18F05.

Lecture 7.

Almost mathematics III.

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$I \subseteq R$ flat and $I^2 = I$.

$\mathrm{Mod}_R^a = \mathrm{Mod}_R / \mathrm{Mod}_{R/I}$.

$$\mathrm{Mod}_R \underset{j_!}{\overset{j_*}{\rightleftarrows}} \mathrm{Mod}_R^a$$

$M \in \mathrm{Mod}_R$

$M_! = j_!(M^a) = I \otimes_R M$.

$M_* = j_*(M^a) = \mathrm{Hom}_R(I, M)$.

$\mathcal{A} \subseteq \mathrm{Mod}_R$, M s.t. $I \otimes_R M \simeq M$.

$\mathrm{Mod}_R \xrightarrow[I\otimes_R -]{} \mathcal{A}$ is a concrete realization of Mod_R^a.

$j_!$ obvious inclusion.

$j_*(M) = \mathrm{Hom}_R(I, M)$.

Internal Homs and $\otimes$-products.

$M \in \mathrm{Mod}_{R/I}$, $N \in \mathrm{Mod}_R \Rightarrow M \otimes_R N$ I-torsion.

So, get a $\otimes$ on Mod_R^a.

Also, $M^a \otimes N^a \simeq (M \otimes N)^a$.

$\mathrm{alHom}(X, Y) \in \mathrm{Mod}_R^a$ for $X, Y \in \mathrm{Mod}_R^a$.

$M, N \in \mathrm{Mod}_R$.

$\mathrm{alHom}(M^a, N^a) \simeq \mathrm{Hom}(M, N)^a$.

Prop. $X, Y, Z \in \mathrm{Mod}_R^a$.

proof. $\mathrm{Hom}_{\mathrm{Mod}_R^a}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathrm{Mod}_R^a}(X, \mathrm{alHom}_{\mathrm{Mod}_R^a}(Y, Z))$

Warning. $\mathrm{alHom}(X, Y) \neq \mathrm{Hom}_{\mathrm{Mod}_R^a}(X, Y)$.

Under concrete realization ($\mathcal{A}$), $\mathrm{Hom}_R(I \otimes_R M, I \otimes_R N) \neq I \otimes_R \mathrm{Hom}_R(M, N)$.

Only true for M f.p.

Doing algebra in Mod_R^a.

$(\mathrm{Mod}_R^a, \otimes)$.

Define algebras, modules, etc.

Q. When is Mod_R^a unipotent?
I guess always.

__Notation__: $\mathrm{Mod}_{R^a} = \mathrm{Mod}_R$.

Warning: sometimes $R^a = R$ in the sense of almost elts.

Lax symmetric monoidal.

$$j_*X \otimes j_*Y \to j_*(X \otimes Y).$$

Preserves algebras. So, every almost alg. comes from an actual algebra.

Almost homological algebra.

$M, N \in \mathrm{Mod}_R$

$$\mathrm{Ext}^i_{R^a}(M,N) := \mathrm{Ext}^i_R(I \otimes_R M, N)$$

- M or M^a is almost flat if $M^a \otimes -$ is almost exact $(\mathrm{Tor}_i^R(M,N)^a = 0,\ i>0)$.
- M or M^a is almost projective if $\mathrm{alHom}(M^a, -)$ is exact or eq. if $\mathrm{Ext}^i_R(M,N)^a = 0$ for all N.

__Claim__. I not f.g., R local. Then R^a is not projective in Mod_R^a.

If R^a is projective, then for any exact seq $0 \to M \to N \to P \to 0$ in Mod_R, get

$$\begin{array}{ccccccc} 0 \to & M_* & \to & N_* & \to & P_* & \to 0. \\ & \| & & \| & & \| & \\ & \mathrm{Hom}_R(I,M) & & \mathrm{Hom}_R(I,N) & & \mathrm{Hom}_R(I,P) & \end{array}$$

$\Rightarrow$ I is projective.
$\Rightarrow$ I is principal, which it's not.

Rem. Mod_R^a has enough injectives: $\mathrm{Hom}_{R^a}(-, E^a) \cong \mathrm{Hom}_R(j_!(\ ), E)\ldots$

In fact, any injective object E in Mod_R^a, $j_* E$ is injective.

Q. Is $E \to j_* E^a$ an iso when E is injective.

Probably not. Injective envelope of R/I? No.

Finiteness. We say M, M^a is <u>almost f.g.</u> if for all $\varepsilon \in I$ there exists a f.g. R-module M_ε and a map $f_\varepsilon : M_\varepsilon \to M$ s.t. $\ker(f_\varepsilon)$, $\mathrm{coker}(f_\varepsilon)$ are <u>ε-torsion</u>.

Same for f.p.

M, M^a is <u>uniformly</u> gen. by n elts if $\exists n \in \mathbb{N}$ s.t. $\forall \varepsilon \in I$, M_ε can taken to be gen. by n elts.

<u>Lemma</u>. M almost f.p., then M almost flat $\iff$ M almost projective.

<u>Prop</u>. S of f. type over R. TFAE $\quad \mu : S \otimes_R S \to S$.

(1) $\Omega^1_{S/R} = 0$,

(2) $\ker(\mu) = \ker(\mu)^2$,

(3) $\ker(\mu)$ gen by an idempotent elt,

(4) $\exists$ an idempotent in $S \otimes_R S$ s.t. $\mu(\varepsilon) = 1$ and $\varepsilon \cdot \ker(\mu) = 0$.

Def. $A \to B$ R^a algebras.

$A \to B$ is <u>almost</u> <u>unramified</u> if there exists

$$e \in (B \otimes_A B)_*$$ s.t.

$$e^2 = e,$$

$$\mu(e) = 1,$$

and $\ker(\mu_*)\, e = 0$.

Def. $A \to B$ is almost finite étale if it is almost unramified and B is an almost f.p. projective A-module.