GEOMETRY/TOPOLOGY PRELIMINARY EXAMINATION, JUNE 2024

INSTRUCTIONS:

- There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas established in class. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **three** of the following five problems.

(1) Let M be a manifold.
  (a) Write the definition of a connection on TM.
  (b) Write the definition of the torsion of a connection ∇ on TM.
  (c) Give an example of a connection with non-zero torsion.

(2) Let M^{n+1} and N^n be closed orientable manifolds of dimensions $n+1$ and n, respectively. Suppose that $f : M^{n+1} \to N^n$ is a submersion. Prove that there is a **nowhere zero** vector field X on M such that, for every $p \in M$, $X(p)$ is tangent to the fiber $f^{-1}(f(p))$.

(3) Let (M, g) be a closed Riemannian manifold and let G be a group acting on M by isometries. Assume that $M^G := \{x \in M \mid (\forall g \in G) g \cdot x = x\}$ is a one-dimensional submanifold. Prove that M^G is a union of finitely many geodesics.

(4) Let M be a closed connected manifold and let $f : M \to M$ be an immersion. Prove that f is onto.

(5) Suppose that $M^n \subseteq \mathbb{R}^{n+2}$ is a submanifold of codimension 2 and that the normal bundle of M^n is trivial. Prove that M is orientable.

Part II

Do **three** of the following five problems.

(1) Recall that the fundamental group of the wedge of two circles is isomorphic to a free group on two generators a and b:

$$\pi_1(S^1 \vee S^1) \cong F\langle a, b\rangle \ .$$

(a) Draw the universal cover E of $S^1 \vee S^1$, and exhibit the map $E \to S^1 \vee S^1$. Compute the homology $\mathrm{H}_*(E, \mathbb{Z})$.

(b) Consider the cyclic subgroup $H := \langle ab\rangle \subset F\langle a, b\rangle$ generated by the element ab. Draw the covering space E_H associated to H, and exhibit the map $E_H \to S^1 \vee S^1$.

(2) Consider the connect sum

$$N_3 := \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \ ,$$

which is also known as von Dyck's surface. Compute:

(a) the fundamental group $\pi_1 N_3$, in terms of generators and relations;

(b) the homology $\mathrm{H}_*(N_3, \mathbb{Z})$;

(c) the mod-2 cohomology ring $\mathrm{H}^*(N_3, \mathbb{Z}/2)$.

(3) Consider the exact sequence of groups $\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$, which determines an exact sequence of chain complexes

$$0 \longrightarrow \mathrm{C}^*(X, \mathbb{Z}/2) \xrightarrow{\ 2\ } \mathrm{C}^*(X, \mathbb{Z}/4) \longrightarrow \mathrm{C}^*(X, \mathbb{Z}/2) \longrightarrow 0$$

for any space X. The Bockstein homomorphism

$$\beta : \mathrm{H}^*(X, \mathbb{Z}/2) \to \mathrm{H}^{*+1}(X, \mathbb{Z}/2)$$

is the connecting homomorphism in the associated long exact sequence of cohomology groups. Compute the map

$$\beta : \mathrm{H}^*(\mathbb{RP}^\infty, \mathbb{Z}/2) \to \mathrm{H}^{*+1}(\mathbb{RP}^\infty, \mathbb{Z}/2)$$

for all values of $*$. Prove your answer.

(4) For each $k \geq 0$, construct a closed 3-manifold M for which there is an isomorphism

$$\mathrm{H}_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \mathbb{Z}/2 \ .$$

(5) Let M and N be connected closed n-manifolds which are oriented, with fundamental classes $[M]$ and $[N]$. Recall that the degree of a map $f : M \to N$ is the integer $\deg(f)$ satisfying the equality

$$f_*[M] = \deg(f) \cdot [N] \ .$$

(a) Let $g : M \to N$ be a k-sheeted covering. Prove $\deg(g) = \pm k$.

(b) If f has degree ± 1, prove that the map $\pi_1(f) : \pi_1 M \to \pi_1 N$ is surjective.

Part III

Do **three** of the following five problems.

(1) Let M be compact n-manifold with boundary ∂M. Prove:
  (a) the natural map
$$\mathrm{H}_{n-1}(\partial M, \mathbb{Z}/2) \to \mathrm{H}_{n-1}(M, \mathbb{Z}/2)$$
  is zero;
  (b) there does not exist a retraction of M onto ∂M.

(2) Let M be an orientable 3-manifold with boundary ∂M a surface of genus g. Prove that the dimension of the $\mathbb{Q}$-vector space
$$\mathrm{Ker}\Big(\mathrm{H}_1(\partial M, \mathbb{Q}) \to \mathrm{H}_1(M, \mathbb{Q})\Big)$$
is equal to g. (This is known as "Half Lives, Half Dies.")

(3) State the Universal Coefficient Theorem for the homology of a space X, and prove that the splitting cannot be chosen so as to be functorial in X. That is, prove that there is map $f : X \to Y$ and an abelian group A such that the map
$$f_* : \mathrm{H}_*(X, A) \longrightarrow \mathrm{H}_*(Y, A)$$
does not preserve any splitting of the source and target given by the UCT.

(4) Let $H = \big\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0 \wedge x^2 + y^2 + z^2 = 1\big\}$ be the upper half unit sphere in $\mathbb{R}^3$ with its standard orientation. Denote the differential form
$$\omega := dy \wedge dz + dx \wedge dz + dx \wedge dy \ .$$
Compute
$$\int_H \omega \ .$$

(5) Prove the following form of the Poincaré Lemma: define an operator
$$K : \Omega^*(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{*-1}(\mathbb{R}^n \times \mathbb{R})$$
and show that K defines a homotopy equivalence of cochain complexes
$$\pi^* : \Omega^*(\mathbb{R}^n) \leftrightarrows \Omega^*(\mathbb{R}^n \times \mathbb{R}) : s^*$$
where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is projection and $s : \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^n \times \mathbb{R}$ is the zero section.