Commutative Algebra of Categories

John D. Berman

University of Virginia

September 22, 2017

K Theory of categories

$\mathcal{C}$ is an (∞-)category.

K Theory of categories

$\mathcal{C}$ is an (∞-)category.
$\mathcal{C}^{\text{iso}}$ is an (∞-)groupoid (space).

K Theory of categories

$\mathcal{C}$ is an (∞-)category.
$\mathcal{C}^{\text{iso}}$ is an (∞-)groupoid (space).

Example 1

$\mathcal{C} = \text{Fin}$, $\mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \coprod_n B\Sigma_n$

$\mathcal{C}$ is an (∞-)category.
$\mathcal{C}^{\text{iso}}$ is an (∞-)groupoid (space).

Example 1

$\mathcal{C} = \text{Fin}$, $\mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \coprod_n B\Sigma_n$

$\mathcal{C}^{\text{iso}}$ inherits extra structure from $\mathcal{C}$.

K Theory of categories

$\mathcal{C}$ is an (∞-)category.
$\mathcal{C}^{\text{iso}}$ is an (∞-)groupoid (space).

Example 1

$\mathcal{C} = \text{Fin}$, $\mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \coprod_n B\Sigma_n$

$\mathcal{C}^{\text{iso}}$ inherits extra structure from $\mathcal{C}$.

Example 2

If $\mathcal{C}^{\oplus}$ is symmetric monoidal, $\mathcal{C}^{\text{iso}}$ inherits $\mathbb{E}_\infty$-space structure.

K Theory of categories

$\mathcal{C}$ is an (∞-)category.
$\mathcal{C}^{\text{iso}}$ is an (∞-)groupoid (space).

Example 1

$\mathcal{C} = \text{Fin}$, $\mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \coprod_n B\Sigma_n$

$\mathcal{C}^{\text{iso}}$ inherits extra structure from $\mathcal{C}$.

Example 2

If $\mathcal{C}^{\oplus}$ is symmetric monoidal, $\mathcal{C}^{\text{iso}}$ inherits $\mathbb{E}_\infty$-space structure.

Example 3

$\coprod_n B\Sigma_n$ inherits *two* $\mathbb{E}_\infty$-space structures from $\amalg, \times$.

An $\mathbb{E}_\infty$-space X is *grouplike* if the commutative monoid $\pi_0(X)$ is an abelian group.

K Theory of categories

An $\mathbb{E}_\infty$-space X is *grouplike* if the commutative monoid $\pi_0(X)$ is an abelian group.

Theorem 4

$\Omega^\infty : Sp \to \mathbb{E}_\infty Top$ determines an equivalence

$$\mathbb{E}_\infty Top_{gp} \simeq Sp^{\geq 0}.$$

An $\mathbb{E}_\infty$-space X is *grouplike* if the commutative monoid $\pi_0(X)$ is an abelian group.

Theorem 4

$\Omega^\infty : Sp \to \mathbb{E}_\infty Top$ *determines an equivalence*

$$\mathbb{E}_\infty Top_{gp} \simeq Sp^{\geqslant 0}.$$

$K(\mathcal{C}^\oplus) =$'group completion' of the $\mathbb{E}_\infty$-space $\mathcal{C}^{\text{iso}}$ (a spectrum).

Examples

Example 5

Perfect modules over a ring spectrum: $\mathcal{C}^{\oplus} = \mathrm{Mod}_R^{\mathrm{perf},\oplus}$

Examples

Example 5

Perfect modules over a ring spectrum: $\mathcal{C}^{\oplus} = \mathrm{Mod}_R^{\mathrm{perf},\oplus}$
$K(\mathcal{C}^{\oplus}) = K(R)$ (definition of higher algebraic K-theory)

Examples

Example 5

Perfect modules over a ring spectrum: $\mathcal{C}^{\oplus} = \mathrm{Mod}_R^{\mathrm{perf},\oplus}$
$K(\mathcal{C}^{\oplus}) = K(R)$ (definition of higher algebraic K-theory)

Example 6

Finite sets: $\mathcal{C}^{\oplus} = \mathrm{Fin}^{\amalg}$

Examples

Example 5

Perfect modules over a ring spectrum: $\mathcal{C}^{\oplus} = \text{Mod}_R^{\text{perf},\oplus}$
$K(\mathcal{C}^{\oplus}) = K(R)$ (definition of higher algebraic K-theory)

Example 6

Finite sets: $\mathcal{C}^{\oplus} = \text{Fin}^{\amalg}$
$K(\mathcal{C}^{\oplus}) \cong \mathbb{S}$ (Barratt-Priddy-Quillen theorem)

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;
- $\otimes$ distributes over $\oplus$.

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;
- $\otimes$ distributes over $\oplus$.

$K(\mathcal{C}^{\oplus})$ should inherit the structure of an $\mathbb{E}_{\infty}$-ring spectrum.

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;
- $\otimes$ distributes over $\oplus$.

$K(\mathcal{C}^{\oplus})$ should inherit the structure of an $\mathbb{E}_\infty$-ring spectrum.

Obstacles to making this precise:

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;
- $\otimes$ distributes over $\oplus$.

$K(\mathcal{C}^{\oplus})$ should inherit the structure of an $\mathbb{E}_\infty$-ring spectrum.

Obstacles to making this precise:

1. What is a 'semiring (∞-)category'?

Ring structure

In each case, $\mathcal{C}^{\oplus}$ is a 'commutative semiring (∞-)category':

- $\mathcal{C}$ has a second symmetric monoidal operation $\otimes$;
- $\otimes$ distributes over $\oplus$.

$K(\mathcal{C}^{\oplus})$ should inherit the structure of an $\mathbb{E}_{\infty}$-ring spectrum.

Obstacles to making this precise:

1. What is a 'semiring (∞-)category'?
2. What is 'group completion'?

Ring structure

Obstacles to making this precise:

1. What is a 'semiring (∞-)category'?
2. What is 'group completion'?

Traditional answers (70's):

Ring structure

Obstacles to making this precise:

1. What is a ‘semiring (∞-)category’?
2. What is ‘group completion’?

Traditional answers (70’s):

1. bipermutative categories

Ring structure

Obstacles to making this precise:

1. What is a 'semiring (∞-)category'?
2. What is 'group completion'?

Traditional answers (70's):

1. bipermutative categories
2. Quillen Q construction, Waldhausen $S_\bullet$ construction

Ring structure

Obstacles to making this precise:

1. What is a 'semiring (∞-)category'?
2. What is 'group completion'?

Traditional answers (70's):

1. bipermutative categories
2. Quillen Q construction, Waldhausen $S_\bullet$ construction

Alternative: categorify ordinary semirings and group completion!

Semirings and group completion

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.
2. $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) is a symmetric monoidal category.

Semirings and group completion

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.
2. $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) is a symmetric monoidal category.
3. Monoids in $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) are rings (semirings).

Semirings and group completion

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.
2. $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) is a symmetric monoidal category.
3. Monoids in $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) are rings (semirings).
4. $\mathbb{Z}$ ($\mathbb{N}$) is the free abelian group (commutative monoid) on one generator.

Semirings and group completion

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.
2. $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) is a symmetric monoidal category.
3. Monoids in $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) are rings (semirings).
4. $\mathbb{Z}$ ($\mathbb{N}$) is the free abelian group (commutative monoid) on one generator.
5. A commutative monoid is an abelian group if and only if it is a $\mathbb{Z}$-module.

1. Given abelian groups (commutative monoids) A, B, there is an abelian group (commutative monoid) $A \otimes B$.
2. $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) is a symmetric monoidal category.
3. Monoids in $\mathrm{Ab}^{\otimes}$ ($\mathrm{ComMon}^{\otimes}$) are rings (semirings).
4. $\mathbb{Z}$ ($\mathbb{N}$) is the free abelian group (commutative monoid) on one generator.
5. A commutative monoid is an abelian group if and only if it is a $\mathbb{Z}$-module.
6. $\mathbb{Z} \otimes_{\mathbb{N}} -$ is group completion.

Semiring categories

(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal ∞-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal ∞-category $\mathcal{C} \otimes \mathcal{D}$.

Semiring categories

(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal ∞-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal ∞-category $\mathcal{C} \otimes \mathcal{D}$.
2. $\mathrm{SymMon}_{\infty}^{\otimes}$ is a (large) symmetric monoidal ∞-category.

(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal ∞-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal ∞-category $\mathcal{C} \otimes \mathcal{D}$.
2. $\mathrm{SymMon}_\infty^\otimes$ is a (large) symmetric monoidal ∞-category.
3. (Definition) Monoids in $\mathrm{SymMon}_\infty^\otimes$ are semiring ∞-categories.

Semiring categories

(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal ∞-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal ∞-category $\mathcal{C} \otimes \mathcal{D}$.
2. $\mathrm{SymMon}_\infty^\otimes$ is a (large) symmetric monoidal ∞-category.
3. (Definition) Monoids in $\mathrm{SymMon}_\infty^\otimes$ are semiring ∞-categories.
4. $\mathrm{Fin}^{\mathrm{iso}}$ is the free symmetric monoidal ∞-category on one generator.

(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal ∞-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal ∞-category $\mathcal{C} \otimes \mathcal{D}$.
2. $\mathrm{SymMon}_\infty^\otimes$ is a (large) symmetric monoidal ∞-category.
3. (Definition) Monoids in $\mathrm{SymMon}_\infty^\otimes$ are semiring ∞-categories.
4. $\mathrm{Fin}^{\mathrm{iso}}$ is the free symmetric monoidal ∞-category on one generator.

Proofs are formal, using higher algebra of presentable ∞-categories.

Examples of commutative semiring ∞-categories:

Examples of commutative semiring ∞-categories:

- closed monoidal categories (Set, Top, Vect, Set_G)

Examples of commutative semiring ∞-categories:

- closed monoidal categories (Set, Top, Vect, Set_G)
- categories built via some constructions ($\mathrm{Set}^{\mathrm{op}}$, $\mathrm{Set}^{\mathrm{iso}}$)

Semiring categories

Examples of commutative semiring ∞-categories:

- closed monoidal categories (Set, Top, Vect, Set_G)
- categories built via some constructions (Set^{op}, Set^{iso})
- connective commutative ring spectra ($\mathbb{S}$, KU, HR)

Group completion

There is a full subcategory inclusion $\mathrm{Sp}^{\geqslant 0} \subseteq \mathrm{SymMon}_{\infty}$.

There is a full subcategory inclusion $\mathrm{Sp}^{\geqslant 0} \subseteq \mathrm{SymMon}_{\infty}$.

Theorem 7 (B.)

- *$Sp^{\geqslant 0} \simeq Mod_{\mathbb{S}}$ (i.e., $\mathcal{C}^{\oplus}$ is an $\mathbb{S}$-module iff it is a spectrum).*

There is a full subcategory inclusion $\mathrm{Sp}^{\geqslant 0} \subseteq \mathrm{SymMon}_{\infty}$.

Theorem 7 (B.)

- *$Sp^{\geqslant 0} \simeq Mod_{\mathbb{S}}$ (i.e., $\mathcal{C}^{\oplus}$ is an $\mathbb{S}$-module iff it is a spectrum).*
- *$\mathbb{S} \otimes \mathcal{C}^{\oplus} \simeq K(\mathcal{C}^{\oplus})$ if $\mathcal{C}$ is a groupoid.*

There is a full subcategory inclusion $\mathrm{Sp}^{\geqslant 0} \subseteq \mathrm{SymMon}_\infty$.

Theorem 7 (B.)

- $Sp^{\geqslant 0} \cong Mod_{\mathbb{S}}$ *(i.e.,* $\mathcal{C}^{\oplus}$ *is an* $\mathbb{S}$*-module iff it is a spectrum).*
- $\mathbb{S} \otimes \mathcal{C}^{\oplus} \cong K(\mathcal{C}^{\oplus})$ *if* $\mathcal{C}$ *is a groupoid.*
- $Hom(\mathbb{S}, \mathcal{C}^{\oplus}) \cong Pic(\mathcal{C}^{\oplus})$.

Group completion

There is a full subcategory inclusion $\mathrm{Sp}^{\geqslant 0} \subseteq \mathrm{SymMon}_\infty$.

Theorem 7 (B.)

- $Sp^{\geqslant 0} \simeq Mod_{\mathbb{S}}$ *(i.e., $\mathcal{C}^{\oplus}$ is an $\mathbb{S}$-module iff it is a spectrum).*
- $\mathbb{S} \otimes \mathcal{C}^{\oplus} \simeq K(\mathcal{C}^{\oplus})$ *if $\mathcal{C}$ is a groupoid.*
- $Hom(\mathbb{S}, \mathcal{C}^{\oplus}) \simeq Pic(\mathcal{C}^{\oplus})$.

If $\mathcal{C}^{\oplus,\otimes}$ not a groupoid but semiadditive (Mod_R),

$$K(\mathcal{C}^{\oplus}) \simeq \mathbb{S} \otimes \mathrm{Fun}^{\oplus,\otimes}(\mathrm{Burn}[\mathrm{Cob}_1^{\mathrm{fr}}], \mathcal{C}).$$

Cartesian monoidal categories

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Cartesian monoidal categories

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.
- $\mathcal{C}$ is cocartesian monoidal if $\oplus = \amalg$.

Cartesian monoidal categories

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.
- $\mathcal{C}$ is cocartesian monoidal if $\oplus = \amalg$.
- $\mathcal{C}$ is semiadditive if $\oplus = \times = \amalg$.

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.
- $\mathcal{C}$ is cocartesian monoidal if $\oplus = \amalg$.
- $\mathcal{C}$ is semiadditive if $\oplus = \times = \amalg$.

Example 9

- Set is cocartesian monoidal.

Cartesian monoidal categories

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.
- $\mathcal{C}$ is cocartesian monoidal if $\oplus = \amalg$.
- $\mathcal{C}$ is semiadditive if $\oplus = \times = \amalg$.

Example 9

- Set is cocartesian monoidal.
- Set^{op} is cartesian monoidal.

$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.

Definition 8

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$.
- $\mathcal{C}$ is cocartesian monoidal if $\oplus = \amalg$.
- $\mathcal{C}$ is semiadditive if $\oplus = \times = \amalg$.

Example 9

- Set is cocartesian monoidal.
- $\mathrm{Set}^{\mathrm{op}}$ is cartesian monoidal.
- Ab (or ComMon) is semiadditive.

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ *is a Fin-module iff* $\oplus = \amalg$)

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ is a Fin-module iff $\oplus = \amalg$)
- $Mod_{Fin^{op}} \cong CartMonCat$

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ is a Fin-module iff $\oplus = \amalg$)
- $Mod_{Fin^{op}} \cong CartMonCat$
- $Mod_{Fin \otimes Fin^{op}} \cong SemiaddCat$

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ *is a Fin-module iff* $\oplus = \amalg$)
- $Mod_{Fin^{op}} \cong CartMonCat$
- $Mod_{Fin \otimes Fin^{op}} \cong SemiaddCat$

Results are true for categories or ∞-categories.

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ *is a Fin-module iff* $\oplus = \amalg$)
- $Mod_{Fin^{op}} \cong CartMonCat$
- $Mod_{Fin \otimes Fin^{op}} \cong SemiaddCat$

Results are true for categories or ∞-categories.
Question: What is $\text{Fin} \otimes \text{Fin}^{\text{op}}$?

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ ($\mathcal{C}^{\oplus}$ is a Fin-module iff $\oplus = \amalg$)
- $Mod_{Fin^{op}} \cong CartMonCat$
- $Mod_{Fin \otimes Fin^{op}} \cong SemiaddCat$

Results are true for categories or ∞-categories.
Question: What is $\mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}}$?

Theorem 11 (Glasman)

The Burnside category is the free semiadditive category on one object.

Cartesian monoidal categories

Theorem 10 (B.)

- $Mod_{Fin} \cong CocartMonCat$ *($\mathcal{C}^{\oplus}$ is a Fin-module iff $\oplus = \amalg$)*
- $Mod_{Fin^{op}} \cong CartMonCat$
- $Mod_{Fin \otimes Fin^{op}} \cong SemiaddCat$

Results are true for categories or ∞-categories.
Question: What is $\mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}}$?

Theorem 11 (Glasman)

The Burnside category is the free semiadditive category on one object.

$\mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}} \cong \mathrm{Burn}$

Modules over semiring categories

Semiring ∞-category $\mathcal{R}$	**$\mathcal{R}$-modules**
$\mathbb{S}$	Spectra
$\mathrm{Fin}^{\mathrm{iso}}$	Symmetric monoidal
Fin	Cocartesian monoidal
$\mathrm{Fin}^{\mathrm{op}}$	Cartesian monoidal
$\mathrm{Fin}^{\mathrm{inj}}$	Symmetric monoidal with initial unit
$\mathrm{Fin}^{\mathrm{inj,op}}$	Symmetric monoidal with terminal unit
Fin_*	Cocartesian monoidal with $0 = 1$
$\mathrm{Fin}_*^{\mathrm{op}}$	Cartesian monoidal with $0 = 1$
Burn	Semiadditive
$\mathrm{Burn}_{\mathrm{gp}}$	Additive

Definition 12

A *PROP* (PROduct and Permutation category) is a symmetric monoidal category $\mathcal{P}^{\oplus}$ generated by one object under $\oplus$.

Definition 12

A *PROP* (PROduct and Permutation category) is a symmetric monoidal category $\mathcal{P}^{\oplus}$ generated by one object under $\oplus$.

Think: objects labeled by finite sets, $\oplus = \amalg$.

Definition 12

A *PROP* (PROduct and Permutation category) is a symmetric monoidal category $\mathcal{P}^{\oplus}$ generated by one object under $\oplus$.

Think: objects labeled by finite sets, $\oplus = \amalg$.

Definition 13

A $\mathcal{P}^{\oplus}$-algebra in $\mathcal{C}^{\otimes}$ is a symmetric monoidal functor

$$\mathrm{Alg}_{\mathcal{P}}(\mathcal{C}^{\otimes}) = \mathrm{Hom}(\mathcal{P}^{\oplus}, \mathcal{C}^{\otimes}).$$

Example 14

- Fin is the PROP for commutative monoids;

Example 14

- Fin is the PROP for commutative monoids;
- Fin^{op} is the PROP for cocommutative comonoids;

Example 14

- Fin is the PROP for commutative monoids;
- $\mathrm{Fin}^{\mathrm{op}}$ is the PROP for cocommutative comonoids;
- $\mathrm{Burn} \cong \mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}}$ is the PROP for commutative-cocommutative bimonoids;

Example 14

- Fin is the PROP for commutative monoids;
- $\mathsf{Fin}^{\mathsf{op}}$ is the PROP for cocommutative comonoids;
- $\mathsf{Burn} \cong \mathsf{Fin} \otimes \mathsf{Fin}^{\mathsf{op}}$ is the PROP for commutative-cocommutative bimonoids;
- $\mathsf{Burn}_{\mathsf{gp}}$ is the PROP for Hopf algebras.

Example 14

- Fin is the PROP for commutative monoids;
- $\mathsf{Fin}^{\mathsf{op}}$ is the PROP for cocommutative comonoids;
- $\mathsf{Burn} \cong \mathsf{Fin} \otimes \mathsf{Fin}^{\mathsf{op}}$ is the PROP for commutative-cocommutative bimonoids;
- $\mathsf{Burn}_{\mathsf{gp}}$ is the PROP for Hopf algebras.

If $\mathcal{P}^{\oplus}$ is cartesian monoidal, $\mathcal{P}^{\mathsf{op}} \subseteq \mathsf{Alg}_{\mathcal{P}}(\mathsf{Set}^{\times})$:

Algebraic theories

Example 14

- Fin is the PROP for commutative monoids;
- $\mathrm{Fin}^{\mathrm{op}}$ is the PROP for cocommutative comonoids;
- $\mathrm{Burn} \cong \mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}}$ is the PROP for commutative-cocommutative bimonoids;
- $\mathrm{Burn}_{\mathrm{gp}}$ is the PROP for Hopf algebras.

If $\mathcal{P}^{\oplus}$ is cartesian monoidal, $\mathcal{P}^{\mathrm{op}} \subseteq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Top}^{\times})$:

Algebraic theories

Example 14

- Fin is the PROP for commutative monoids;
- $\mathrm{Fin}^{\mathrm{op}}$ is the PROP for cocommutative comonoids;
- $\mathrm{Burn} \cong \mathrm{Fin} \otimes \mathrm{Fin}^{\mathrm{op}}$ is the PROP for commutative-cocommutative bimonoids;
- $\mathrm{Burn}_{\mathrm{gp}}$ is the PROP for Hopf algebras.

If $\mathcal{P}^{\oplus}$ is cartesian monoidal, $\mathcal{P}^{\mathrm{op}} \subseteq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Top}^{\times})$:
Subcategory of finitely generated free objects.

Algebraic theories

Example 14

- Fin is the PROP for commutative monoids;
- $\mathsf{Fin}^{\mathsf{op}}$ is the PROP for cocommutative comonoids;
- $\mathsf{Burn} \cong \mathsf{Fin} \otimes \mathsf{Fin}^{\mathsf{op}}$ is the PROP for commutative-cocommutative bimonoids;
- $\mathsf{Burn}_{\mathsf{gp}}$ is the PROP for Hopf algebras.

If $\mathcal{P}^{\oplus}$ is cartesian monoidal, $\mathcal{P}^{\mathsf{op}} \subseteq \mathsf{Alg}_{\mathcal{P}}(\mathsf{Top}^{\times})$: Subcategory of finitely generated free objects.

Definition 15

A *Lawvere theory* is a cartesian monoidal PROP $\mathcal{L}^{\times}$. Algebras are taken in $\mathsf{Set}^{\times}$ (1-categories) or $\mathsf{Top}^{\times}$ (∞-categories):

$$\mathsf{Alg}_{\mathcal{L}} = \mathsf{Alg}_{\mathcal{L}}(\mathsf{Top}^{\times}) \simeq \mathsf{Hom}(\mathcal{L}^{\times}, \mathsf{Top}^{\times}).$$

Example 16

Lawvere theory	Set-algebras	Top-algebras
Fin^{op}	Set	Top

Example 16

Lawvere theory	Set-algebras	Top-algebras
$\mathsf{Fin}^{\mathsf{op}}$	Set	Top
$\mathsf{Burn} = \mathsf{Span}(\mathsf{Fin})$	Ab	$\mathsf{Sp}^{\geqslant 0}$

Example 16

Lawvere theory	Set-algebras	Top-algebras
$\mathrm{Fin}^{\mathrm{op}}$	Set	Top
$\mathrm{Burn} = \mathrm{Span}(\mathrm{Fin})$	Ab	$\mathrm{Sp}^{\geqslant 0}$
$\mathrm{Poly} = \mathrm{Bispan}(\mathrm{Fin})$	ComRing	? $(\mathrm{ComRingSp}^{\geqslant 0})$

Example 16

Lawvere theory	Set-algebras	Top-algebras
$\mathrm{Fin}^{\mathrm{op}}$	Set	Top
$\mathrm{Burn} = \mathrm{Span}(\mathrm{Fin})$	Ab	$\mathrm{Sp}^{\geqslant 0}$
$\mathrm{Poly} = \mathrm{Bispan}(\mathrm{Fin})$	ComRing	? $(\mathrm{ComRingSp}^{\geqslant 0})$

Theorem 17 (B.)

- *A PROP is a cyclic Fin^{iso}-module.*

Lawvere theories

Example 16

Lawvere theory	Set-algebras	Top-algebras
$\mathrm{Fin}^{\mathrm{op}}$	Set	Top
$\mathrm{Burn} = \mathrm{Span}(\mathrm{Fin})$	Ab	$\mathrm{Sp}^{\geqslant 0}$
$\mathrm{Poly} = \mathrm{Bispan}(\mathrm{Fin})$	ComRing	? $(\mathrm{ComRingSp}^{\geqslant 0})$

Theorem 17 (B.)

- *A PROP is a cyclic $\mathrm{Fin}^{\mathrm{iso}}$-module.*
- *A Lawvere theory is a cyclic $\mathrm{Fin}^{\mathrm{op}}$-module.*

Lawvere theories

Example 16

Lawvere theory	Set-algebras	Top-algebras
Fin^{op}	Set	Top
$\mathsf{Burn} = \mathsf{Span}(\mathsf{Fin})$	Ab	$\mathsf{Sp}^{\geqslant 0}$
$\mathsf{Poly} = \mathsf{Bispan}(\mathsf{Fin})$	ComRing	? $(\mathsf{ComRingSp}^{\geqslant 0})$

Theorem 17 (B.)

- *A PROP is a cyclic Fin^{iso}-module.*
- *A Lawvere theory is a cyclic Fin^{op}-module.*
- *If $\mathcal{P}, \mathcal{P}'$ are PROPs/Lawvere theories, so is $\mathcal{P} \otimes \mathcal{P}'$.*

Lawvere theories

Example 16

Lawvere theory	Set-algebras	Top-algebras
Fin^{op}	Set	Top
$\mathsf{Burn} = \mathsf{Span}(\mathsf{Fin})$	Ab	$\mathsf{Sp}^{\geqslant 0}$
$\mathsf{Poly} = \mathsf{Bispan}(\mathsf{Fin})$	ComRing	? $(\mathsf{ComRingSp}^{\geqslant 0})$

Theorem 17 (B.)

- *A PROP is a cyclic* Fin^{iso}*-module.*
- *A Lawvere theory is a cyclic* Fin^{op}*-module.*
- *If* $\mathcal{P}, \mathcal{P}'$ *are PROPs/Lawvere theories, so is* $\mathcal{P} \otimes \mathcal{P}'$*.*
- *If* $\mathcal{P}^{\oplus}$ *is a PROP, the associated Lawvere theory is* $\mathcal{P}^{\oplus} \otimes \mathsf{Fin}^{op}$*:*

$$Alg_{\mathcal{P}}(Top^{\times}) \simeq Alg_{\mathcal{P} \otimes Fin^{op}}(Top^{\times}).$$

Application: equivariant homotopy theory

Definition 18 (B.)

An *equivariant Lawvere theory* is a cyclic $\mathrm{Fin}_G^{\mathrm{op}}$-module $\mathcal{L}^\times$.

$$\mathrm{Alg}_{\mathcal{L}} = \mathrm{Hom}(\mathcal{L}^\times, \mathrm{Top}^\times).$$

Application: equivariant homotopy theory

Definition 18 (B.)

An *equivariant Lawvere theory* is a cyclic $\mathrm{Fin}_G^{\mathrm{op}}$-module $\mathcal{L}^\times$.

$$\mathrm{Alg}_{\mathcal{L}} = \mathrm{Hom}(\mathcal{L}^\times, \mathrm{Top}^\times).$$

Theorem 19 (Elmendorf)

Fin_G^{op} is the equivariant Lawvere theory for Top_G.

Application: equivariant homotopy theory

Definition 18 (B.)

An *equivariant Lawvere theory* is a cyclic $\mathrm{Fin}_G^{\mathrm{op}}$-module $\mathcal{L}^\times$.

$$\mathrm{Alg}_{\mathcal{L}} = \mathrm{Hom}(\mathcal{L}^\times, \mathrm{Top}^\times).$$

Theorem 19 (Elmendorf)

Fin_G^{op} is the equivariant Lawvere theory for Top_G.

Theorem 20 (Guillou-May)

$Burn_G = Span(Fin_G)$ is the equivariant Lawvere theory for $Sp_G^{\geq 0}$.

Application: equivariant homotopy theory

Definition 18 (B.)

An *equivariant Lawvere theory* is a cyclic $\mathrm{Fin}_G^{\mathrm{op}}$-module $\mathcal{L}^\times$.

$$\mathrm{Alg}_{\mathcal{L}} = \mathrm{Hom}(\mathcal{L}^\times, \mathrm{Top}^\times).$$

Theorem 19 (Elmendorf)

Fin_G^{op} is the equivariant Lawvere theory for Top_G.

Theorem 20 (Guillou-May)

$Burn_G = Span(Fin_G)$ is the equivariant Lawvere theory for $Sp_G^{\geqslant 0}$.

Conjecture

$\mathrm{Poly}_G = \mathrm{Bispan}(\mathrm{Fin}_G)$ is the equivariant Lawvere theory for $\mathrm{CRingSp}_G^{\geqslant 0}$.

Application: operads

Operad $\mathcal{O}$:

- given a finite set X, set $\mathcal{O}(X)$ of ways to multiply objects of X

Application: operads

Operad $\mathcal{O}$:

- given a finite set X, set $\mathcal{O}(X)$ of ways to multiply objects of X
- composition maps

Application: operads

Operad $\mathcal{O}$:

- given a finite set X, set $\mathcal{O}(X)$ of ways to multiply objects of X
- composition maps
- associative

Operad $\mathcal{O}$:

- given a finite set X, set $\mathcal{O}(X)$ of ways to multiply objects of X
- composition maps
- associative

Example 21

Commutative operad $\mathrm{Comm}(X) = *$.

Application: operads

Symmetric monoidal category $\mathrm{Env}(\mathcal{O})^{\amalg}$:

- Objects are finite sets.

Application: operads

Symmetric monoidal category $\mathrm{Env}(\mathcal{O})^{\amalg}$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn X into Y using operations in $\mathcal{O}$.

Application: operads

Symmetric monoidal category $\mathrm{Env}(\mathcal{O})^{\amalg}$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn X into Y using operations in $\mathcal{O}$.
- Symmetric monoidal operation is $\amalg$.

Application: operads

Symmetric monoidal category $\mathrm{Env}(\mathcal{O})^{\amalg}$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn X into Y using operations in $\mathcal{O}$.
- Symmetric monoidal operation is $\amalg$.

$\mathrm{Env}(\mathcal{O})^{\amalg}$ is a PROP; algebras are $\mathcal{O}$-algebras

$$\mathrm{Hom}(\mathrm{Env}(\mathcal{O})^{\amalg}, \mathcal{C}^{\otimes}) \cong \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\otimes}).$$

Symmetric monoidal category $\mathrm{Env}(\mathcal{O})^{\amalg}$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn X into Y using operations in $\mathcal{O}$.
- Symmetric monoidal operation is $\amalg$.

$\mathrm{Env}(\mathcal{O})^{\amalg}$ is a PROP; algebras are $\mathcal{O}$-algebras

$$\mathrm{Hom}(\mathrm{Env}(\mathcal{O})^{\amalg}, \mathcal{C}^{\otimes}) \cong \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\otimes}).$$

Example 22

$\mathrm{Env}(\mathrm{Comm})^{\amalg} = \mathrm{Fin}^{\amalg}$.

Applications: operads

$$\begin{pmatrix}\text{Operads}\\ \mathcal{O}\end{pmatrix} \to \begin{pmatrix}\text{PROPs}\\ \text{Env}(\mathcal{O})\end{pmatrix} \to \begin{pmatrix}\text{Lawvere Theories}\\ \text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}\end{pmatrix}$$

Applications: operads

$$\begin{pmatrix}\text{Operads}\\ \mathcal{O}\end{pmatrix} \to \begin{pmatrix}\text{PROPs}\\ \text{Env}(\mathcal{O})\end{pmatrix} \to \begin{pmatrix}\text{Lawvere Theories}\\ \text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}\end{pmatrix}$$

Theorem 23

Given an operad $\mathcal{O}$, $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$ is:

- *the Lawvere theory associated to $\mathcal{O}$;*

Applications: operads

$$\begin{pmatrix}\text{Operads}\\ \mathcal{O}\end{pmatrix} \to \begin{pmatrix}\text{PROPs}\\ \text{Env}(\mathcal{O})\end{pmatrix} \to \begin{pmatrix}\text{Lawvere Theories}\\ \text{Env}(\mathcal{O})\otimes \text{Fin}^{\text{op}}\end{pmatrix}$$

Theorem 23

Given an operad $\mathcal{O}$, $Env(\mathcal{O}) \otimes Fin^{op}$ is:

- *the Lawvere theory associated to $\mathcal{O}$;*
- *the PROP for $\mathcal{O} - Comm-$bialgebras;*

Applications: operads

$$\begin{pmatrix} \text{Operads} \\ \mathcal{O} \end{pmatrix} \to \begin{pmatrix} \text{PROPs} \\ \text{Env}(\mathcal{O}) \end{pmatrix} \to \begin{pmatrix} \text{Lawvere Theories} \\ \text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \end{pmatrix}$$

Theorem 23

Given an operad $\mathcal{O}$, $Env(\mathcal{O}) \otimes Fin^{op}$ is:

- *the Lawvere theory associated to $\mathcal{O}$;*
- *the PROP for $\mathcal{O} - Comm-$bialgebras;*
- *an explicit span construction.*

Applications: operads

$$\begin{pmatrix} \text{Operads} \\ \mathcal{O} \end{pmatrix} \to \begin{pmatrix} \text{PROPs} \\ \text{Env}(\mathcal{O}) \end{pmatrix} \to \begin{pmatrix} \text{Lawvere Theories} \\ \text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \end{pmatrix}$$

Theorem 23

Given an operad $\mathcal{O}$, $Env(\mathcal{O}) \otimes Fin^{op}$ is:

- *the Lawvere theory associated to $\mathcal{O}$;*
- *the PROP for $\mathcal{O} - Comm-$bialgebras;*
- *an explicit span construction.*

Conjecture

The PROP for $\mathcal{O} - \mathcal{O}'-$bialgebras can be computed via a span construction.

Future work

Push/pull square of rings:

$$\begin{array}{ccc} \mathsf{Fin}^{\mathsf{iso}} & \longrightarrow & \mathsf{Fin} \\ \downarrow & & \downarrow \\ \mathsf{Fin}^{\mathsf{op}} & \longrightarrow & \mathsf{Burn} \end{array}$$

Future work

Push/pull square of rings:

$$\begin{array}{ccc}
\mathrm{SymMon}_\infty & \xrightarrow{-\otimes \mathrm{Fin}} & \mathrm{CocartMon}_\infty \\
{\scriptstyle -\otimes \mathrm{Fin}^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle -\otimes \mathrm{Burn}} \\
\mathrm{CartMon}_\infty & \xrightarrow[-\otimes \mathrm{Burn}]{} & \mathrm{SemiaddCat}_\infty
\end{array}$$

Future work

Push/pull square of rings:

$$\begin{array}{ccc} \mathrm{Fin}^{\mathrm{iso}} & \longrightarrow & \mathrm{Fin} \\ \downarrow & & \downarrow \\ \mathrm{Fin}^{\mathrm{op}} & \longrightarrow & \mathrm{Burn} \end{array}$$

Descent: Can $\mathcal{C}^{\otimes} \in \mathrm{SymMon}_{\infty}$ be reconstructed from $\mathcal{C} \otimes \mathrm{Fin}$ and $\mathcal{C} \otimes \mathrm{Fin}^{\mathrm{op}}$?

Future work

Push/pull square of rings:

$$\begin{array}{ccc} \mathrm{Fin}^{\mathrm{iso}} & \longrightarrow & \mathrm{Fin} \\ \downarrow & & \downarrow \\ \mathrm{Fin}^{\mathrm{op}} & \longrightarrow & \mathrm{Burn} \end{array}$$

Descent: Can $\mathcal{C}^{\otimes} \in \mathrm{SymMon}_{\infty}$ be reconstructed from $\mathcal{C} \otimes \mathrm{Fin}$ and $\mathcal{C} \otimes \mathrm{Fin}^{\mathrm{op}}$?
Answer: Not always!

Future work

Push/pull square of rings:

$$\begin{array}{ccc} \mathsf{Fin}^{\mathsf{iso}} & \longrightarrow & \mathsf{Fin} \\ \downarrow & & \downarrow \\ \mathsf{Fin}^{\mathsf{op}} & \longrightarrow & \mathsf{Burn} \end{array}$$

Descent: Can $\mathcal{C}^{\otimes} \in \mathsf{SymMon}_{\infty}$ be reconstructed from $\mathcal{C} \otimes \mathsf{Fin}$ and $\mathcal{C} \otimes \mathsf{Fin}^{\mathsf{op}}$?

Answer: Not always!

Example 24

$\mathbb{S} \otimes \mathsf{Fin} \simeq \mathbb{S} \otimes \mathsf{Fin}^{\mathsf{op}} \simeq 0$, but $\mathbb{S} \not\simeq 0$.

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \cong \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \simeq \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin} \simeq \mathrm{Fin}$

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \simeq \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin} \simeq \mathrm{Fin}$
- $\mathcal{L}_{\mathcal{O}} \otimes \mathrm{Burn} \simeq \mathrm{Burn}$

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \simeq \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin} \simeq \mathrm{Fin}$
- $\mathcal{L}_{\mathcal{O}} \otimes \mathrm{Burn} \simeq \mathrm{Burn}$

Conjecture

There is an equivalence of (∞-)categories between unital (∞-)operads and cyclic $\mathrm{Fin}^{\mathrm{op}}$-modules with trivialization over Burn.

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \simeq \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin} \simeq \mathrm{Fin}$
- $\mathcal{L}_{\mathcal{O}} \otimes \mathrm{Burn} \simeq \mathrm{Burn}$

Conjecture

There is an equivalence of (∞-)categories between unital (∞-)operads and cyclic $\mathrm{Fin}^{\mathrm{op}}$-modules with trivialization over Burn.

Applications:

- earlier conjecture on operadic bialgebras

Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}$ and $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}}$?

- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin}^{\mathrm{op}} \cong \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Fin} \cong \mathrm{Fin}$
- $\mathcal{L}_{\mathcal{O}} \otimes \mathrm{Burn} \cong \mathrm{Burn}$

Conjecture

There is an equivalence of (∞-)categories between unital (∞-)operads and cyclic $\mathrm{Fin}^{\mathrm{op}}$-modules with trivialization over Burn.

Applications:

- earlier conjecture on operadic bialgebras
- equivariant ∞-operads