{m}_{X',\varphi(p)} & \longrightarrow & \mathcal{O}_{X,p}/\mathfrak{m}_{X',\varphi(p)}\mathcal{O}_{X,p} & \longrightarrow & N/\mathfrak{m}_{X'.\varphi(p)} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
k & \xrightarrow{\cong} & \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \cong k & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

By Nakayama's lemma, $\mathcal{O}_{X',\varphi(p)} \to \mathcal{O}_{X,p}$ is a surjection of $\mathcal{O}_{X',\varphi(p)}$-modules.

$$T^*_{\varphi(p)}|D|^* \to T^*_{\varphi(p)}X' \to T^*_p X$$

so it is enough to show that

$$T^*_{\varphi(p)}|D|^* \to T_p(X)$$

is surjective. Let $x_0, \cdots, x_n$ be coordinates on $|D|^*$, where $\varphi(p) = (1, 0, \cdots, 0)$. Let's recall what rational functions look like. Let $(f_0, \cdots, f_n)$ be a basis of $\mathcal{L}(D)$. Then $\varphi(p) =$

$(f_0(p), \cdots, f_n(p))$. Then $\varphi^* \frac{x_i}{x_0} = \frac{f_i}{f_0} \in \mathcal{O}_{X,p}$; this is actually in $\mathfrak{m}_{X,p}$ because f_i vanishes at p, since each $\frac{x_i}{x_0}$ vanishes at $\varphi(p)$. But the x_i span the cotangent space $T^*_{\varphi(p)|D|^*}$. If we define $x_0 = 0$ then the rest are local parameters at that point.

To show $T^*_{\varphi(p)}|D|^* \to T^*_p(X) = \mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}$ is surjective, it suffices to show that there is some i for which $\frac{f_i}{f_0} \notin \mathfrak{m}^2_{X,p}$.

$$\mathcal{L}(D-2p) \subsetneq \mathcal{L}(D-p) \subsetneq \mathcal{L}(D)$$

Let $f \in \mathcal{L}(D-p) \backslash \mathcal{L}(D-2p)$. Choose $p \notin Supp(D)$ such that $(f) + D = p + D$. Then $\frac{f}{f_0} = \lambda_0 + \lambda_1 \frac{f_1}{f_0} + \cdots + \lambda_n \frac{f_n}{f_0}$ implies that $\frac{f}{f_0}$ has a zero of order 2 at p, which is a contradiction.

Summary: use $p = q$ case to show that φ is injective on tangent spaces; then use commutative algebra to show it is a closed embedding. ⊟

What is $Z_{|K_x|} : X \dashrightarrow \mathbb{P}^{g-1}$? Let's consider $g = 2$. We get $\varphi : X \to \mathbb{P}^1$: a map between two curves. Assume characteristic zero. Exercise: show that φ is separable (this is equivalent to it being nonconstant). If φ is a separable map, then $K_x - \varphi^* K_{\mathbb{P}^1}$ is the branched divisor. Also, $\varphi^*([0]) = K_X$. By taking degrees, we get $\deg \varphi = 2g - 2 = 2$; taking degrees in the previous equation gives $\deg B - 6$ where B is the branched divisor. So the morphism is generically a double cover, which means $[k(X) : k(\mathbb{P}^1)] = 2$. So you can write this as a quadratic extension

$$k(X) = k(X)[y]/y^2 - f(x)$$

Exercise: The support of the branched divisor B consists of the points over the zeroes of $f(x)$ of odd order, and the points over ∞ if $\deg f$ is odd. For example, if f has simple zeroes at six points, then these points are all the branched points.

DEFINITION 22.10. A curve X is called hyperelliptic if there exists $\varphi : X \to \mathbb{P}^1$ of degree 2.

COROLLARY 22.11. *Every genus 2 curve is hyperelliptic. If $g \geq 3$, most curves are not hyperelliptic.*

EXAMPLE 22.12. Let $g \geq 3$. If X is not hyperelliptic then the morphism

$$Z_{|K_X|} : X \to \mathbb{P}^{g-1}$$

is a closed embedding. (You have to check that this linear system satisfies the conditions of the theorem.)

THEOREM 22.13 (Riemann-Hurwitz formula). *Let $\varphi : X \to Y$ be separable of degree n. Let $K_X - \varphi^* K_Y = B$ be the branched divisor. Then*

$$2g_X - 2 = n(2g_Y - 2) + \deg B$$

Let $x \in X$, $y = \varphi(x) \in Y$. Let t_x, t_y be local parameters. Then $\varphi^*(t_y) = t_x^{e_x} u$ where e_x is some power called the ramification index of φ at x, and $y \in \mathcal{O}^*_{X,x}$ is some invertible element. In general, the branched divisor is a determinant of the Jacobian matrix. But here, we can just take

$$\frac{d\varphi^* t_y}{dt_x} = e t_x^{e-2} u + t_x^{e_x} \frac{du}{dt_x}$$

DEFINITION 22.14. φ is said to be tame at x if $char(k) \nmid e$.

COROLLARY 22.15. *If φ is tame at x then $\operatorname{ord}_x \left(\frac{d\varphi^* t_y}{dt_x}\right) = e-1$. This gives the coefficients of the branched divisor. So if φ is tame everywhere, then $B = \sum_{x\in X}(e_x - 1)$ and*

$$\deg B = \sum_{x\in X}(e_x - 1).$$

23. November 29

Presentations Thursday 2:30-4, Friday 2-4, SC 530, papers due December 10

23.1. Sketch of the proof of Riemann-Roch. Let X be a smooth projective curve over k. If $x \in X$ then consider $\widehat{\mathcal{O}}_{X,x} \cong k[[t]]$, where we can choose t to be a local parameter. Then the maximal ideal $\widehat{\mathfrak{m}}_{x,x}$ corresponds to $tk[[t]]$. Let K_x be the fractional field of $\widehat{\mathcal{O}}_{X,x}$. (Write $\mathcal{O}_{X,x} =: \mathcal{O}_x$ because the curve will be fixed here.) $\widehat{K}_x \cong k((t_x))$ is the set of Laurent series $\sum_{i\geq -N}^{\infty} a_i t^i$.

$$\begin{tikzcd}
\mathcal{O}_x \arrow[r, hook] \arrow[d, hook] & k(X) = K \arrow[d, hook] \\
\widehat{\mathcal{O}}_x \arrow[r, hook] & \widehat{K_x}
\end{tikzcd}$$

These morphisms are given by taking the Taylor expansion of a function in $\mathcal{O}_x$.

DEFINITION 23.1. The adèles of X is

$$\mathbb{A}_X = \prod_{x\in X}{}' \widehat{K}_x$$

where $\prod'$ is the restricted product: more precisely, elements in $\mathbb{A}_X$ are $(f_x)_{x\in X}$, where $f_x \in \widehat{K}_x$ and all but finitely many f_x are in $\widehat{\mathcal{O}}_x$.

First observe that there is a natural k-algebra homomorphism $K \hookrightarrow \mathbb{A}_X$ given by using the inclusion of f in each $\widehat{K}_x$, and then taking $f \mapsto (f_x)_{x\in X}$. Let $D = \sum n_p \cdot p$ be a divisor on X. We defined

$$\mathbb{A}_X(D) = \{(f_x)_{x\in X} \in \prod \widehat{K}_x \,:\, f_x \in \widehat{\mathfrak{m}}_x^{-n_x}\widehat{\mathcal{O}}_x\} \subset \mathbb{A}_x$$

(There are only finitely many x such that $n_x \neq 0$.) If $\widehat{\mathfrak{m}_x} = t_x \cdot k[[t_x]] =\subset \widehat{\mathcal{O}}_x$ then we can find $\widehat{\mathfrak{m}}_x^{-1} = t_x^{-1} \cdot k[[t_x]] \subset \widehat{K}_x$ such that $\widehat{\mathfrak{m}_x}^{-1} \cdot \widehat{\mathfrak{m}}_x = \widehat{\mathcal{O}}_x$. If $D \leq D'$ then $\mathbb{A}_X(D) \hookrightarrow \mathbb{A}_X(D')$ and $\mathbb{A}_X = \varinjlim_{\leq} \mathbb{A}_X(D)$

Given D consider $K \oplus \mathbb{A}_X(D) \overset{d_D}{\to} \mathbb{A}_X$ where $(f, 0) \mapsto (f_x)_{x \in X}$ and $(0, (f_x)_{x \in X}) \mapsto (-f_x)_{x \in X}$ (aside from the minus sign, we're just using the natural embeddings here). The kernel is

$$\ker d_D = \mathcal{L}(D) = \{f \in K^\times : (f) + D \geq 0\} \cup \{0\}$$

The cokernel is a vector space which we call $H^1(X, D)$.

If $D \leq D'$ then we have a natural commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K \oplus \mathbb{A}_X(D) \arrow[r, "d_D"] \arrow[d, hook] & \mathbb{A}_X \arrow[d, "\sim"] \\
K \oplus \mathbb{A}_X(D') \arrow[r, "d_{D'}"] \arrow[d, two heads] & \mathbb{A}_X \arrow[d] \\
V \arrow[r] \arrow[d] & 0 \\
0 &
\end{tikzcd}$$

where V is the cokernel of the first vertical chain of maps, and $\dim_k V = \deg D' - \deg D$. Now use the snake lemma to get a sequence

$$0 \longrightarrow \ker d_D \longrightarrow \ker d_{D'} \longrightarrow V \longrightarrow H^1(X, D) \longrightarrow H^1(X, D') \longrightarrow 0$$

The key point is to prove that $H^1(X, D)$ is a finite-dimensional vector space. We know that $\ker d_D$ is finite, because it is just $\mathcal{L}(D)$. Let $D_m \in \mathcal{L}_X(m) = \mathbb{P}(S(X)_m)$. We can choose D_m for $m \in \mathbb{Z}$ such that $D_m \leq D_{m+1}$. Now, if $m \gg 0$ consider the short exact sequence

$$0 \to \mathcal{L}(D_m) \to \mathcal{L}(D_{m+1}) \to V \to H^1(X, D_m) \to H^1(X, D_{m+1} \to 0)$$

$\dim_k V = \deg D_{m+1} - \deg D_m = \deg D_1 = \deg X$ (note that $D_m \in |m \cdot D_1|$ and therefore $\deg D_m = m \cdot \deg D_1$, as degree does not change under linear equivalence). If m is large enough, then the linear system is complete, and $\ell(D_{m+1}) = P_X(m+1)$, and $\ell(D_m) = P_X(m)$ so the difference is

$$\ell(D_{m+1}) - \ell(D_m) = P_X(m+1) - P_X(m) = \deg X$$

(The last equality is because the Hilbert polynomial is $P_X(t) = (\deg X)t + (1 - p_a)$.) In particular, the canonical map $H^1(X, D_m) \overset{\sim}{\to} H^1(X, D_{m+1})$ is an isomorphism.

CLAIM 23.2. *For m large enough, $H^1(X, D_m) = 0$.*

PROOF. Let $(f_x) \in \mathbb{A}_X \to H^1(X, D_m) := \mathbb{A}_X / K \oplus \mathbb{A}_X(D_m)$. We just said that this maps isomorphically to $\mathbb{A}_X / K \oplus \mathbb{A}_X(D_{m+1})$. We can choose n, and $D_{m+n} \in \mathcal{L}_X(m+n)$ such that $f_x \in \mathbb{A}_X(D_{m+1})$. Then $(f_x) = 0$ in $H^1(X, D_{n+m})$, so $(f_x) = 0$ in $H^1(X, D_m)$. $\boxminus$

Summary: We have proved: if $m \gg 0$ then for any $D_m \in \mathcal{L}_X(m)$ and $D_{m+1} \in \mathcal{L}_X(m+1)$ such that $D_{m+1} \geq D_m$, then $H^1(X, D_m) \cong H^1(X, D_{m+1})$. Now we have claimed that these spaces are zero.

COROLLARY 23.3. *For any divisor D, $\dim_k H^1(X, D) < \infty$.*

PROOF. Find some $D_m > D$ with m sufficiently large.

$$V \to H^1(X, D) \to H^1(X, D_m) \to 0$$

$H^1(X, D_m) = 0$, so because V is finite-dimensional, so is $H^1(X, D)$. ⊟

DEFINITION 23.4. $\chi(D) := \ell(D) - \dim_k H^1(X, D)$

THEOREM 23.5 (Cheap Riemann-Roch).

$$\chi(D) = 1 - p_a + \deg D$$

PROOF. If $D' = D + x$, then we have a sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D') \to V \to H^1(X, D) \to H^1(X, D') \to 0$$

This implies $\chi(D') = \chi(D) + 1$. So it suffices to prove this for hyperplane sections D_m, for m very large. We have already done this. For D_m,

$$\chi(D_m) = \ell(D_m) = P_X(m) = 1 - p_a + m \cdot \deg X = 1 - p_a + \deg D_m$$

⊟

COROLLARY 23.6.

$$p_a = \dim_k H^1(X, D = 0)$$

In particular, $p_a \geq 0$.

The other part of Riemann-Roch follows from Serre duality:

THEOREM 23.7 (Serre duality). *There is a canonical isomorphism $H^1(X, D)^\times \xrightarrow{\sim} \mathcal{L}(K_X - D)$ where K_X is the canonical divisor.*

COROLLARY 23.8. *$p_a = g$ (geometric genus)*

PROOF. Use $D = 0$. Also, $\chi(D) = \ell(D) - \ell(K_x - D)$ so we get Riemann-Roch. ⊟

Finally, we will give a sketch of the proof of Serre duality.

DEFINITION 23.9. Let $\Omega_{\widehat{K}_x} = \Omega_K \otimes_K \widehat{K}_x$. Then $\Omega_{\widehat{\mathcal{O}}_x} = \widehat{\Omega_{\mathcal{O}_x}} = \Omega_{\mathcal{O}_x} \otimes_{\mathcal{O}_x} \widehat{\mathcal{O}}_x$. Elements of this are $f(t)dt$ for $f(t) \in k[[t]]$, and elements of $\Omega_{\widehat{K}_x}$ have the form $f(t)dt$ for $f(t) \in k((t))$.

THEOREM 23.10. *There exists a k-linear map*

$$res_x : \Omega_{\widehat{K}_x} \to k$$

that satisfies:

(a) $res_x|_{\Omega_{\widehat{\mathcal{O}}_x}} = 0$
(b) $res_x(f^n df) = 0$ *for all* $f \in K^\times$ *and all* $n \neq -1$
(c) $res_x(f^{-1}df) = \mathrm{ord}_x(f)$

COROLLARY 23.11. *If* $\omega \in \Omega_{\widehat{K}_x}$, *then* $\omega = \sum a_i t_i dt$ *with* $res_x \omega = a_{-1}$.

This is hard in characteristic p. If $k = \mathbb{C}$, then $res_x \omega = \frac{1}{2\pi i} \oint \omega$. In this case, it is clear that a_{-1} is independent of the choice of local coordinates. In general:

THEOREM 23.12 (Residue theorem). *Let* $\omega \in \Omega_K$. *Then*

$$\sum_{x \in X} res_x \omega = 0$$

(This is due to Tate.)

NOTATION 23.13. If D is a divisor, then

$$\Omega_X(D) := \{\omega \in \Omega_K \,:\, (\omega) + D \geq 0\} = \mathcal{L}(K_X + D)$$

Consider the pairing $\Omega_X(-D) \times \mathbb{A}_X \to k$ that sends $(\omega, (f_x)) \mapsto \sum_{x \in X} res_x(f_x \omega)$. Clearly, this k-bilinear map factors through $\Omega_X(-D) \times \mathbb{A}_x / K \oplus \mathbb{A}_X(D) \to k$ (that is, it vanishes on $K \oplus \mathbb{A}_X(D)$). Since $K \oplus \mathbb{A}_X(D) = H^1(X, D)$, we get

$$\Omega_X(-D) \otimes H^1(X, D)$$

The claim is that this is a perfect pairing:

THEOREM 23.14. *This pairing induces* $H^1(X, D)^\vee \xrightarrow{\sim} \Omega_X(-D)$.

All of this is contained in a paper of Tate.

THE END.