Hesselholt.

THH and arithmetic: Bökstedt periodicity.

$$HH_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{F}_p\langle x\rangle \text{ free divided power algebra}$$

$HH_{\text{even}}(A/\mathbb{Z})$ always a divided power algebra.

$$\Big\uparrow$$

$$HH_*(\mathbb{F}_p/\mathbb{S}) \cong \mathbb{F}_p[x]$$

$\uparrow$

Bökstedt periodicity.

$\Longrightarrow$ Bott periodicity, but not the other way around.

$$\begin{array}{ccccc} \mathbb{N} & \rightsquigarrow & \mathbb{Z} & \rightsquigarrow & \text{algebra} \\ \Uparrow & & | & & | \\ \text{Fin} & & \mathbb{S} & & \text{higher algebra} \end{array}$$

Counting antelopes 20,000 years ago.

Lars: the divided powers come from the $n!$ ways of labelling $\{n\}$ which we forced to be the same when we passed from Fin to $\mathbb{N}$.

$A = E_\infty$-ring.

$$A \rightrightarrows A\otimes A \;\overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}}\; A\otimes A\otimes A$$

$$\pi_*(A) \rightrightarrows \pi_*(A\otimes A) \;\overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}}\; \pi_*(A\otimes A\otimes A)$$

$\uparrow \simeq$ if flat

$$\pi_*(A\otimes A) \underset{\pi_*(A)}{\otimes} \pi_*(A\otimes A)$$

cogroupoid in graded commutative rings.

Hopf algebroid.

$A = \mathbb{F}_p$. $\pi_* A = \mathbb{F}_p$.

So, get a Hopf algebra, or cogroup since there is one object.

Milnor $(p=2)$:

$$\pi_*(\mathbb{F}_p \otimes \mathbb{F}_p) \cong S_{\mathbb{F}_p}\{\xi_1, \xi_2, \dots\}$$

$$|\xi_i| = 2^i - 1.$$

Image by $\pi_*(BC_p \otimes \mathbb{F}_p) \longrightarrow \pi_*(\Sigma\mathbb{F}_p \otimes \mathbb{F}_p)$

$$\pi_{2i}(BC_p \otimes \mathbb{F}_p) \cong H_{2i}(BC_p, \mathbb{F}_p) \cong \mathbb{Z}/p.$$

$$\xi := t + \sum \xi_i t^i$$

$$\gamma(\xi) = t + \sum \bar{\xi}_i t^i$$

$$d^1(\xi) = (\xi\otimes 1)\circ(1\otimes \xi)$$

composition of power series.

$$\gamma(\xi)\circ\xi = t = \xi\circ\gamma(\xi).$$

Power operations.

R, A $\mathbb{E}_\infty$-rings.

$e \in \pi_k\left(\left((\Sigma^m \mathbb{S})^{\otimes n}\right)_{h\Sigma_n} \otimes A\right)$

$x \in \pi_m(R \otimes A) \xrightarrow{Q_e} \pi_k(R \otimes A) \ni Q_e(x)$

extended powers.

$$\begin{tikzcd}
\Sigma^k \mathbb{S} \arrow[r, "e"] \arrow[ddd, bend right, "Q_e(x)"'] & \left((\Sigma^m \mathbb{S})^{\otimes n}\right)_{h\Sigma_n} \otimes A \arrow[d, "x^{\otimes n} \otimes \mathrm{id}"] \\
 & \left((R \otimes A)^{\otimes n}\right)_{h\Sigma_n} \otimes A \arrow[d, "\mu_n \otimes \mathrm{id}"] \\
 & R \otimes A \otimes A \arrow[d, "\mathrm{id} \otimes \mu"] \\
 & R \otimes A
\end{tikzcd}$$

$A = \mathbb{F}_p$, $p = 2$.
Araki–Kudo
Dyer–Lashof (p odd).

$\underline{p=2}$

$$E^2_{ij} = \begin{cases} H_i(B\Sigma_p, \mathbb{F}_p) & j = pm \\ 0 & \text{otherwise} \end{cases} \Rightarrow \pi_{i+j}\left((\Sigma^m \mathbb{S})^{\otimes p}\right)_{h\Sigma_p} \otimes \mathbb{F}_p$$

1-dim for all $i+j \geq pm$.
Since the s.s. collapses.

So, first operation is $2m$.

$$A = \mathbb{F}_p \quad \begin{cases} \pi_m(R \otimes A) \xrightarrow{Q^m} \pi_{2m}(R \otimes A) & \text{squaring} \\ \pi_m(R \otimes A) \xrightarrow{Q^{m+1}} \pi_{2m+1}(R \otimes A) \end{cases}$$

Relations are Nishida.

R, A $\mathbb{E}_\infty$-rings, R an A-algebra.

$$R \longrightarrow R \otimes A \rightleftarrows R \otimes A \otimes A \cdots$$

split equalizer

$$\pi_*(R) \longrightarrow \pi_*(R \otimes A) \rightrightarrows \pi_*(R \otimes A \otimes A)$$

$\wr$ | flatness

$$\pi_*(R \otimes A) \otimes_{\pi_*(A)} \pi_*(A \otimes A)$$

Understand $\pi_*(R)$ by its A-homology and the action of the Hopf-algebroid.

$A = \mathbb{F}_p$

$R = THH(\mathbb{F}_p) \simeq S^1 \otimes \mathbb{F}_p \simeq \Delta^1/\partial\Delta^1 \otimes \mathbb{F}_p \simeq \left| [n] \longmapsto \mathbb{F}_p^{\otimes \Delta^1[n]/\partial\Delta^1[n]} \right|$

free S^1-$\mathbb{E}_\infty$-ring on $\mathbb{F}_p$.

$\wr$ $\mathbb{F}_p^{\otimes n}$

Using that the $\otimes$ commutes with colimits $\boxed{\text{and}}$ that Δ^{op} is sifted so that $\mathrm{Mod}_S \longleftarrow \mathrm{CAlg}_S$ commutes w/ sifted colimits.

Skeleton spectral sequence.

$$E^2 = HH_*(\pi_*(R\otimes A)/\pi_*(A)) \Rightarrow \pi_*(R\otimes A).$$

Since $\pi_*(A\otimes A)$ is a connected Hopf algebra,

$$E^2 \cong \pi_*(A\otimes A) \otimes_{\pi_*(A)} \mathrm{Tor}^{\pi_*(A\otimes A)}_{*,*}(\pi_*(A), \pi_*(A))$$

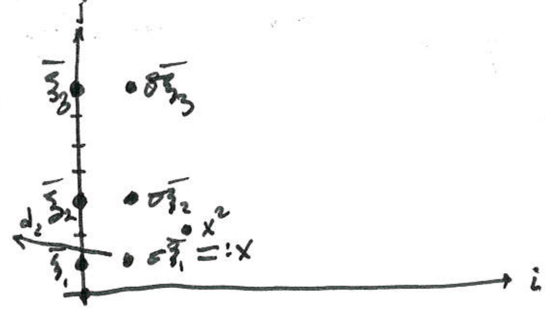
$$\cong \pi_*(A\otimes A) \otimes_{\pi_*(A)} \Lambda_{\pi_*(A)}\{\sigma\xi_1, \sigma\xi_2, \dots\}$$

Using Milnor.

$|\sigma\xi_i| = (1, 2^i - 1)$

Homological Serre grading.

Just writing alg. gens.



Spectral sequence is multiplicative.

All diffs vanish on gens for degree reasons (p=2).

So, the entire s.s. collapses.

$x^2 = 0$ in the ass. graded, but there is a filtration shift.

<u>Thm</u> (Steinberger). In $\pi_*(A\otimes A)$, $Q^{2^i}(\bar{\xi}_i) = \bar{\xi}_{i+1}$ for all $i \geq 1$.

Would like to conclude that

$$(\sigma\bar{\xi}_i)^2 = \sigma\bar{\xi}_{i+1}.$$

But $(\sigma\bar{\xi}_i)^2 = Q^{2^i}(\sigma\bar{\xi}_i) \overset{?}{=} \sigma Q^{2^i}(\bar{\xi}_i) \underset{\text{Steinberger}}{=} \sigma\bar{\xi}_{i+1}$.

Circle action on R induces

$$\text{A-homology of } \mathbb{T}_+ \to \mathbb{T}_+ \otimes A \xrightarrow{\sigma} R$$

Is σ $\mathbb{E}_\infty$? Apparently not clear.

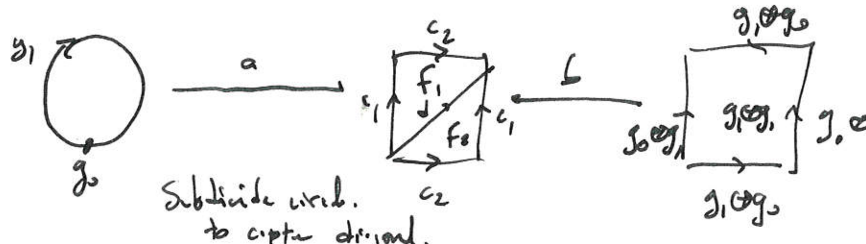
$$[\mathbb{T}] \otimes x \longmapsto \sigma(x)$$

↑ fundamental class of circle.

$$\begin{tikzcd}
\mathbb{T}_+ \otimes A \otimes A \arrow[rr, "\sigma \otimes id"] & & R \otimes A \\
\mathbb{T}_+ \otimes \left((A \otimes A)^{\otimes p} \right)_{h\Sigma_p} \otimes A \arrow[u] \arrow[d, "\Delta \otimes id"] & \text{Claim.} & \\
\left((\mathbb{T}_+ \otimes A \otimes A)^{\otimes p} \right)_{h\Sigma_p} \otimes A \arrow[rr] & & \left((R \otimes A)^{\otimes p} \right)_{h\Sigma_p} \otimes A \arrow[uu, "\mu_p"]
\end{tikzcd}$$

Claim. The diagonal Δ maps class of $[\mathbb{T}] \otimes e_i \otimes x^{\otimes p}$ to $e_{i-(p-1)} \otimes ([\mathbb{T}] \otimes x)^{\otimes p}$. Only true in homotopy. Need a homotopy.

proof ($p=2$).



$[\mathbb{T}]$ rep. by g_1

$$g_1 \otimes e_i \otimes x \otimes x \longmapsto d \otimes e_i \otimes x \otimes x$$

homologous?

$$e_{i-1} \otimes (g_1 \otimes x) \otimes (g_1 \otimes x) \xmapsto{b} e_{i-1} \otimes (f_1 + f_2) \otimes x \otimes x$$

$$\begin{aligned}
\partial(e_i \otimes f_1 \otimes x \otimes x) &= e_{i-1} \otimes N(f_1 \otimes x \otimes x) + e_i \otimes \partial(f_1 \otimes x \otimes x) \\
&= e_{i-1} \otimes (f_1 + f_2) \otimes x \otimes x + e_i \otimes (c_1 + c_2 + d) \otimes x \otimes x.
\end{aligned}$$

$$\partial(e_{i+1} \otimes c_1 \otimes x \otimes x) = e_i \otimes N(c_1 \otimes x \otimes x) + e_{i+1} \otimes \partial(c_1 \otimes x \otimes x)$$

$$= e_i \otimes (c_1 + c_2) \otimes x \otimes x \quad \blacksquare$$

This proves homology.
Get homotopy by descent.

Q. How does it imply Bott?

$$THH_*(\mathbb{F}_p) \simeq \mathbb{F}_p[x] \rightsquigarrow THH(\mathcal{O}_C, \mathbb{Z}_p) \cong \mathcal{O}_C[x]$$

$$C = \mathbb{C}_p.$$

$$\rightsquigarrow TP_*(\mathcal{O}_C, \mathbb{Z}_p) \cong A_{\inf}[x] \ni \mu \cdot x$$

$$K_*(\mathcal{O}_C, \mathbb{Z}_p) \cong \mathbb{Z}_p[\beta] \ni \beta$$