Gijs Heuts.

Tate diagonals.

Basic observation: spaces have diagonals. $X \in S_*$.

$$X \longrightarrow X \times \cdots \times X.$$

$$\begin{tikzcd}
\Sigma^\infty X \arrow[r] \arrow[dr] & (\Sigma^\infty X)^{\otimes n} \\
& \left((\Sigma^\infty X)^{\otimes n}\right)^{h\Sigma_n} \arrow[u]
\end{tikzcd}$$

Makes $\Sigma^\infty X$ into an $\mathbb{E}_\infty$-coalgebra.

In general, there are not such diagonal maps for spectra.

More precisely: $\mathrm{Map}\left(\mathrm{id}_{Sp}, (\mathrm{id}_{Sp}^{\otimes n})^{h\Sigma_n}\right) \simeq *$.
This is an <u>exercise</u>.

However there is a stable shadow, the Tate diagonals.

<u>Recall</u>: [redacted] E is a spectrum w/action of a finite group:

$$E_{hG} \xrightarrow{Nm} E^{hG} \longrightarrow E^{tG}$$

$\boxed{\text{analogue of the norm map in algebra } M_G \to M^G,\ [x] \mapsto \Sigma gx}$

Tate construction is the cofiber.

<u>John Klein</u>. For functors $Sp^G \to Sp$, Nm is characterized as a nat. transformation $F(E) \xrightarrow{\phi} E^{hG}$ by
(i) F preserves colimits,
(ii) ϕ is an $\simeq$ on induced spectra w/G-action.
Or $\simeq$ on [redacted] $(Sp^G)^\omega$. $\boxed{X \wedge G_+}$

That is, $F \simeq (\ )_{hG}$ and $\phi \simeq Nm$. Note: $(\ )^{tG}$ vanishes on compacts and induced objects.

Prop. The functor $Sp \longrightarrow Sp$

$$E \longmapsto (E^{\otimes p})^{tC_p}$$

is exact.

Rem. Goes back to the 70s, Jones and someone.

proof. First, check that it preserves direct sums:

$$((X\oplus Y)^{\otimes p})^{tC_p} \simeq (X^{\otimes p} \oplus Y^{\otimes p} \oplus \underbrace{\text{cross terms}}_{\text{induced up: divisible by } p.})^{tC_p}$$

$$\simeq (X^{\otimes p})^{tC_p} \oplus (Y^{\otimes p})^{tC_p} \oplus \underbrace{(\text{cross})^{tC_p}}_{\simeq 0 \text{ by above.}}$$

Actual proof. $X \to Y \to Z$ a cofiber seq.

Smashing $X \to Y$ with itself p times. Get

$$N\mathcal{P}(\{1,\dots,p\}) \xrightarrow{\ c\ } Sp$$
$$S \longmapsto X^{\otimes \{1,\dots,p\}\setminus S} \otimes Y^{\otimes S}$$

Filter $Y^{\otimes p}$ as

$$F_j Y^{\otimes p} \simeq \operatorname{colim} C|_{\text{subsets of cardinality at most } j.}$$

$$F_0 Y^{\otimes p} = X^{\otimes p} \to F_1 Y^{\otimes p} \to \cdots \to F_p Y^{\otimes p} = X^{\otimes p}$$

$$F_j Y^{\otimes p} / F_{j-1} Y^{\otimes p} \simeq \bigoplus_{|S|=j} X^{\otimes\{1\dots\}\setminus S} \otimes Z^{\otimes S}$$

Typo in talk.

If $0<j<p$, this is induced.

So, get a cofiber seq.

$$(X^{\otimes p})^{tC_p} \to (Y^{\otimes p})^{tC_p} \to (Z^{\otimes p})^{tC_p}.$$

Consequence.

$$\begin{array}{ccc} \Sigma^\infty X & \longrightarrow & (\Sigma^\infty X^{\otimes p})^{hC_p} \\ & \searrow^{\tau_{\Sigma^\infty X}} & \downarrow \\ & & (\Sigma^\infty X^{\otimes p})^{tC_p} \end{array}$$

extends to a nat trans.

$$E \xrightarrow{\ \tau_E\ } (E^{\otimes p})^{tC_p}.$$

Via the next lemma.

$F : Sp \to Sp$

<u>Lemma</u>. $\mathrm{Map}(\mathrm{id}_{Sp}, F) \simeq \mathrm{Map}(\mathbb{S}, F(\mathbb{S}))$.

<u>proof</u>. $\varinjlim_n \Omega^n \Sigma^\infty \Omega^\infty \Sigma^n \simeq \mathrm{id}_{Sp}$

$$\Rightarrow \mathrm{Nat}(\mathrm{id}_{Sp}, F) \simeq \varprojlim \mathrm{Nat}(\Omega^n \Sigma^\infty \Omega^\infty \Sigma^n, F)$$
$$\simeq \varprojlim \mathrm{Nat}(\Omega^\infty \Sigma^n, \Omega^\infty \Sigma^n F)$$
$$\simeq \varprojlim \Omega^\infty \Sigma^n F(S^{-n})$$
$$\simeq \Omega^\infty F(S^0).$$

<u>Exs</u> (i) $(\mathbb{S}^{\otimes p})^{tC_p} \simeq \mathbb{S}_p^\wedge$.

Segal conjecture for C_p.

$\to$ For finite E, $(E^{\otimes p})^{tC_p} \simeq E_p^\wedge$.

The map $E \xrightarrow{T_E} E_p^\wedge$ is p-completion in this case.

(Nikolaus–Scholze). True for bounded below spectra.

(ii) R is an $\mathbb{E}_\infty$-ring spectrum.

$$\begin{tikzcd} R \arrow[r, "T_R"] \arrow[rr, bend right, "\varphi_p"'] & (R^{\otimes p})^{tC_p} \arrow[r, "m^{tC_p}"] & R^{tC_p} \end{tikzcd}$$

φ_p Frobenius, or

Tate-valued Frobenius.

(iii) $R = KU$. $\pi_*(KU^{tC_p}) \simeq KU_*((x))/[p](x) \simeq KU_*((x))/((1+x)^p - 1)$

$\simeq KU_* \otimes \mathbb{Q}_p(\zeta_p)$.

Similar for KU-cohomology of X. but

$$\pi_*(KU^X) \longrightarrow \pi_*((KU^X)^{tC_p})$$

$\alpha \longmapsto \psi^p \alpha$, image in $KU_{\mathbb{Q}_p(\zeta_p)}$-cohomology.

(iv) $R = \mathbb{F}_2$. $\pi_*(\mathbb{F}_2^{tC_2}) \simeq \mathbb{F}_2((x))$, $|x| = 1$.

$\mathbb{F}_2^{tC_2} \simeq \prod_{n \in \mathbb{Z}} \Sigma^n \mathbb{F}_2$. $\phi : \mathbb{F}_2 \longrightarrow \mathbb{F}_2^{tC_2}$ is $\prod Sq^n$.

What about $(X^{\otimes n})^{t\Sigma_n}$. No longer exact for $n \geq 2$.

But, it is (n-1)-excisive.

Is $(\ )^{tG}$ polynomial for every finite G? Using e.g. $G \hookrightarrow \Sigma_n$?

$$X \longrightarrow (X^{\otimes S})^{tG}$$

S = finite G-set.

Def. $H: Sp \to Sp$ is n-homogeneous if

$$H(X) \simeq (C \otimes X^{\otimes n})_{h\Sigma_n}.$$

$F: Sp^\omega \to Sp$ is ~~an~~ n-excisive or polynomial of degree n if there is a fiber seq

$$H \longrightarrow F \longrightarrow G$$

H: n homogeneous; G: $(n-1)$-excisive.

idea:

$$\underbrace{(X^{\otimes n})_{h\Sigma_n}}_{n\text{-homogeneous}} \longrightarrow \underbrace{(X^{\otimes n})^{h\Sigma_n}}_{n\text{-excisive}} \longrightarrow (X^{\otimes n})^{t\Sigma_n}.$$

Approximations to S_*.

$$\begin{tikzcd}
 & & P_3 S_*^\omega \arrow[d] \\
 & & P_2 S_*^\omega \arrow[d] \\
S_*^\omega \arrow[rruu] \arrow[rru, "\Sigma_2^\infty"] \arrow[rr, "\Sigma^\infty"'] & & Sp^\omega \simeq P_1 S_*^\omega.
\end{tikzcd}$$

Object of $P_2 S_*^\omega$. It is a finite spectrum E equipped with a lift

$$\begin{tikzcd}
 & (E^{\otimes 2})^{hC_2} \arrow[d] \\
E \arrow[r] \arrow[ur, dashed, "\delta_2"] & (E^{\otimes 2})^{tC_2}
\end{tikzcd}$$

"Lift of Frobenius"

These are like 2-truncated coalgebras.

Any suspension spectrum has this structure.

Get 2-excisive approximations to id_{S_*}.

Ind $P_2 S_*^\omega$ universal functor from S_* which is 2-excisive and colim-preserving.

Objects here have

$$E \xrightarrow[T_E^3]{} (E^{\otimes 3})^{t\Sigma_3}.$$

A Σ_3-Tate diagonal.

Objects of $P_3 S_+^{\omega}$. It is an object $E \in P_2 S_+^{\omega}$ with a lift

$$\begin{tikzcd} E \arrow[r] \arrow[dr, dashed] & (E^{\otimes 3})^{t\Sigma_3} \arrow[r] & (\qquad)^{t\Sigma_3} \\ & (E^{\otimes 3})^{h\Sigma_3} \arrow[u] \arrow[r, "\text{Use } S_2"] & \left((E \otimes E)\otimes E \oplus \text{cyclic perms}\right)^{h\Sigma_3} \arrow[u] \end{tikzcd}$$

Again, any $\Sigma^\infty X$ has this structure.

$\vdots$

Generally: object of $P_n S_+^{\omega}$ will be an n-truncated coalgebra compatible with all Tate diagonals.

Object of $P_\infty S_+^{\omega} \simeq \lim P_n S_+^{\omega}$ is a Tate coalgebra. It will be $\mathbb{E}_\infty$ for S_+^{ω}.

Thm $(S_+^{\omega})^{\geq 2} \xrightarrow{\sim} (P_\infty S_+^{fin})^{\geq 2}$.

Nilpotent version too.

Rationally, Tate disappears