Proof. If $S \cup \{v\}$ is linearly dependent, then there are vectors $u_1, u_2, \ldots, u_n$ in $S \cup \{v\}$ such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$ for some nonzero scalars $a_1, a_2, \ldots, a_n$. Because S is linearly independent, one of the u_i's, say u_1, equals v. Thus $a_1v + a_2u_2 + \cdots + a_nu_n = 0$, and so

$$v = a_1^{-1}(-a_2u_2 - \cdots - a_nu_n) = -(a_1^{-1}a_2)u_2 - \cdots - (a_1^{-1}a_n)u_n.$$

Since v is a linear combination of $u_2, \ldots, u_n$, which are in S, we have $v \in \operatorname{span}(S)$.

Conversely, let $v \in \operatorname{span}(S)$. Then there exist vectors $v_1, v_2, \ldots, v_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $v = b_1v_1 + b_2v_2 + \cdots + b_mv_m$. Hence

$$0 = b_1v_1 + b_2v_2 + \cdots + b_mv_m + (-1)v.$$

Since $v \neq v_i$ for $i = 1, 2, \ldots, m$, the coefficient of v in this linear combination is nonzero, and so the set $\{v_1, v_2, \ldots, v_m, v\}$ is linearly dependent. Therefore $S \cup \{v\}$ is linearly dependent by Theorem 1.6. ∎

Linearly independent generating sets are investigated in detail in Section 1.6.

EXERCISES

1. Label the following statements as true or false.

- **(a)** If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.
- **(b)** Any set containing the zero vector is linearly dependent.
- **(c)** The empty set is linearly dependent.
- **(d)** Subsets of linearly dependent sets are linearly dependent.
- **(e)** Subsets of linearly independent sets are linearly independent.
- **(f)** If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

2.[3] Determine whether the following sets are linearly dependent or linearly independent.

- **(a)** $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(R)$
- **(b)** $\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\}$ in $M_{2\times 2}(R)$
- **(c)** $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(R)$

[3]The computations in Exercise 2(g), (h), (i), and (j) are tedious unless technology is used.

- **(d)** $\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\}$ in $P_3(R)$
- **(e)** $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in R^3
- **(f)** $\{(1, -1, 2), (2, 0, 1), (-1, 2, -1)\}$ in R^3
- **(g)** $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}\right\}$ in $M_{2\times 2}(R)$
- **(h)** $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}\right\}$ in $M_{2\times 2}(R)$
- **(i)** $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + 3x^3 + 4x^2 - x + 1, x^3 - x + 2\}$ in $P_4(R)$
- **(j)** $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x\}$ in $P_4(R)$

3. In $M_{3\times 2}(F)$, prove that the set

$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$$

is linearly dependent.

4. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

5. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

6. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ is linearly independent.

7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

8. Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.

- **(a)** Prove that if $F = R$, then S is linearly independent.
- **(b)** Prove that if F has characteristic 2, then S is linearly dependent.

9.[†] Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

10. Give an example of three linearly dependent vectors in R^3 such that none of the three is a multiple of another.

11. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent subset of a vector space V over the field Z_2. How many vectors are there in span(S)? Justify your answer.

12. Prove Theorem 1.6 and its corollary.

13. Let V be a vector space over a field of characteristic not equal to two.
    (a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
    (b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

14. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ in S such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \le k < n)$.

16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

17. Let M be a square upper triangular matrix (as defined in Exercise 12 of Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

18. Let S be a set of nonzero polynomials in P(F) such that no two have the same degree. Prove that S is linearly independent.

19. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $\mathsf{M}_{n\times n}(F)$, then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

20. Let $f, g, \in \mathcal{F}(R, R)$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(R, R)$.

1.6 BASES AND DIMENSION

We saw in Section 1.5 that if S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent. A linearly independent generating set for W possesses a very useful property—every vector in W can be expressed in one and only one way as a linear combination of the vectors in the set. (This property is proved below in Theorem 1.8.) It is this property that makes linearly independent generating sets the building blocks of vector spaces.

Definition. *A* **basis** β *for a vector space* V *is a linearly independent subset of* V *that generates* V. *If* β *is a basis for* V, *we also say that the vectors of* β *form a basis for* V.

Example 1

Recalling that span($\varnothing$) $= \{0\}$ and $\varnothing$ is linearly independent, we see that $\varnothing$ is a basis for the zero vector space. ◆

Example 2

In F^n, let $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$; $\{e_1, e_2, \ldots, e_n\}$ is readily seen to be a basis for F^n and is called the **standard basis** for F^n. ◆

Example 3

In $\mathsf{M}_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Then $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $\mathsf{M}_{m\times n}(F)$. ◆

Example 4

In $\mathsf{P}_n(F)$ the set $\{1, x, x^2, \ldots, x^n\}$ is a basis. We call this basis the **standard basis** for $\mathsf{P}_n(F)$. ◆

Example 5

In P(F) the set $\{1, x, x^2, \ldots\}$ is a basis. ◆

Observe that Example 5 shows that a basis need not be finite. In fact, later in this section it is shown that no basis for P(F) can be finite. Hence not every vector space has a finite basis.

The next theorem, which is used frequently in Chapter 2, establishes the most significant property of a basis.

Theorem 1.8. *Let* V *be a vector space and* $\beta = \{u_1, u_2, \ldots, u_n\}$ *be a subset of* V. *Then* β *is a basis for* V *if and only if each* $v \in$ V *can be uniquely expressed as a linear combination of vectors of* β, *that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof. Let β be a basis for V. If $v \in$ V, then $v \in \text{span}(\beta)$ because $\text{span}(\beta) =$ V. Thus v is a linear combination of the vectors of β. Suppose that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \quad \text{and} \quad v = b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.
(f) The dimension of $\mathsf{P}_n(F)$ is n.
(g) The dimension of $\mathsf{M}_{m\times n}(F)$ is $m+n$.
(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.
(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.
(j) Every subspace of a finite-dimensional space is finite-dimensional.
(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

2. Determine which of the following sets are bases for R^3.

(a) $\{(1,0,-1),(2,5,1),(0,-4,3)\}$
(b) $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$
(c) $\{(1,2,-1),(1,0,2),(2,1,1)\}$
(d) $\{(-1,3,1),(2,-4,-3),(-3,8,2)\}$
(e) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$

3. Determine which of the following sets are bases for $\mathsf{P}_2(R)$.

(a) $\{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$
(b) $\{1+2x+x^2, 3+x^2, x+x^2\}$
(c) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$
(d) $\{-1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2\}$
(e) $\{1+2x-x^2, 4-2x+x^2, -1+18x-9x^2\}$

4. Do the polynomials $x^3-2x^2+1, 4x^2-x+3$, and $3x-2$ generate $\mathsf{P}_3(R)$? Justify your answer.

5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of R^3? Justify your answer.

6. Give three different bases for F^2 and for $\mathsf{M}_{2\times 2}(F)$.

7. The vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$, and $u_5=(-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for R^3.

8. Let W denote the subspace of R^5 consisting of all the vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1&=(2,-3,4,-5,2), & u_2&=(-6,9,-12,15,-6),\\ u_3&=(3,-2,7,-9,1), & u_4&=(2,-8,2,-2,6),\\ u_5&=(-1,1,2,1,-3), & u_6&=(0,-3,-18,9,12),\\ u_7&=(1,0,-2,3,-2), & u_8&=(2,-1,1,-9,7)\end{aligned}$$

generate W. Find a subset of the set $\{u_1,u_2,\ldots,u_8\}$ that is a basis for W.

9. The vectors $u_1=(1,1,1,1)$, $u_2=(0,1,1,1)$, $u_3=(0,0,1,1)$, and $u_4=(0,0,0,1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1,a_2,a_3,a_4) in F^4 as a linear combination of u_1, u_2, u_3, and u_4.

10. In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points.

(a) $(-2,-6)$, $(-1,5)$, $(1,3)$
(b) $(-4,24)$, $(1,9)$, $(3,3)$
(c) $(-2,3)$, $(-1,-6)$, $(1,0)$, $(3,-2)$
(d) $(-3,-30)$, $(-2,7)$, $(0,15)$, $(1,10)$

11. Let u and v be distinct vectors of a vector space V. Show that if $\{u,v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.

12. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u,v,w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

13. The set of solutions to the system of linear equations

$$\begin{aligned} x_1-2x_2+x_3&=0\\ 2x_1-3x_2+x_3&=0\end{aligned}$$

is a subspace of R^3. Find a basis for this subspace.

14. Find bases for the following subspaces of F^5:

$$\mathsf{W}_1=\{(a_1,a_2,a_3,a_4,a_5)\in\mathsf{F}^5\colon a_1-a_3-a_4=0\}$$

and

$$\mathsf{W}_2=\{(a_1,a_2,a_3,a_4,a_5)\in\mathsf{F}^5\colon a_2=a_3=a_4 \text{ and } a_1+a_5=0\}.$$

What are the dimensions of W_1 and W_2?

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(F)$ (see Example 4 of Section 1.3). Find a basis for W. What is the dimension of W?

16. The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$ (see Exercise 12 of Section 1.3). Find a basis for W. What is the dimension of W?

17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$ (see Exercise 28 of Section 1.3). Find a basis for W. What is the dimension of W?

18. Find a basis for the vector space in Example 5 of Section 1.2. Justify your answer.

19. Complete the proof of Theorem 1.8.

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.
    (a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)
    (b) Prove that S contains at least n vectors.

21. Prove that a vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

22. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

23. Let $v_1, v_2, \ldots, v_k, v$ be vectors in a vector space V, and define $W_1 = \mathrm{span}(\{v_1, v_2, \ldots, v_k\})$, and $W_2 = \mathrm{span}(\{v_1, v_2, \ldots, v_k, v\})$.
    (a) Find necessary and sufficient conditions on v such that $\dim(W_1) = \dim(W_2)$.
    (b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case that $\dim(W_1) \neq \dim(W_2)$.

24. Let $f(x)$ be a polynomial of degree n in $P_n(R)$. Prove that for any $g(x) \in P_n(R)$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x),$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

25. Let V, W, and Z be as in Exercise 21 of Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

26. For a fixed $a \in R$, determine the dimension of the subspace of $P_n(R)$ defined by $\{f \in P_n(R) : f(a) = 0\}$.

27. Let W_1 and W_2 be the subspaces of $P(F)$ defined in Exercise 25 in Section 1.3. Determine the dimensions of the subspaces $W_1 \cap P_n(F)$ and $W_2 \cap P_n(F)$.

28. Let V be a finite-dimensional vector space over C with dimension n. Prove that if V is now regarded as a vector space over R, then $\dim V = 2n$. (See Examples 11 and 12.)

Exercises 29–34 require knowledge of the sum and direct sum of subspaces, as defined in the exercises of Section 1.3.

29. (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. *Hint:* Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots v_m\}$ for W_1 and to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots w_p\}$ for W_2.
    (b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

30. Let

$$V = M_{2\times 2}(F), \quad W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\},$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.
    (a) Prove that $\dim(W_1 \cap W_2) \leq n$.
    (b) Prove that $\dim(W_1 + W_2) \leq m + n$.

32. (a) Find an example of subspaces W_1 and W_2 of R^3 with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 \cap W_2) = n$.
    (b) Find an example of subspaces W_1 and W_2 of R^3 with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 + W_2) = m + n$.