Teena Gerhardt.

An overview of computations in algebraic K-theory via trace methods III.

Pointed monoid algebras.

$$K(A(\Pi)) \simeq THH(A) \wedge N^{cyc}(\Pi).$$

$$\pi_q\left(THH(A(?))^{C_{p^n}}\right) \rightsquigarrow \pi_{q-\lambda}\left(THH(A)^{C_{p^n}}\right).$$

<u>Problem #1.</u>

Understand how to build $N^{cy}(\Pi)$ out of rep spheres.

<u>Problem #2.</u>

Compute $\pi_\alpha(THH(A)^{C_{p^n}})$.

$A = \mathbb{F}_p$,

$A = \mathbb{Z}$.

$\alpha = [\beta] - [\gamma]$.

usually $q - \lambda$.

$T = THH(A)$

Replace THH w/ $THH(A) \wedge S^\lambda$ $\quad\quad \alpha' = \rho_p^*(\alpha^{C_p})$

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_q(T \wedge S^\lambda)_{hC_{p^n}} \arrow[r] \arrow[d] & \pi_{q-\lambda}(T^{C_{p^n}}) \arrow[r] \arrow[d] & \pi_{q-\lambda}(T^{C_{p^{n-1}}}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_q(T \wedge S^\lambda)_{hC_{p^n}} \arrow[r] & \pi_q(T \wedge S^\lambda)^{hC_{p^n}} \arrow[r] & \pi_q((T \wedge S^\lambda)^{tC_{p^n}}) \arrow[r] & \cdots
\end{tikzcd}$$

Want: $\pi_{q-\lambda}(THH(\mathbb{Z})^{C_{p^n}})$.

[Angeltveit-G] Computed $\pi_\alpha(THH(\mathbb{Z})^{C_{p^n}}, \mathbb{Z}/p)$.

$$E^1 = \pi_t\left((THH(\mathbb{Z}) \wedge S^{\lambda^{(n-s)}})_{hC_{p^s}}, \mathbb{Z}/p\right) \Rightarrow \pi_{t-\lambda}\left(THH(\mathbb{Z})^{C_{p^n}}, \mathbb{Z}/p\right).$$

<u>Thm</u> (Angeltveit-G-Hesselholt).

$\pi_{2i-\lambda} THH(\mathbb{Z})^{C_{p^n}}$ is free ab. of rank equal to the number of integers $0 \le s \le n$ s.t. $i = \dim_{\mathbb{C}} \lambda^{C_{p^s}}$.

Idea: LES. Homotopy orbit s.s.

$$E^2_{st} = H_s(BC_{p^n}, \pi_{t-|\lambda|} THH(\mathbb{Z})) \Rightarrow \pi_{s+t}(THH \wedge S^\lambda)_{hC_{p^n}}.$$

Mod p calcs show that ranks account for everything.

Can use this to get some K-theory calcs.

Ex. $\Pi_m = \{0, 1, x, \dots, x^{m-1}\}$, $x^m = 0$.

$$\mathbb{Z}(\Pi_m) \simeq \mathbb{Z}[x]/(x^m).$$

$N^{cyc}(\Pi_m)$.

Let $N^{cy}(\Pi_m, i)$ denote the cyclic subset of total x-degree i.

HM. $S^{\lambda d} \wedge S^1/C_{i/m+} \to S^{\lambda d} \wedge S^1/C_{i+} \to N^{cy}(\Pi_m, i)$

$d = \lfloor \frac{i-1}{m} \rfloor$

$\lambda d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \dots \oplus \mathbb{C}(d)$

$$\pi_q THH(\mathbb{Z}[x]/(x^m))^{C_{p^n}} = [S^q \wedge S^1/C_{p^n+}, THH(\mathbb{Z}[x]/(x^m))]_{S^1}$$

$$\simeq [S^q \wedge S^1/C_{p^n}, THH(\mathbb{Z}) \wedge N^{cy}(\Pi_m)]_{S^1}$$

$\sim$ Computations like $[S^q \wedge S^1/C_{p^n+}, THH(\mathbb{Z}) \wedge S^{\lambda d}]_{S^1} = \pi_{q-\lambda_d}(THH(\mathbb{Z})^{C_{p^n}})$.

Recall: Thm (Geller-Roberts). $K_2(\mathbb{Z}[x]/(x^2), (x)) \cong \mathbb{Z}/2$.

Thm (Soulé, Stasheff). rank $K_q(\mathbb{Z}[x]/(x^m), (x)) = m-1$ q odd, 0 q even.

Thm (AGH).

$$K_{2i+1}(\mathbb{Z}[x]/(x^m), (x)) \cong \mathbb{Z}^{m-1}$$

$$\# K_{2i}(\qquad) = (mi)!\,(i!)^{m-2}.$$

Thm (AG)

$$K_{2i}(\mathbb{Z}[x,y]/(xy), (x,y)) \cong \mathbb{Z}$$

$$\# K_{2i+1}(\qquad) = (i!)^2.$$

$K(\mathbb{Z}[C_2])$.

$\Pi_{C_2} = \{0,1,x\}$, $x^2=1$

$\mathbb{Z}(\Pi_{C_2}) = \mathbb{Z}[C_2]$

Π_2 is similar but $x^2=0$.

Known:

$K_0(\mathbb{Z}[C_2]) \cong \mathbb{Z}$ (Rim, 1958).

$K_1(\qquad) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (Rim, Milnor 1960).

$K_2(\qquad) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (Dunwoody '75).

Alg. defs of K-groups.

Ranks are known.

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

Example of a Milnor square.

$\mathbb{Z}[x]/(x^2=1)$

$\mathbb{Z}[x]/(x+1)(x-1)$

Two points away from (2), Nilpotent at (2).

$\mathbb{Z}_{(2)}[x]/(x+1)(x-1)$

Get MV up through deg 2. Not higher.

What about KH?

How badly does M-V fail?

Birelative $K(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$ is iterated fiber of

$$\begin{tikzcd}
K(\mathbb{Z}[C_2]) \arrow[r] \arrow[d] & K(\mathbb{Z}) \arrow[d] \\
K(\mathbb{Z}) \arrow[r] & K(\mathbb{Z}/2)
\end{tikzcd}$$

$$\cdots \to K_{n+1}(R/I+J) \oplus K_n(R,I,J) \to K_n(R) \to K_n(R/I) \oplus K_n(R/J) \to \cdots$$

$$\cdots \to K_{n+1}(\mathbb{Z}/2) \oplus K_n(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}) \to K_n(\mathbb{Z}[C_2]) \to K_n(\mathbb{Z}) \oplus K_n(\mathbb{Z}) \to \cdots$$

Need to understand this.

$e_i \mapsto 1$

$A = \mathbb{Z}[C_2]$, $B = \mathbb{Z}$

$$\mathrm{trc}: K_n(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}) \cong TC_n(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}).$$

$I = \ker(\mathbb{Z}[C_2] \to \mathbb{Z})$

$e_1 \mapsto 1$

$e_{-1} \mapsto -1$

$\boxed{\text{Cortiñas, Geisser – Hesselholt.}}$ Interesting.

Thus, need birelative TC.

Milnor squares are pullbacks and pushouts.

Categories of perfect complexes: get pullback.

$$THH(\mathbb{Z}[C_2]) \simeq THH(\mathbb{Z}) \wedge N^{cy}(\Pi_{C_2})$$

Filt $F^i N^{cy}(\Pi_{C_2})$.

$THH(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$

Associated graded is same as for Π_2.

$N^{cy}(\Pi_2, i)$.

$$\begin{tikzcd}
THH(\mathbb{Z}) \wedge \widetilde{N}^{cy}() \arrow[r] \arrow[d] & THH(\mathbb{Z}) \wedge N^{cy}(\Pi_{C_2}) \arrow[r] \arrow[d] & THH(\mathbb{Z}) \arrow[d] \\
D \arrow[r] & THH(\mathbb{Z}) \arrow[r] & THH(\mathbb{F}_p)
\end{tikzcd}$$

Thm (A-G). With these methods.

Recover K_0, K_1, K_2.

$K_3 \cong \mathbb{Z}/16 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$.

$K_5 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus H$, $|H| \mid 8$.

$$\boxed{TC(\mathbb{Z}[C_2])}$$