Theorem. Let $f : X \to Y$ be a pointed map with homotopy fiber $P_* f$. Then, there is a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(Y) \to \pi_n(P_* f) \to \pi_n(X) \to \pi_n(Y) \to \cdots$$

$$\cdots \to \pi_1(Y) \to \pi_0(P_* f) \to \pi_0(X) \to \pi_0(Y).$$

proof. Step 1. It is enough to prove that if $X \to Y$ is a fibration with fiber F, then there is a long exact sequence

$$\cdots \to \pi_{n+1}(Y) \to \pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \cdots.$$

Indeed, the sequence in the theorem comes from

$$\begin{tikzcd} P_* f \arrow[r] \arrow[d, equal] & Pf \arrow[r] \arrow[d, "\wr"] & Y \arrow[d, equal] \\ P_* f \arrow[r] & X \arrow[r] & Y. \end{tikzcd}$$

Step 2. Let $p : X \to Y$ be a fibration with fiber F. Recall that $P_* f$ (x, ϕ), $x \in X$, ϕ a map from $p(x)$ to y, the basepoint of Y. Hence, there is a natural map $F \xrightarrow{f} P_* F$ that is given by taking (x, c_y), $x \in F$, c_y the constant path at y.

$$\begin{array}{c} (x, \phi, 0) \longmapsto x \\ \begin{tikzcd} P_* f \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ P_* f \times I \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd} \\ (x, \phi, t) \longmapsto \phi(t) \end{array}$$

Define $h : P_* f \times I \to P_* f$ by $(x, \phi, t) \longmapsto \left(g(x, \phi, t), \phi|_{[t,1]} \right)$.
Then, $h_0 = \mathrm{id}_{P_* f}$, while $h_1(x, \phi) \in F$ for all x, ϕ.
So, h gives a homotopy from $\mathrm{id}_{P_* F}$ to $f \circ h_1$, and $h|_{F \times I}$ gives a homotopy from id_F to $h_1 \circ f$. Hence, f is a weak equivalence.

Step 3. If $f : W \to Y$, and if $p : X \to Y$ is a (pointed) fibration, then $W \times_Y X \to W$ is a (pointed) fibration.

Step 4. Even though we reduced to the case when f is a fibration, we'll still use the homotopy fiber as well. Consider the diagram

$$\begin{tikzcd}
P_* j \arrow[r] \arrow[d, "\simeq"'] & P_* i \arrow[r, "j"] \arrow[d, "\simeq"'] & P_* f \arrow[r, "i"] \arrow[d, "\simeq"'] & X \arrow[r, "f", "\text{a fibration}"'] \arrow[d, equal] & Y \arrow[d, equal] \\
\Omega X \arrow[r, dashed] & \Omega Y \arrow[r, dashed] & F \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Now, i is itself a fibration, since $P_* F \to X$ is the pullback to X of the standard fibration $P_* Y \to Y$. Hence, $P_* i$ is homotopy equivalent to the actual fiber of i, which is ΩY.

Similarly, $P_* j$ is homotopy equivalent to the actual fiber of the fibration $P_* i \to P_* f$. Now, $P_* i$ consists of triples

(x, ϕ, γ),

where $x \in X$, $\phi(0) = f(x)$, $\phi(1) = y$, γ is a path in X with $\gamma(0) = x$, $\gamma(1) =$ basepoint of X. So, again, the actual fiber is ΩX.

Step 5. One can show that $\Omega X \dashrightarrow \Omega Y$ is homotopy equivalent to Ωf, and one can show that in general the homotopy fiber of $\Omega X \xrightarrow{\Omega f} \Omega Y$ is the pointed loopspace of the homotopy fiber of $X \xrightarrow{f} Y$.

Step 6. Iterating the construction, one obtains

$$\cdots \to \Omega^2 Y \to \Omega F \to \Omega X \to \Omega Y \to F \to X \to Y.$$

Now, take π_0.

Cor. This gives the long exact sequence of a pair from Lecture 6.

Definition. A map $f: X \to Y$ is a weak homotopy equivalence if $\pi_n(f)$ is a bijection for all $n \geq 0$. I'll write these as $\simeq_w$.

Cor. If $F \to X \to Y$ is a fibration with X contractible, then $\Omega Y \simeq_w F$.