Symmetric power structures on algebraic K-theory

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Thanks.

This is basically all joint work with Barwick, Mathew and Nikolaus.

This talk will be about a subject very close to my heart: power operations on algebraic K-groups, and how one might lift them to space- or spectrum-level operations.

Let's start by defining Lawvere theories. These provide a very general framework for parametrising algebraic structure.

Definition 1. A Lawvere theory is a category which looks like a category of free finitely generated algebraic structures. Formally, it's a category $\mathbf{T}$ - all categories are ∞-categories, by the way - equipped with a coproduct-preserving functor from the category $\mathcal{F}$ of finite sets which gives a bijection on equivalence classes of objects.

If S is a finite set, then $\mathbf{T}(S)$ will denote the image of S under the structural functor $\mathcal{F} \to \mathbf{T}$. The point is that $\mathbf{T}(S, U) := \mathrm{Map}_{\mathbf{T}}(\mathbf{T}_S, \mathbf{T}_U)$ should be thought of as the operations with U inputs and S outputs in our theory.

The category of fg free objects in your favorite category of algebraic structures - monoids, groups, rings, you name it - is an example of this kind of structure.

We can talk about "models" for a Lawvere theory:

Definition 2. A *model* for a Lawvere theory $\mathbf{T}$ in a category $\mathbf{C}$ with finite products is a product-preserving functor $F : \mathbf{T}^{\mathrm{op}} \to \mathbf{C}$.

If $X = F(\mathbf{T}(1))$ (where, like a prehistoric human, I'm denoting by 1 the essentially unique one-element set), then this gives us maps

$$\mathrm{Map}_{\mathbf{T}}(S, U) \to \mathrm{Map}_{\mathbf{C}}(X^U, X^S).$$

The idea of this is as follows. If X is, say, a group, then we can think of the underlying set of X as the hom-set from the free group on one generator, $\mathrm{Grp}(\mathbb{Z}, X)$. More generally, if S is a finite set, then as sets,

$$X^S \simeq \mathrm{Grp}(F_S, X),$$

where F_S is the free group on an S-indexed set of generators. If we have a homomorphism $F_S \to F_T$, then we have a set map - not necessarily a group homomorphism - from X^T to X^S. So in particular, X is a set with 1-ary operations $\mathrm{Map}(\mathbb{Z},\mathbb{Z})$ - "take an element to an integral power" - and 2-ary operations $\mathrm{Map}(\mathbb{Z},F_2)$ - such as $1 \mapsto ab$ and $1 \mapsto ba$, giving "multiply elements in either order". You can see from this picture that $\mathbf{T}(1,S)$ is isomorphic to the underlying set of the free object generated by S.

Example 3. E_∞ spaces, grouplike E_∞ spaces (connective spectra) and E_∞ ring spaces are all modelled by Lawvere theories. Spectra aren't - if you try that, you just get connective spectra.

So let's talk about power operations. If E is a connective commutative ring spectrum, then we traditionally think of the homotopy groups of the free commutative E-algebra on a generator $\pi_*E[x]$ as power operations living on π_0 of commutative E-algebras. Indeed, if R is a commutative E-algebra, $x \in \pi_0(R)$ and $\alpha \in \pi_r(E[x])$, then x corresponds to an essentially unique E-algebra map $f_x : E[x] \to R$. Thinking of α as an operation, we write $\alpha(x) := f_x(\alpha) \in \pi_r(R)$.

Lawvere theories offer us a "spacification" of this: $\Omega^\infty(R)$ is a model for the Lawvere theory of commutative E-algebras in spaces, and so α doesn't just give us a set map $\pi_0(R) \to \pi_r(R)$ but an element of $\pi_r(\mathrm{Map}(\Omega^\infty(R),\Omega^\infty(R)))$. Think about this when $r = 0$. And these are definitely not spectrum maps - they include things like squaring.

In this talk I want to discuss the analog of this story for algebraic K-theory.

First let R be a commutative ring. By $K_0(R)$, we mean the abelian group whose elements are formal differences of fg projective R-modules $[P_1] - [P_2]$ (subject to some predictable equivalence relation), and the relation that $[P_1] + [P_2] = [P_1 \oplus P_2]$. $K_0(R)$ is a commutative ring by setting $[P_1][P_2] = [P_1 \otimes P_2]$. Grothendieck realized this isn't all the structure there is: you can take exterior powers of modules, defining

$$\lambda^k[P] = [\Lambda^k(P)],$$

and by some elementary linear algebra,

$$\lambda^k([P] + [Q]) = \Sigma_{j=0}^k \lambda^j([P])\lambda^{k-j}([Q]).$$

This formula can be used to recursively extend the λ^k to formal differences of projective modules, and thus to an operation on $K_0(R)$. In other words, $K_0(R)$ has the structure of a λ-ring:

Definition 4. A λ ring is a ring R together with set maps $\lambda^k : R \to R$ satisfying various axioms: the Cartan formula above, $\lambda^0(r) = 1$, $\lambda^1(r) = r$, and appropriate formulas (which can be recursively computed) for $\lambda^k(rs)$ and $\lambda^j(\lambda^k(r))$.

The easiest way to define this structure is using the language of plethories, which I don't have time to talk about today, but you should think of me as wearing a badge that says "Ask Me About Plethories".

You could ask, why did we take exterior powers - couldn't we just as well have taken symmetric powers? It turns out that in K_0, the symmetric powers can be written as polynomials in the exterior powers, so we aren't missing anything. It also turns out that in the Lawvere theory of λ-rings, $\mathcal{L}_\lambda$, the 1-ary operations are, as a ring,

$$\mathbb{Z}[\lambda_1, \lambda_2, \cdots]$$

So any natural operation on λ-rings is a polynomial in the exterior powers, and there are no polynomial relations between these.

So what? So there's a fancy spectrum version of K_0 called, simply, K-theory and denoted $K(R)$, such that

$$\pi_0(K(R)) \simeq K_0(R),$$

and we'd like the additional power bestowed by space-level λ-operations acting on $\Omega^\infty(K(R))$ in a Lawvere-theoretic fashion. Let me try to at least sketch a definition of $K(R)$. Really, K-theory isn't an invariant of a *ring*, it's an invariant of a *category*: I produced $K_0(R)$ not from R itself but from the category Proj_R of fg projective R-modules.

Definition 5. A category $\mathbf{C}$ is *additive* if it has a zero object, finite direct sums - this means that finite coproducts and finite products exist and coincide - and the mapping spaces, which are automatically commutative monoids, are actually grouplike. For convenience, we'll also assume our additive categories to be idempotent complete.

Definition 6. If $\mathbf{C}$ is additive, let $\iota\mathbf{C}$ be the maximal subgroupoid of $\mathbf{C}$ - that is, the category whose objects are those of $\mathbf{C}$ and whose morphisms are equivalences. Direct sum of objects makes $\iota\mathbf{C}$ into a commutative monoid space. The K-theory space $K(\mathbf{C})$ is the homotopy-theoretic group completion of $\iota\mathbf{C}$.

This is a fairly primitive definition of K-theory, but $K(\mathrm{Proj}_R)$ in this sense does recover Quillen's K-theory of R.

Now, one always makes things easier by categorifying, and we might expect there's some category-level structure that gives rise to the λ-structure on $K(R)$. Here we'll have to make a digression.

Let $\mathbf{C}$ and $\mathbf{D}$ be additive categories. We'd like to pull out a class of *polynomial* functors from $\mathbf{C}$ to $\mathbf{D}$. There are various ways of approaching this, all similar in spirit, but today we'll stick with one of the classics, due to Eilenberg and Mac Lane.

Definition 7. We say a functor $F : \mathbf{C} \to \mathbf{D}$ is *polynomial of degree* ≤ 0 if it's constant. We'll say a functor is *poly of degree* $\leq n$ if for all $X \in \mathbf{C}$, the functor

$$F_X : Y \mapsto \mathrm{fib}(F(Y \oplus X) \to F(Y))$$

(Note: It looks like I'm assuming this fiber exists. That's true if $\mathbf{D}$ is idempotent complete.) which we can think of as the derivative of F at X, is polynomial of degree $\leq n-1$. A functor is polynomial if it's polynomial of some degree.

So that's an inductive definition.

Example 8. Suppose **C** and **D** are both abelian groups, say, or spectra. Then functors like $X \mapsto X^{\otimes n}$, $X \mapsto \Lambda^n X$, $X \mapsto \mathrm{Sym}^n X$ are polynomial functors of degree n.

We'll access our K-theoretic operations by first defining a Lawvere theory that acts on categories of the form Proj_R, for R a commutative ring. As a preliminary, let's note that any fg $\mathbb{Z}$-lattice M can be thought of as a scheme:

$$M = \mathrm{Spec}\ (\mathrm{Sym}^*(M^\vee)),$$

in such a way that

$$\mathrm{Hom}(\mathrm{Spec}\ R, M) \simeq M \otimes R$$

as sets. Maps of schemes from M_1 to M_2 should be thought of as polynomial maps. For any $m \in \mathbb{N}$, we can therefore think of $\mathrm{Proj}_{\mathbb{Z}}$ as a scheme-enriched category.

Definition 9. A *strict polynomial functor* is a scheme-enriched functor between scheme-enriched categories. A strict polynomial functor from $\mathrm{Proj}_{\mathbb{Z}}^m$ to $\mathrm{Proj}_{\mathbb{Z}}^n$ is equivalently a polynomial functor $\mathrm{Proj}_R^m \to \mathrm{Proj}_R^n$ for each commutative ring R, compatible with base change.

Definition 10. The Lawvere theory $\mathcal{L}_{sp}$ of strict polynomial functors is the category whose objects are the categories $\mathrm{Proj}_{\mathbb{Z}}^n$ as n varies and whose morphisms are strict polynomial functors.

Clearly $\mathcal{L}_{sp}$ acts on Proj_R for any R. To pass through to K-theory, we need a new theorem, which is one of the main theorems of our forthcoming paper:

Theorem 11 (BGMN)**.** A polynomial functor $F : \mathbf{C} \to \mathbf{D}$ between additive categories induces a map of K-theory spaces $K(F) : \Omega^\infty(K(\mathbf{C})) \to \Omega^\infty(K * (\mathbf{D}))$. (In fact, more is true: it gives a polynomial map of spectra, but we won't need that today.)

Corollary 12. Let $\mathcal{K}_{sp}$ be the Lawvere theory obtained by applying K-theory pointwise to the mapping categories in $\mathcal{L}_{sp}$. Then $\mathcal{K}_{sp}$ acts on $\Omega^\infty K(R)$ for any commutative ring R.

This is only going to get us somewhere if we understand something about this Lawvere theory. Fortunately, we have another theorem about that:

Theorem 13 (BGMN)**.** The spectrum of 1-ary operations in $\mathcal{K}_{sp}$ is given, as a $K(\mathbf{Z})$-module, by

$$\mathcal{K}_{sp}(1,1) \simeq K(\mathbf{Z})\{\lambda_1, \lambda_2, \lambda_3, \cdots\}.$$

The proof of this theorem is based on Henning Krause's work on highest weight categories. So the exterior power operations, and all polynomial combinations of them, such as symmetric powers and Adams operations, do exist on the K-theory space. But we don't know the composition structure of this algebra. For instance, if $\alpha \in K_*(\mathbf{Z})$, we know there's a formula for $\lambda_n \circ \alpha$ as a polynomial in the λ_k with coefficients in $K_*(\mathbf{Z})$, but I at least don't know how to get at that formula.

So the bottom line is that we now have space-level lambda-operations on algebraic K-theory spectra.

STATUS: HIGHLY SPECULATIVE

In the last few minutes, I'd like to sketch a picture of how these polynomial considerations might interact with the trace. So the burning question here is: is THH polynomially functorial? Well, it's polynomially functorial in polynomial maps of rings, but not of categories. Let me illustrate the distinction here. In the linear setting, maps of rings correspond to maps of module categories pretty closely, but that's not true for polynomial maps. For instance, the map of rings

$$f : \mathbb{Z}/2 \to \mathbb{Z}/4,\ f(1) = 1$$

is a quadratic map. (It's the universal one, in fact.) As such it gives rise to a map of spaces $\Omega^\infty(THH(\mathbb{Z}/2)) \to \Omega^\infty(THH(\mathbb{Z}/4))$. However, at least as far as I can tell, it doesn't give a quadratic map of module categories $\mathrm{Proj}_{\mathbb{Z}/4} \to \mathrm{Proj}_{\mathbb{Z}/2}$. So the functorialities don't match up.

However, here's the conjecture I want to make:

Conjecture 14 (50% confidence)**.** Suppose $F \in \mathcal{L}_{sp}(1,1)$ is a strict polynomial functor of degree n. Then for each commutative ring R, we get a map of spaces

$$THH(F) : \Omega^\infty(THH(R))^{C_n} \to \Omega^\infty THH(R)$$

and more generally $\Omega^\infty(THH(R))^{C_{dn}} \to \Omega^\infty(THH(R))^{C_d}$, for any d. These maps can be expressed as polynomials in R, F, and their various compositions. They are compatible with R, and taking the limit, we find that TR is polynomially functorial and enjoys the structure of a spectral λ-ring. The cyclotomic trace $K \to TR$ is a homomorphism of λ-rings. I don't know what happens for TC.

If this is true, then the trace suddenly has a lot more structure which we should be able to exploit. The obvious place to start is by trying to prove an "Adams = Adams" theorem that relates the Adams operations coming from the λ-ring structure to those coming from coverings of the circle. The only thing we know in this direction at the moment is a rational result from Kantorovitz's thesis in the 90s.

Thanks.