CHAPTER 14

The Van Kampen Theorem

14a. G-Coverings from the Universal Covering

In this section X will denote a connected, locally path-connected, and semilocally simply connected space, so X has a universal covering, denoted $u: \tilde{X} \to X$. All spaces will have base points, and all maps will be assumed to take base points to base points. The base point of X is denoted x, and the base point of $\tilde{X}$ over x is denoted $\tilde{x}$.

We have seen in §13d that for every G-covering $p: Y \to X$, with Y connected, and with base point y, there is a surjective homomorphism of $\pi_1(X,x)$ onto G. If H is the kernel of this homomorphism, so $G \cong \pi_1(X,x)/H$, Y is the quotient of $\tilde{X}$ by the action of H, with y the image of $\tilde{x}$. We want to extend this correspondence to G-coverings that may not be connected. In this case there will only be a homomorphism from $\pi_1(X,x)$ to G, which need not be surjective. Here we will set up this correspondence between G-coverings and homomorphisms directly and rather briefly, omitting some verifications. Other ways to carry this out, with a more general context for these constructions, together with more details about the verifications, are described §16d and §16e.

Suppose $\rho: \pi_1(X,x) \to G$ is a homomorphism from the fundamental group of X to any group G. We will construct from ρ a G-covering $p_\rho: Y_\rho \to X$, together with a base point y_ρ in Y_ρ over x. Give G the discrete topology, so the Cartesian product $\tilde{X} \times G$ is a product of copies of $\tilde{X}$, one for each element in G. The group $\pi_1(X,x)$ acts on the left on $\tilde{X} \times G$ by the rule

$$[\sigma] \cdot (z \times g) \;=\; [\sigma] \cdot z \times g \cdot \rho([\sigma]^{-1}) \;=\; [\sigma] \cdot z \times g \cdot \rho([\sigma])^{-1},$$

for $[\sigma] \in \pi_1(X,x)$, $z \in \tilde{X}$, $g \in G$. Here $[\sigma] \cdot z$ is the action of $\pi_1(X,x)$ on $\tilde{X}$ that was described in §13b, and $g \cdot \rho([\sigma]^{-1})$ is the product in the group G. Define Y_ρ to be the quotient of $\tilde{X} \times G$ by this action of $\pi_1(X,x)$:

$$Y_\rho = \tilde{X} \times G/\pi_1(X,x),$$

and let y_ρ be the image of the point $\tilde{x} \times e$ in Y_ρ. Let $\langle z \times g \rangle$ denote the image in Y_ρ of the point $z \times g$ in $\tilde{X} \times G$. Note that, by the above action of $\pi_1(X,x)$ on $\tilde{X} \times G$, we have, for z in $\tilde{X}$, g in G, and $[\sigma]$ in $\pi_1(X,x)$,

$$\langle [\sigma] \cdot z \times g \rangle = \langle z \times g \cdot \rho([\sigma]) \rangle .$$

Define $p_\rho: Y_\rho \to X$ by taking $\langle z \times g \rangle$ to $u(z)$.

The group G acts on Y_ρ by the formula $h \cdot \langle z \times g \rangle = \langle z \times h \cdot g \rangle$, for h and g in G and z in X. (Note that using the right side of G for the left action of $\pi_1(X,x)$ frees up the left side of G for a left action of G!) We claim that this is an even action, making $p_\rho: Y_\rho \to X$ a G-covering. To prove this, let N be any open set in X over which the universal covering $u: \tilde{X} \to X$ is trivial. By Lemma 11.18 there is an isomorphism of $u^{-1}(N)$ with the product covering $N \times \pi_1(X,x)$, on which $\pi_1(X,x)$ acts on the left on the second factor. This gives homeomorphisms

$$p_\rho^{-1}(N) \cong (N \times \pi_1(X,x)) \times G/\pi_1(X,x) \cong N \times G,$$

the latter homeomorphism by $\langle (u \times [\sigma]) \times g \rangle \mapsto u \times g \cdot \rho([\sigma])$. (The map back takes $u \times g$ to $\langle (u \times e) \times g \rangle$.) These homeomorphisms are compatible with the projections to N, and it follows that, over N, the action of G is even and the covering is a G-covering. Since X is covered by such open sets N, the same is true for the map p_ρ from Y_ρ to X.

Conversely, suppose $p: Y \to X$ is a G-covering, with a base point y over x. From this we construct a homomorphism ρ from $\pi_1(X,x)$ to G. For each $[\sigma]$ in $\pi_1(X,x)$ the element $\rho([\sigma])$ in G is determined by the formula

$$\rho([\sigma]) \cdot y = y * \sigma,$$

where $y * \sigma$ is the endpoint of the lift of the path σ that starts at y. We will need two facts about this operation:

(i) $(z * \sigma) * \tau = z * (\sigma \cdot \tau)$ for $z \in p^{-1}(x)$, σ a loop at x, and τ a path starting at x;

(ii) $g \cdot (z * \gamma) = (g \cdot z) * \gamma$ for $g \in G$, $z \in p^{-1}(x)$, and γ a path starting at x.

The first of these facts is immediate from the definition. The second follows from the fact that if $\tilde{\gamma}$ is lifting of γ starting at z, then the path $t \mapsto g \cdot \tilde{\gamma}(t)$, $0 \leq t \leq 1$, is a lifting of γ that starts at $g \cdot z$. The endpoint of this path, which is $(g \cdot z) * \gamma$ by definition, is $g \cdot \tilde{\gamma}(1)$, and since $\tilde{\gamma}(1) = z * \gamma$, (ii) follows.

We claim now that the ρ defined above is a homomorphism. This is a calculation, using (ii) and (i):

$$\begin{aligned}(\rho([\sigma]) \cdot \rho([\tau])) \cdot y &= \rho([\sigma]) \cdot (\rho([\tau]) \cdot y) = \rho([\sigma]) \cdot (y * \tau) \\ &= (\rho([\sigma]) \cdot y) * \tau = (y * \sigma) * \tau \\ &= y * (\sigma \cdot \tau) = \rho([\sigma] \cdot [\tau]) \cdot y .\end{aligned}$$

Proposition 14.1. *The above constructions determine a one-to-one correspondence between the set of homomorphisms from* $\pi_1(X,x)$ *to the group* G *and the set of* G*-coverings with base point, up to isomorphism*:

$$\mathrm{Hom}(\pi_1(X,x), G) \leftrightarrow \{G\text{-coverings}\}/\text{isomorphism}.$$

Proof. Given a G-covering $p\colon Y \to X$ with base points, from which we constructed a homomorphism ρ, we must now show that the given covering is isomorphic to the covering $p_\rho\colon Y_\rho \to X$ constructed from ρ. To map Y_ρ to Y, we need to map $\tilde{X} \times G$ to Y, and show that orbits by $\pi_1(X,x)$ have the same image. For this we identify the universal covering $\tilde{X}$ as the space of homotopy classes of paths in X starting at x. Define a map

$$\tilde{X} \times G \to Y, \qquad [\gamma] \times g \mapsto g \cdot (y * \gamma) \;=\; (g \cdot y) * \gamma .$$

This is easily checked to be continuous. We must check that an equivalent point $([\sigma] \cdot [\gamma]) \times (g \cdot \rho([\sigma])^{-1})$ maps to the same point. By (i) and (ii), this point maps to

$$\begin{aligned}(g \cdot \rho([\sigma])^{-1}) \cdot (y * (\sigma \cdot \gamma)) &= (g \cdot \rho([\sigma])^{-1}) \cdot ((y * \sigma) * \gamma) \\ &= ((g \cdot \rho([\sigma])^{-1}) \cdot (y * \sigma)) * \gamma \\ &= (g \cdot ((y * \sigma) * \sigma^{-1})) * [\gamma] \\ &= (g \cdot (y * (\sigma \cdot \sigma^{-1}))) * [\gamma] = (g \cdot y) * [\gamma] ,\end{aligned}$$

as required. Since the map takes the same values on equivalent points, it gives a mapping from the quotient Y_ρ to Y, which is a mapping of covering spaces of X. This is easily checked to be a mapping of G-coverings, from which it follows that it must be an isomorphism.

Conversely, starting with a homomorphism ρ, we constructed a G-

covering $Y_\rho \to X$, from which we constructed another homomorphism, say $\tilde{\rho}$. We must verify that $\tilde{\rho} = \rho$. Now for $[\sigma]$ in $\pi_1(X,x)$,

$$\begin{aligned}\tilde{\rho}([\sigma]) \cdot \langle \tilde{x} \times e \rangle &= \langle \tilde{x} \times e \rangle * \sigma = \langle \tilde{x} * \sigma \times e \rangle \\ &= \langle [\sigma] \cdot \tilde{x} \times e \rangle = \langle \tilde{x} \times e \cdot \rho([\sigma]) \rangle = \langle \tilde{x} \times \rho([\sigma]) \rangle \\ &= \rho([\sigma]) \cdot \langle \tilde{x} \times e \rangle .\end{aligned}$$

This shows that $\tilde{\rho}([\sigma]) = \rho([\sigma])$, which concludes the proof. □

Exercise 14.2. If p: $Y \to X$ is the G-covering corresponding to a homomorphism ρ: $\pi_1(X,x) \to G$, and X' is a subspace of X that also has a universal covering, with x in X', show that the restriction $p^{-1}(X') \to X'$ of this covering to X' is the G-covering corresponding to the composite homomorphism $\rho \circ i_*$, where i_*: $\pi_1(X',x) \to \pi_1(X,x)$ is induced by the inclusion i of X' in X.

Exercise 14.3. Show that, if base points are ignored, two G-coverings Y_ρ and $Y_{\rho'}$ are isomorphic G-coverings if and only if the homomorphisms ρ and ρ' are *conjugate*, i.e., there is some g in G such that

$$\rho'([\sigma]) = g \cdot \rho([\sigma]) \cdot g^{-1} \qquad \text{for all} \quad [\sigma] \in \pi_1(X,x).$$

14b. Patching Coverings Together

Suppose X is a union of two open sets U and V. A covering of X restricts to coverings of U and V, which are isomorphic over $U \cap V$. Conversely, suppose we have coverings p_2: $Y_1 \to U$ and p_2: $Y_2 \to V$, and we have an isomorphism of coverings

$$\vartheta: p_1^{-1}(U \cap V) \to p_2^{-1}(U \cap V)$$

of $U \cap V$. Then one may patch (or "glue," or "clutch") these together to get a covering p: $Y \to X$, together with isomorphisms of coverings

$$\varphi_1: Y_1 \xrightarrow{\cong} p^{-1}(U), \qquad \varphi_2: Y_2 \xrightarrow{\cong} p^{-1}(V)$$

of U and of V, so that, over $U \cap V$, $\vartheta = \varphi_2^{-1} \circ \varphi_1$.

One can construct Y as the quotient space of the disjoint union $Y_1 \sqcup Y_2$, by the equivalence relation that identifies a point y_1 in $p_1^{-1}(U \cap V)$ with the point $\vartheta(y_1)$ in $p_2^{-1}(U \cap V)$. (See Appendix A3.) Since ϑ is compatible with maps to X, one gets a mapping p from Y to X. Since the map from Y_1 to Y is a homeomorphism onto its image $p^{-1}U$, which is open in Y, and similarly Y_2 maps homeomorphically onto $p^{-1}V$, one sees that the restriction of p to the inverse image of

U is isomorphic to $Y_1 \to U$, and the restriction over V is isomorphic to $Y_2 \to V$. From this it follows in particular that p is a covering map.

If each of $Y_1 \to U$ and $Y_2 \to V$ is a G-covering, for a fixed group G, and ϑ is an isomorphism of G-coverings, then $Y \to X$ gets a unique structure of a G-covering in such a way that the maps from Y_1 and Y_2 commute with the action of G.

Occasionally the following generalization is useful. Suppose we have a collection X_α of open sets, $\alpha \in \mathcal{A}$, whose union is X, and a collection $p_\alpha: Y_\alpha \to X_\alpha$ of covering maps. Suppose, for each α and β, we have an isomorphism

$$\vartheta_{\beta\alpha}: p_\alpha^{-1}(X_\alpha \cap X_\beta) \to p_\beta^{-1}(X_\alpha \cap X_\beta)$$

of coverings of $X_\alpha \cap X_\beta$. Assume these are compatible, i.e.,

(1) $\vartheta_{\alpha\alpha}$ is the identity on Y_α; and
(2) $\vartheta_{\gamma\alpha} = \vartheta_{\gamma\beta} \circ \vartheta_{\beta\alpha}$ on $p_\alpha^{-1}(X_\alpha \cap X_\beta \cap X_\gamma)$ for all α, β, $\gamma \in \mathcal{A}$.

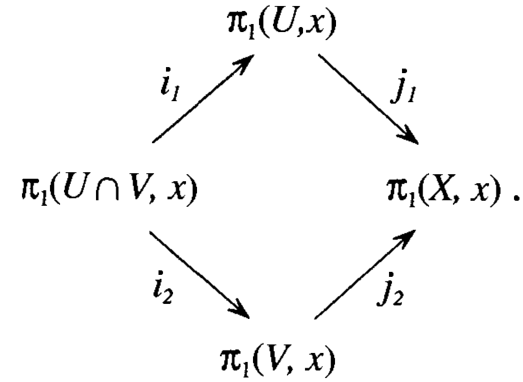
Then one can patch these coverings together to obtain a covering $p: Y \to X$. One has isomorphisms $\varphi_\alpha: Y_\alpha \to p^{-1}(X_\alpha)$ of coverings of X_α, such that $\vartheta_{\beta\alpha} = \varphi_\beta^{-1} \circ \varphi_\alpha$ on $p_\alpha^{-1}(X_\alpha \cap X_\beta)$. In addition, the space Y is the union of the open sets $\varphi_\alpha(Y_\alpha)$.

One constructs Y as the quotient space $\sqcup_{\alpha \in \mathcal{A}} Y_\alpha / R$ of the disjoint union of the Y_α by the equivalence relation determined by the $\vartheta_{\beta\alpha}$'s. The assertions about Y and the φ_α are general facts about patching spaces together, as proved in Appendix A3. The map p is determined by the equations $p \circ \varphi_\alpha = p_\alpha$ on Y_α. Since φ_α is a homeomorphism of Y_α onto $p^{-1}(X_\alpha)$, it follows that p is a covering map.

If each $p_\alpha: Y_\alpha \to X_\alpha$ is a G-covering, with fixed G, and each $\theta_{\beta\alpha}$ is an isomorphism of G-coverings, then there is a unique action of G on Y so that each φ_α commutes with the action of G, i.e., $\varphi_\alpha(g \cdot y_\alpha) = g \cdot \varphi_\alpha(y_\alpha)$ for g in G and y_α in Y_α. This gives the patched covering $p: Y \to X$ the structure of a G-covering, so that each φ_α is an isomorphism of G-coverings.

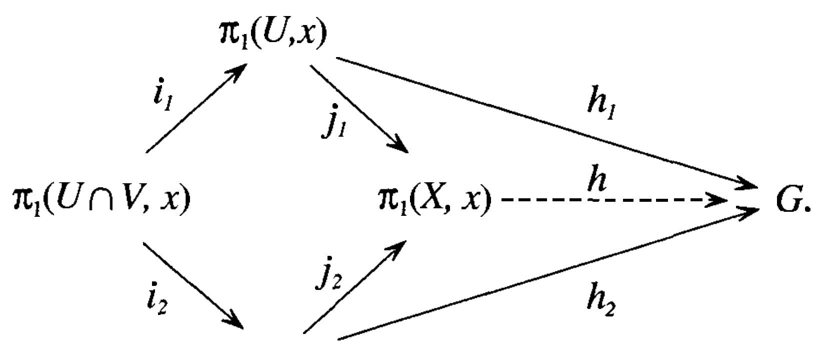
14c. The Van Kampen Theorem

The Van Kampen theorem describes the fundamental group of a union of two spaces in terms of the fundamental group of each and of their intersection, under suitable hypotheses. Let X be a space that is a union of two open subspaces U and V. Assume that each of the spaces U, V and their intersection $U \cap V$ is path-connected, and let x be a point in the intersection. Assume also that all these spaces X, U, V, and $U \cap V$ have universal covering spaces; this is the case, for example, if X is locally simply connected. We have a commutative diagram of homomorphisms of fundamental groups:



The maps are induced by the inclusions of subspaces, and commutativity means that $j_1 \circ i_1 = j_2 \circ i_2$.

We will describe how $\pi_1(X,x)$ is determined by the other groups (and the above maps between them). The description will not be direct, but will be by a *universal property*. Note that any homomorphism h from $\pi_1(X,x)$ to a group G determines a pair of homomorphisms $h_1 = h \circ j_1$ from $\pi_1(U,x)$ to G and $h_2 = h \circ j_2$ from $\pi_1(V,x)$ to G; the two homomorphisms $h_1 \circ i_1$ and $h_2 \circ i_2$ from $\pi_1(U \cap V,x)$ to G determined by these are the same. The Van Kampen theorem says that $\pi_1(X,x)$ is the "universal" group with this property.



Theorem 14.4 (Seifert–van Kampen). *For any homomorphisms*

$$h_1\colon \pi_1(U,x) \to G \quad \textit{and} \quad h_2\colon \pi_1(V,x) \to G,$$

such that $h_1 \circ i_1 = h_2 \circ i_2$, *there is a unique homomorphism*

$$h\colon \pi_1(X,x) \to G,$$

such that $h \circ j_1 = h_1$ *and* $h \circ j_2 = h_2$.

Exercise 14.5. Show that $\pi_1(X,x)$, together with the homomorphisms

j_1 and j_2, is determined up to canonical isomorphism by the universal property.

Exercise 14.6. Use the universal property to show that $\pi_1(X,x)$ is generated by the images of $\pi_1(U,x)$ and $\pi_1(V,x)$. Can you prove this assertion directly?

A version of the Van Kampen theorem was found first by Seifert, and the theorem is also known as the Seifert–Van Kampen theorem. The version given here, via universal properties, was given by Fox, see Crowell and Fox (1963). The usual proof of the Van Kampen theorem (without the hypotheses that the spaces all have universal coverings) is rather technical, and for it we refer to Crowell and Fox (1963) or Massey (1991). Here we will give a quick proof, due to Grothendieck (see Godbillon (1971)), using the correspondence between homomorphisms from fundamental groups to a group G and G-coverings. The assumptions assure that each of the spaces X, U, V, and $U \cap V$ has a universal covering space, and that homomorphisms from their fundamental groups to a group G correspond to G-coverings.

In particular, the homomorphisms h_1 and h_2 determine G-coverings $Y_1 \to U$ and $Y_2 \to V$, together with base points y_1 and y_2 over x. The fact that $h_1 \circ i_1$ is equal to $h_2 \circ i_2$ means that the restrictions of these coverings to $U \cap V$ are isomorphic G-coverings, and since $U \cap V$ is connected, there is a unique isomorphism between these G-coverings that maps the base point y_1 to the base point y_2. (The uniqueness is a special case of Exercise 11.24.) By the construction of the preceding section, these two coverings patch together, using this isomorphism over the intersection. This gives a G-covering $Y \to X$ that restricts to the two given G-coverings (and has the same base point). This G-covering corresponds to a homomorphism h from $\pi_1(X,x)$ to G, and the fact that the restricted coverings agree means precisely that $h \circ j_1 = h_1$ and $h \circ j_2 = h_2$. □

Corollary 14.7. *If U and V are simply connected, then X is simply connected.*

Note the important hypothesis in all these theorems, that all spaces, including the intersection $U \cap V$, are connected. It does not apply to the annulus, written as a union of two sets homeomorphic to disks!

Exercise 14.8. If V is simply connected, show that $j_1\colon \pi_1(U,x) \to \pi_1(X,x)$

is surjective, with kernel the smallest normal subgroup of $\pi_1(X,x)$ that contains the image of i_1: $\pi_1(U \cap V, x) \to \pi_1(U,x)$.

Corollary 14.9. *If $U \cap V$ is simply connected, then, for any G,*

$$\mathrm{Hom}(\pi_1(X,x),G) = \mathrm{Hom}(\pi_1(U,x),G) \times \mathrm{Hom}(\pi_1(V,x),G).$$

This means that $\pi_1(X,x)$ is the *free product* of $\pi_1(U,x)$ and $\pi_1(V,x)$.

Exercise 14.10. If $U \cap V$ is simply connected, show that the inclusion mappings j_1 and j_2 are one-to-one.

The following is a useful generalization of Van Kampen's theorem, which can be used to compute the fundamental group of an increasing union of spaces, each of whose fundamental groups in known. The proof is identical to that of the preceding theorem, using the general patching construction of the preceding section.

Suppose a space X is a union of a family of open subspaces X_α, $\alpha \in \mathcal{A}$, with the property that the intersection of any two of these subspaces is in the family. Assume that X and each X_α is path-connected and has a universal covering, and that the intersection of all the X_α contains a point x. When X_β is contained in X_α let $i_{\alpha\beta}$ be the map from $\pi_1(X_\beta,x)$ to $\pi_1(X_\alpha,x)$ determined by the inclusion, and let j_α be the map from $\pi_1(X_\alpha,x)$ to $\pi_1(X,x)$ determined by inclusion.

Theorem 14.11. *With these hypotheses, $\pi_1(X,x)$ is the* direct limit *of the groups $\pi_1(X_\alpha,x)$. That is, for any group G, and any collection of homomorphisms h_α from $\pi_1(X_\alpha,x)$ to G such that $h_\beta = h_\alpha \circ i_{\alpha\beta}$ whenever $X_\beta \subset X_\alpha$, there is a unique homomorphism h from $\pi_1(X,x)$ to G such that $h_\alpha = h \circ j_\alpha$ for all α.* □

The preceding theorem is recovered by taking the family to consist of U, V, and $U \cap V$.

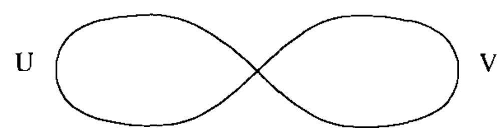
Although this version of Van Kampen's theorem is stated with each subspace X_α open in X, it can often be applied to subspaces that are not open. For example, if each X_α is contained in an open set U_α, of which it is a deformation retract, with $U_\beta \subset U_\alpha$ whenever $X_\beta \subset X_\alpha$, and the hypotheses of Theorem 14.11 apply to these U_α, then $\pi_1(X,x)$ is the direct limit of the groups $\pi_1(X_\alpha,x)$. This follows from the fact that each $\pi_1(X_\alpha,x) \to \pi_1(U_\alpha,x)$ is an isomorphism. Without some such hypotheses, however, the theorem is false. For example, if A and B are copies of a cone over a clamshell (see Exercise 13.19), joined together at the one point where all the circles are tangent, then the spaces A and B are simply connected, and $A \cap B$ is a point, but $A \cup B$ is not simply connected. (In fact, $A \cup B$ is an example of a space that is not simply connected but which has no nontrivial connected coverings.)

Exercise 14.12. Show if a space X is a union of a family of open subspaces X_α such that the intersection of any two sets in the family is also in the family, then H_1X is the direct limit of the groups H_1X_α.

14d. Applications: Graphs and Free Groups

One simple application of the Van Kampen theorem is a result we looked at earlier: the n-sphere S^n is simply connected if $n \geq 2$. To see this now, write the sphere as a union of two hemispheres each homeomorphic to n-dimensional disks, with the intersection homeomorphic to S^{n-1}. It follows from Corollary 14.7 that the fundamental group of S^n is trivial. (The assumption $n \geq 2$ is used to confirm that S^{n-1} is connected.)

Consider next a figure 8:



This is the union X of two circles U and V meeting at a point x. Let γ_1 and γ_2 be loops, one around each circle. The fundamental group of each circle is infinite cyclic, generated by the classes of these loops. It follows that to give a homomorphism from $\pi_1(X,x)$ to a group G is the same as specifying arbitrary elements g_1 and g_2 in G: there is a unique homomorphism from $\pi_1(X,x)$ to G mapping $[\gamma_1]$ to g_1 and $[\gamma_2]$ to g_2. This means that $\pi_1(X,x)$ is the *free group* on the generators $[\gamma_1]$ and $[\gamma_2]$.

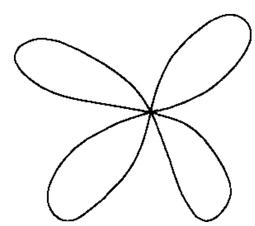
Exercise 14.13. Let $a = [\gamma_1]$ and $b = [\gamma_2]$. Show that every element in $\pi_1(X,x)$ has a unique expression in the form

$$a^{m_0} \cdot b^{m_1} \cdot a^{m_2} \cdot \ . \ . \ . \ \cdot b^{m_r},$$

where the m_i are integers, all nonzero except perhaps the first and last. The identity element is $e = a^0 b^0$.

The free group on two generators a and b can be constructed directly (and algebraically) as the set of all words of this form, with products defined by juxtaposition of words, canceling to get a legitimate word. It is straightforward (if a little awkward) to show by hand that this forms a group, and to see that it satisfies the above universal property. With the use of the Van Kampen theorem, we can avoid this, by constructing the free group as the fundamental group of this figure 8 space.

The free group F_n on n generators $a_1, \ldots, a_n$ is defined similarly: it is generated by these elements, and, for any group G and any elements $g_1, \ldots, g_n$ in G, there is a unique homomorphism from F_n to G taking a_i to g_i for $1 \leq i \leq n$. Again, it can be constructed purely algebraically using words in these letters, or as a fundamental group:



Exercise 14.14. Verify that the fundamental group of the space obtained by joining n circles at a point is the free group on n generators. Use this to show that the fundamental group of the complement of n points in the plane is free on n generators.

Exercise 14.15. Let X be a connected finite graph. (a) Show that, for any edge between two distinct vertices, X is homotopy equivalent to the graph obtained by removing the edge and identifying its two endpoints. (b) Show that X is homotopy equivalent to the graph obtained by joining n circles at a point, where, if the graph has v vertices and e edges, $n = e - v + 1$. (c) Show that n is the "connectivity" of the graph, i.e., the largest number of edges one can remove from the graph (leaving the vertices), so that what is left remains connected.

One can use this result to give a simple proof of a rather surprising fact about free groups.

Proposition 14.16. *If G is a free group on n generators, and H is a subgroup of G that has finite index d in G, then H is a free group, with $dn - d + 1$ generators.*

Proof. Take G to be the fundamental group of a connected graph X that has v vertices and e edges, with $n = e - v + 1$. For simplicity we assume each edge of X connects two distinct vertices. The subgroup H corresponds to a connected covering $p: Y \to X$ with d sheets, with the fundamental group of Y isomorphic to H. It is not hard to verify that Y is a connected graph. In fact, the d points over each of the vertices of X can be taken as vertices of Y, and (since a covering is trivial over an interval), the d components of the inverse image of each edge are edges of Y. Since Y is a graph, its fundamental group is free, with

$$de - dv + 1 \;=\; d(e - v + 1) - d + 1 \;=\; dn - d + 1$$

generators. □

If U is the plane domain that is the complement of two points, then U has the figure 8 as a deformation retract. So U has the same fundamental group. In particular, this is not an abelian group. For example, the path $\gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1}$ is not homotopic to a constant path (although all integrals of all closed 1-forms over this path are trivial).

Problem 14.17. Use the Van Kampen theorem to compute the fundamental groups of the complement of n points (or small disks) in: (1) a two-sphere; (2) a torus; and (3) a projective plane.

Problem 14.18. Describe the fundamental group of $\mathbb{R}^2 \setminus Z$, where Z is the set $\mathbb{Z}$ of all integers, or the set $\mathbb{Z}^2$ of lattice points, or any infinite discrete set.

Problem 14.19. Show that a free group on two generators contains a subgroup that is not finitely generated, in fact, a subgroup that is a free group on an infinite number of variables.

Problem 14.20. Use the Van Kampen theorem to compute the fundamental groups of: (1) the sphere with g handles; (2) the complement of n points in the sphere with g handles; and (3) the sphere with h crosscaps.