The homotopy category, H-groups, and H-cogroups.

<u>Definition</u>. The homotopy category of based spaces $hTop_*$ has the same objects as Top_*, but

$$\mathrm{Hom}_{hTop_*}(X,Y) = [X,Y]_* .$$

<u>Remarks</u>. (1) There is an unbased version.

(2) When working "up to homotopy," one is working in $hTop_*$ or a similar category.

(3) The functors $[X,-]_*$, $[-,Y]_*$ factor through $Top_* \to hTop_*$, $Top_*^{op} \to hTop_*^{op}$.

(4) In $hTop_*$, $\mathbb{R}^n$ is <u>isomorphic</u> to a point. Naturally, any homotopy invariant cannot distinguish $\mathbb{R}^n$ from $*$.

(5) Consider the functors

$$S^1 \wedge : hTop_* \to hTop_*,$$
$$\Omega : hTop_* \to hTop_* .$$

These are adjoint:

$$\begin{array}{ccc} [S^1 \wedge X, Y]_* & \xrightarrow{\cong} & [X, \Omega Y]_* \\ \| & & \| \\ \mathrm{Hom}_{hTop_*}(S^1 \wedge X, Y) & \xrightarrow{\cong} & \mathrm{Hom}_{hTop_*}(X, \Omega Y) . \end{array}$$

$S^1 \wedge$ is the left adjoint, while Ω is the right adjoint.

__Definition.__ An __H-group__ is a based space (G, e) with a map $m : G \times G \longrightarrow G$ called multiplication and a map [redacted] $u : G \longrightarrow G$ called the inverse such that the diagrams

$$\begin{tikzcd} G \arrow[r, "e \times id_G"] \arrow[dr, "id_G"'] & G \times G \arrow[d, "m"'] & G \arrow[l, "id_G \times e"'] \arrow[dl, "id_G"] \\ & G & \end{tikzcd}$$

(unit)

$$\begin{tikzcd} G \times G \times G \arrow[r, "id_G \times m"] \arrow[d, "m \times id_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

(associativity)

$$\begin{tikzcd} G \arrow[r, "id_G \times u"] \arrow[dr, "e"'] & G \times G \arrow[d, "m"] & G \arrow[l, "u \times id_G"'] \arrow[dl, "e"] \\ & G & \end{tikzcd}$$

(inverse)

commute up to homotopy.

If, in addition,

$$\begin{tikzcd} G \times G \arrow[rr, "\psi_{G,G}"] \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

commutes up to homotopy, G is called a __homotopy commutative__ H-group.

__Remark.__ There are of course strictly commuting versions, called topological (abelian) groups.

Definition. An H-cogroup is a based space (C, f) with a comultiplication $c: C \to C \vee C$ and a coinverse $v: C \to C$ such that

$$\begin{tikzcd} C & C \vee C \arrow[l, "f \vee id_C"'] \arrow[r, "id_C \vee f"] & C \\ & C \arrow[ul, "id_C"] \arrow[u, "c"] \arrow[ur, "id_C"'] & \end{tikzcd}$$

(counit)

$$\begin{tikzcd} C \vee C \vee C & C \vee C \arrow[l, "id_C \vee c"'] \\ C \vee C \arrow[u, "c \vee id_C"] & C \arrow[l, "c"] \arrow[u, "c"'] \end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd} C & C \vee C \arrow[l, "id_C \vee v"'] \arrow[r, "v \vee id_C"] & C \\ & C \arrow[ul, "f"] \arrow[u, "m"] \arrow[ur, "f"'] & \end{tikzcd}$$

(coinverse)

commute up to homotopy.

And, (C, f) is homotopy cocommutative if

$$\begin{tikzcd} C \vee C & C \vee C \arrow[l, "\psi"'] \\ & C \arrow[ul, "c"] \arrow[u, "c"'] \end{tikzcd}$$

commutes up to homotopy.

Examples. (1) $GL_n(\mathbb{R})$ is an H-group. It is even strict.

(2) ΩX is an H-group for any based X. The multiplication $m: \Omega X \times \Omega X \to \Omega X$ is by composition of loops, and the inverse $\upsilon: \Omega X \to \Omega X$ sends a loop to the same loop but run backwards.

Concretely, view ΩX as maps from I' to X sending $\partial I'$ to $x \in X$. Let $a, b \in \Omega X$.

$$m(a,b): I' \longrightarrow X$$

$$m(a,b)(t) = \begin{cases} a(2t) & t \in [0,\tfrac{1}{2}], \\ b(2(t-\tfrac{1}{2})) & t \in [\tfrac{1}{2},1], \end{cases}$$

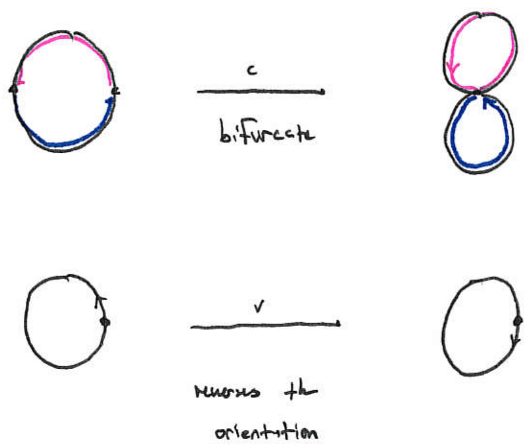
$$\upsilon(a)(t) = a(1-t).$$

(3) $\Sigma X = S' \wedge X$ is an H-cogroup for any based (X, x). Write $S' \wedge X = I' \times X / \partial I' \times X \cup I' \times \{x\}$. Write $[t,y]$ for the image of (t,y) in $S' \wedge X$.

$$c([t,y]) = \begin{cases} ([2t,y], x) & t \in [0,\tfrac{1}{2}] \\ (x, [2(t-\tfrac{1}{2}),y]) & t \in [\tfrac{1}{2},1], \end{cases}$$

$$r([t,y]) = [1-t, y].$$

When $X = S^0$, $\Sigma X = S^1$.



Recall that this is precisely what's used to prove that π_1 is a group.

<u>Proposition.</u> (1) If G is an H-group, then there is a natural factorization

$$\begin{tikzcd} hTop_+^{op} \arrow[rr, "{[-,G]_+}"] \arrow[dr] & & Sets_+ \\ & Groups \arrow[ur] & \end{tikzcd},$$

which factors further through Ab if G is homotopy commutative.
The unit element of $[X,G]_+$ is the map from X to $e \in G$.
Multiplication sends $f,g : X \to G$ to the composition $X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} G \times G \xrightarrow{m} G$,

$$[X,G]_+ \times [X,G]_+ \cong [X, G\times G]_+ \xrightarrow{m} [X,G]_+ .$$

The axioms to be a group follow from those for G to be an H-space. If $f : X \to Y$ is a map,

$$[Y,G]_+ \longrightarrow [X,G]_+$$

is a group homomorphism.

(2) If C is an H-cogroup, then is a natural factorization

$$\begin{tikzcd} hTop_+ \arrow[rr, "{[C,-]_+}"] \arrow[dr] & & Sets_+ \\ & Groups \arrow[ur] & \end{tikzcd},$$

with similar remarks as above.