Rozenblyum.

A geometric approach to the cyclotomic spectrum.

Joint w/ Ayala and Mazel-Gee.

$\mathcal{C}$ stable ∞-cat.

$$\begin{tikzcd} K(\mathcal{C}) \arrow[r, "\text{topological}", "\text{Dennis tr}"'] & THH(\mathcal{C}) \\ \mathcal{C} \arrow[u] \arrow[ur] \end{tikzcd}$$

Enough to give this map.

Trace of holonomy around infinitesimal loops.

Universal trace-like map out of K-theory.

But, other properties of the trace: it is $\mathbb{T}$-invariant.

<u>Other</u> <u>property</u>.

R a ring, $M \in M_n(R)$, $r \in \mathbb{N}$.

$tr(M^{\otimes r})$, $tr(M)^{\otimes r}$ $\in$ $R^{\otimes r}$.

Are these the same? No, of course not. But, they are related. First, they are in $(R^{\otimes r})^{C_r}$.

<u>Observation</u>: $tr(M^{\otimes r}) - tr(M)^{\otimes r}$ $\in$ norms inside $(R^{\otimes r})^{C_r}$ over $\overset{\text{prop}}{\text{a}}$ subgroups of C_r.

(*) <u>Upshot</u>: $tr(M^{\otimes r}) \equiv tr(M)^{\otimes r}$ modulo norms.

<u>Intuition</u>: $TC(\mathcal{C})$ is built from THH by forcing $\mathbb{T}$-invariance and (*) in a homotopy coherent way.

Suppose G is a group.

G-Sp genuine G-Sp.

Sp^{hG} : $Fun(BG, Sp)$ Borel spectra.

$$\begin{tikzcd} G\text{-}Sp \arrow[r, shift left] \arrow[d, "\Phi^G"'] & Sp^{hG} \arrow[l, shift left, "B"] \\ Sp \end{tikzcd}$$

Essential Borel-complete, Fully faithful,

$$Sp^{hG} \xrightarrow{(\ )^{\tau G}} Sp$$

$\tau G : \Phi^G \circ B$.

Ex. (i) $G = C_p$, $\tau C_p = t C_p$

(ii) $G = C_{p^2}$, $X^{\tau G} = \left(X^{hC_p}\right)^{tC_{p^2}/C_p}$ ← zero for X bounded above by Tate fixed point action.

(iii) $G = C_6$ $X^{\tau C_6} = \mathrm{cofib}\left((X^{tC_2})_{hC_3} \longrightarrow (X^{hC_3})^{tC_2} \right)$.

Possibly zero?
Thomas says so.

Consider. $Sp^{h\mathbb{T}}$.

$(\ )^{\tau C_r}$ defines a left-lax action of $\mathbb{N}^\times$ on $Sp^{h\mathbb{T}}$

$Y \in Sp^{h\mathbb{T}} \longmapsto Y^{\tau C_r} \in Sp^{h\mathbb{T}/C_r} \simeq Sp^{h\mathbb{T}}$.

$Y^{\tau C_r} \longrightarrow (Y^{\tau C_r})^{\tau C_s}$

Thm (AMR). The ∞-cat of cyclotomic spectra à la B+M is equivalent to $(Sp^{h\mathbb{T}})^{h\mathbb{N}^\times}$.

Rem. Explicitly, a cyclotomic spectrum is $X \in Sp^{h\mathbb{T}}$ with

(i) $\forall r$ $X \xrightarrow{\phi_r} X^{\tau C_r}$

(ii) $\forall r, s$

$$\begin{tikzcd} X \arrow[r, "\phi_r"] \arrow[d, "\phi_{rs}"'] & X^{\tau C_r} \arrow[d, "(\phi_s)^{\tau C_r}"] \\ X^{\tau C_{rs}} \arrow[r, "\text{left-lax}"'] & (X^{\tau C_s})^{\tau C_r} \end{tikzcd}$$

$\mathbb{T}$-equivariant maps.

The commutativity is part of the data.

Rem. When X is bounded below, this is an $\simeq$, so no extra data, and everything reduces to (i).

Rem. Obvious def. of cyclotomic spectra w/ Frobenius lifts.

Some Tate diagonals.

Are these the 1st derivatives of $X \mapsto (X^{\otimes r})^{hC_r}$?

$X \in Sp.$

$X \longmapsto (X^{\otimes r})^{tC_r}$ is an exact functor.

Same proof as G. Heuts when r is prime, using that proper induceds are killed.

Then are diagonal maps

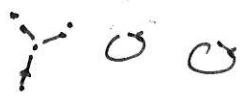
$$X \longrightarrow (X^{\otimes r})^{tC_r}$$

as in Gijs' talk. Reduce to $X = \mathbb{S}$, when it factors through hC_r.

Basic question: how to obtain this structure on $THH(\mathcal{C})$?

<u>Factorization homology</u>. Relevant ∞-cat (joint with Ayala-Francis).

$\mathcal{M} = \begin{cases} \text{Objects: oriented stratified 1-d manifolds.} \\ \qquad \coprod_{\text{finite}} \text{directed graphs and oriented circles.} \\ \text{Morphisms:} \\ \quad ? \end{cases}$



$\mathcal{D}$ = ∞-cat of "disk refinements" of objects of $\mathcal{M}$.

An object of $\mathcal{M}$ with a refinement of the stratification so that each stratum is contractible.

<u>Example of a morphism</u>.

a splits in two
b disappears

$\mathcal{D} \longrightarrow \mathcal{M}$ forgetful functor.

$$\begin{tikzcd}
\Delta^{op} \arrow[r, "\text{cofinal}"] & \Delta^{op}_{\circ} \arrow[r] \arrow[rrr, bend left=10] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Lambda^{op} \arrow[rr] \arrow[d] & & \mathcal{D} \arrow[d] \\
& \{S^1\} \arrow[r] \arrow[rrr, bend right=10] & BAut\, S^1 \simeq S^1 \arrow[rr] & & \mathcal{M}
\end{tikzcd}$$

paracyclic category

full subgroupoid of $\mathcal{M}$ at S^1.

Just points on S^1.

epicyclic

$$\begin{tikzcd}
\Delta^{op} \arrow[r] & \Delta_{\mathbb{D}}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Lambda^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{\Lambda}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{D} \arrow[d] \\
 & \{S^1\}_{S^1}^{\Delta^\circ} \arrow[r] & B\mathbb{T} \arrow[r] \arrow[d, bend left] \arrow[dr, phantom, "\lrcorner"] & BW \arrow[r] \arrow[d, bend left] & \mathcal{M} \\
 & & * \arrow[r] \arrow[u] & B\mathbb{N}^{\times} \arrow[u] &
\end{tikzcd}$$

full subgroupoid of $\mathcal{M}$ on S^1

B monoid full subcat of $\mathcal{M}$ on S^1

rotate or tile covering maps.

$\mathcal{C}$ a stable ∞-category.

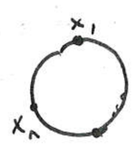
Gepner–Haugseng: $\mathcal{C}$ is an ∞-cat enriched in Sp.

$\iota\mathcal{C} \subseteq \mathcal{C}$ maximal subgroupoid.

<u>Def.</u> $\mathrm{Conf}_{\Delta_{\mathbb{D}}^{op}}(\iota\mathcal{C})$ = points on S^1 labelled by elts of $\iota\mathcal{C}$.

$$\begin{array}{c} \mathrm{Conf}_{\Delta_{\mathbb{D}}^{op}}(\iota\mathcal{C}) \\ \downarrow \\ \Delta_{\mathbb{D}}^{op} \end{array}$$

Functor $H_{\Delta_{\mathbb{D}}^{op}} : \mathrm{Conf}_{\Delta^{op}}(\iota\mathcal{C}) \longrightarrow Sp$



$\longmapsto \mathrm{Map}(x_1, x_n) \otimes \mathrm{Map}(x_4, x_3) \otimes \cdots \otimes \mathrm{Map}(x_2, x_1).$

<u>Def.</u> $\int_{S^1} \mathcal{C} = \operatorname{colim}_{\Delta_{\mathbb{D}}^{op}} H_{\Delta_{\mathbb{D}}^{op}}$.

<u>Fact.</u> $\int_{S^1} \mathcal{C} \simeq THH(\mathcal{C})$.

<u>Proof.</u> Left Kan extend to Δ^{op}.

Then go. looks like bar complex.

Idea: study functoriality of $\int_{S^1}\mathcal{C}$ w.r.t. S^1.

Obvious: functorial for $\mathrm{Aut}_M(S^1)$ since this acts on $\mathrm{Conf}_{\Delta^{op}_{\circlearrowleft}}(\iota\mathcal{C})$.

But: $\int_{S^1}\mathcal{C}$ is not functorial w.r.t. covering maps.

Issue: no diagonal maps. Would be OK in ~~[illegible]~~ spaces.

Upshot. Take diagonals:

$M_1 \longrightarrow M_2$ a C_r-covering.

Natural maps

$$\int_{M_2}\mathcal{C} \longrightarrow \left(\int_{M_1}\mathcal{C}\right)^{tC_r}$$

Taking $M_1 = M_2 = S^1$, get cyclotomic structure.

Ideally would have — does not exist.

$$\mathrm{Conf}_{\tilde\Lambda^{op}}(\iota\mathcal{C}) \longrightarrow Sp.$$

Instead, work relatively.

$$\begin{tikzcd}
\mathrm{Conf}_{\tilde\Lambda^{op}}(\iota\mathcal{C}) \arrow[r, dashed, "\exists"] \arrow[d] & \int_{\tilde\Lambda^{op}} BSp \arrow[dl] \\
\tilde\Lambda^{op} \arrow[r] & BW \arrow[r] & B\mathbb{N}^{\times}
\end{tikzcd}$$

$\int_{\tilde\Lambda^{op}} BSp$: Grothendieck construction of $\tilde\Lambda^{op} \longrightarrow Sp$


$\longmapsto Sp^{\#\text{ of intervals}}$

$\longmapsto [\mathrm{Map}(x_1, x_2) \times \cdots, \mathrm{Map}(x_n, x_1)].$

General construction:

$$\begin{tikzcd}
S_0 \arrow[r] \arrow[dd] & S \arrow[d] & \text{some category} \arrow[l] \\
& BW \arrow[d] & \\
* \arrow[r] & B\mathbb{N}^{\times} &
\end{tikzcd}$$

Get left-lax action on $\mathrm{Fun}(S_0, Sp)$:

$$(r \in \mathbb{N}^{\times}, F) \longmapsto (s \longmapsto F(rs)^{tC_r}).$$

Tate diagonals give a right-lax homotopy fixed point of

$$\mathrm{Fun}\Big(\int_{\Lambda^{op}} B\mathrm{Sp}, \mathrm{Sp}\Big) \circlearrowleft \mathbb{N}^{\times} \leftarrow \text{Non-formal input from Spectra.}$$

Only uses symmetric monoidal structure.

$\Rightarrow$ a homotopy right-lax fixed point

$$\mathrm{Fun}(\mathrm{Conf}_{\Lambda^{op}}(\iota\mathcal{C}), \mathrm{Sp}).$$

Take fibrewise colimit along

$$\mathrm{Conf}_{\Lambda^{op}}(\iota\mathcal{C}) \to BW$$

Gives a homotopy right-lax fixed point of

$$\mathrm{Fun}(B\mathbb{T}, \mathrm{Sp}).$$

This is exactly a cyclotomic spectrum.