Royer / Yeakel.

The Dundas–McCarthy Theorem I.

Sarah starts.

Thm. (Dundas–Goodwillie–McCarthy) $f : R \to S$ a map of connective $\mathbb{E}_1$-rings such that $\pi_0 R \to \pi_0 S$ is surjective with nilpotent kernel. Then,

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(S) \arrow[r] & TC(S) \end{tikzcd} \quad \Big\} \text{ Integral TC.}$$

is cocartesian.

① Taylor tower.

Let X be a space. $S_{/X}$. spaces over X.

$$F : S_{/X} \to S_* \text{ or } Sp.$$

Goodwillie defines a tower of functors

$$F \to \quad \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to P_0 F \simeq F(X \xrightarrow{id} X)$$

$P_n F$: n-excisive approximation

Like a Taylor approximation.

Def. F is analytic if $F(Y) \simeq \lim_n P_n F(Y)$ for suff. connected Y.

Def. F is 1-excisive if it takes pushout squares to pullback squares.

Ex. $\Sigma^\infty : S_* \to Sp$ is 1-excisive.

Ex. $id : S_* \to S_*$ is not 1-excisive.

Blakers–Massey controls failure.

$$\begin{tikzcd} X \arrow[r, "k_0\text{-conn}"] \arrow[d, "k_1\text{-conn}"'] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$X \to$ pullback is (k_0+k_1-1)-connected.

Def. F is stably 1-excisive if $\exists c, k$ s.t. Apply F to a square with $k_0, k_1 \geq k$, get (k_0+k_1-c)-cartesian. $E_1(c,k)$.

Ex. id_{S_*} is $E_1(1,-1)$.

Def. F is 2-excisive if F takes strongly cocartesian 3-cubes to cart cubes. [every face is cocart]

Ex. Σ^∞_+ is 2-excisive.

Higher Blakers–Massey.

Fiber is $(k_0+k_1+k_2-2)$-conn.

Make def. for stably 2-excisive. $E_2(c,k)$.

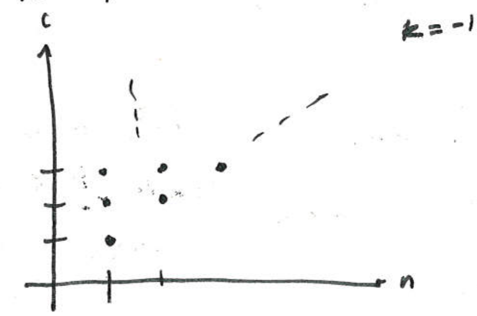
Ex. $\Sigma^\infty X^{\otimes 2}$ is 2-excisive but not 1-excisive.

Ex. Id is $E_2(2,-1)$. In fact $E_n(n,-1)$.

Obvious definition of stably n-excisive.

$$E_n(c,k) \Rightarrow E_n(d,k') \quad \text{for } c \le d,\ k \le k'.$$

(3) "Analyticity is stable n-excisivity for all n."



Def. F is ρ-analytic if there exist ρ, q s.t. F is $E_n(\rho n - q, \rho + 1)$.

Ex. id_{S_*} is $\rho = 1$-analytic.

Thm (Goodwillie). If F is p-analytic and $Y \to X$ is $(p+1)$-connected, then
$$F(Y) \xrightarrow{\sim} \lim P_n F(Y).$$

"Radius of convergence for the Taylor tower."

For Dundas-McCarthy, focus on

$$\begin{tikzcd} F \arrow[r] & P_1F \arrow[r] & P_0F, \\ D_1F \arrow[ur] & & \end{tikzcd}$$

D_1F = differential.

D_1F is 1-homogeneous.

We know that $D_1F(Y) \simeq Y \otimes \partial_1 F(X)$ (or Ω^∞) for some $\partial_1 F(X)$.

Thm [Goodwillie]. "Analytic continuation." Calc II.

If $F, G : S_* \to Sp$ are p-analytic with $F \to G$ inducing $D_1F \xrightarrow{\sim} D_1G$ for all spaces X, then

This can be replaced by $\partial_1 F(X) \simeq \partial_1 G(X)$ $\forall X$.

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(X) \arrow[r] & G(X) \end{tikzcd}$$

for $Y \to X$ $(p+1)$-connected.

Strategy. $K, TC :$ [redacted] $S_* \longrightarrow Sp$ (via $\mathbb{E}_1$-rings)

Show they are (-1)-[redacted] analytic and $\partial_1 K \simeq \partial_1 TC$.

Awesome!

Gives the result for split square-zero extensions.
Then, make another argument.

Aaron takes over. //

$$F: \mathcal{S}_* \longrightarrow \mathcal{S}_*.$$

Assume F preserves filtered colimits.

Q. How do you compute $D_1 F$?

Reduction. Assume $F(*) \simeq *$. Then, $P_0 F \simeq *$.
So, $D_1 F \simeq P_1 F$.

Prop (Calc I).

$$D_1 F(X) \simeq \operatorname{colim}_{n \to \infty} \Omega^n F(\Sigma^n X).$$

Ex. $D_1 \mathrm{id}_{\mathcal{S}_*}(X) \simeq \Omega^\infty \Sigma^\infty X$.

Generalize to other settings, esp. ∞-cats. Appears in Higher Algebra.

$\mathcal{C}$ an ∞-cat w/ finite limits.

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^\omega, \mathcal{C})$$

reduced excisive functors.

I guess this means 1-excisive.

This is "the ∞-cat of $\mathcal{C}$-valued homology theories on $\mathcal{S}_*^\omega$".

Prop. (i) $\mathrm{Sp}(\mathcal{C})$ is a stable ∞-cat.

(ii) If $\mathcal{C}$ is presentable (or just accessible?), there is an adjunction
$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \rightleftarrows \mathrm{Sp}(\mathcal{C}) : \Omega^\infty_{\mathcal{C}}.$$

(iii) $\Sigma^\infty_{\mathcal{C}}$ is initial among colim-preserving functors to a stable ∞-cat, if $\mathcal{C}$ is presentable.

Exs (i) $\mathrm{Sp}(\mathcal{S}) \simeq \mathrm{Sp}$.

(i') $\mathrm{Sp}(\mathcal{S}_{/X}) \simeq$ stable ∞-cat of spectra over X
$\simeq \mathrm{Fun}(X^{op}, \mathrm{Sp})$.

(ii) $\mathcal{C} = \mathrm{Alg}_{/A}$. $\mathrm{Sp}(\mathcal{C}) = \mathrm{Sp}(\mathrm{Alg}_{/A}) \simeq {}_A\mathrm{Mod}_A$. Basterra–Mandell.

$$\Omega^\infty_A : {}_A\mathrm{Mod}_A \to \mathrm{Alg}_{/A}$$
$$M \longmapsto A \oplus M \to A.$$

In thinking about D_1K, $D_1TC : \mathbb{E}_1\text{-rings} \to Sp$, naturally led to think about trivial square zero exts.

$$M \in {}_A\mathrm{Mod}_A \simeq \mathrm{Exc}_*(S^\omega_*, \mathrm{Alg}_{/A}) \quad \cancel{A \oplus M[-]: S_*}$$

$$\tilde{K}(A \oplus M[-]) \to K(A \oplus M[-]) \to K(A)$$

$$A \oplus M[\cdot] : S^\omega_* \to \mathrm{Alg}_{/A}$$

Similarly for TC. End up showing that

$$\begin{gathered}\tilde{K}(A \oplus M[\cdot]) \\ \wr\vert \\ \widetilde{TC}(A \oplus M[\cdot])\end{gathered}$$

$$A \oplus M[X] \simeq \cancel{\quad} A \oplus (M \otimes \Sigma^\infty X).$$

$$\begin{tikzcd} Sp(\mathcal{C}) \arrow[r, dashed, "\text{Does this exist?}"] \arrow[d, "\Omega^\infty_{\mathcal{C}}"'] & Sp(\mathcal{D}) \arrow[d, "\Omega^\infty_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

__Defn__ $\partial F : Sp(\mathcal{C}) \to Sp(\mathcal{D})$. It will be exact and have a nat transform

$$F \circ \Omega^\infty_{\mathcal{C}} \to \Omega^\infty_{\mathcal{D}} \circ \partial F$$

and initial.

TC of a square-zero extension?

In particular, $\Omega^\infty_{\mathcal{D}} \circ \partial F \simeq DF \circ \Omega^\infty_{\mathcal{C}}$.

$$D_1K \circ \Omega^\infty_A(M) \simeq \operatorname{colim} \Sigma^{-n} \tilde{K}(A \oplus \Sigma^n M).$$

$$\begin{tikzcd} M \arrow[d] \arrow[r, phantom, "\simeq"] & M \arrow[d] \\ R \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r, "d"', "\text{derivation}"] & S \oplus \Sigma M \end{tikzcd}$$

What can we say about Milnor squares here?

Birelativity?

⚠ In the remaining lectures. We prove $\mathcal{I}\oplus M$ $S \oplus M \to S$. Two strategies for general proof. 1: reduce by hand to square-zero. Then reduce to trivial square zero. 2: prove analytic continuation/analyticity for K, TC thought of as functors $\mathrm{Alg} \to Sp$. Hard but tractable.