PRELIMINARY EXAM IN ANALYSIS
JUNE 2024

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Fix a measure space $(X, \mathcal{M}, \mu)$. Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.
  (a) Show that $\int f = \lim \int f_n$.
  (b) Show that $f_n \to f$ in L^1.
  (c) Provide a counterexample showing that $f_n \to f$ in measure does not always imply $f_n \to f$ in L^1.

(2) Let m be Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $m(E) > 0$. Show that for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

(3) Suppose $F, G \in NBV$ and $-\infty < a < b < \infty$. Show that

$$\int_{[a,b]} \frac{F(x) + F(x-)}{2}\, dG(x) + \int_{[a,b]} \frac{G(x) + G(x-)}{2}\, dF(x)$$
$$= F(b)G(b) - F(a-)G(a-).$$

(Here we are using the convention that $dF = d\mu_F$. Hint: it suffices to assume F and G are increasing; consider a product measure of the set $\{(x, y) \in \mathbb{R}^2 : a \leq x \leq y \leq b\}$.)

(4) Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every nonempty subinterval I of $[0, 1]$. Let $F(x) = m([0, x] \cap A)$ for $x \in [0, 1]$. Show that F is absolutely continuous and strictly increasing on $[0, 1]$, but $F' = 0$ on a set of positive measure.

(5) Fix a measure space $(X, \mathcal{M}, \mu)$. Let $f \in L^p \cap L^\infty$ for some $p < \infty$. Show that $f \in L^q$ for all $q > p$ and $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

Part II. Functional Analysis

Note: You may use any (consistent) normalization that you prefer for Fourier transforms and Fourier series.

(1) (a) Compute the Fourier transform of the function $f(x) = e^{-|x|}$ on $\mathbb{R}$.
 (b) For what s does f lie in the Sobolev space $H^s(\mathbb{R})$?

(2) For f a measurable function on $[0,1]$, M_f denotes the operator given by $M_f(g) = fg$.
 (a) Show that M_f is bounded on $L^2([0,1])$ iff $f \in L^\infty([0,1])$, and $\|M_f\| = \|f\|_\infty$.
 (b) Find (with proof) all $f \in L^\infty([0,1])$ such that M_f maps $L^2([0,1])$ *onto* $L^2([0,1])$.

(3) Let X be a Hilbert space and $K : X \to X$ a compact operator. Let $Y \subset X$ be a closed subspace with $Y \subset \mathrm{Ran}(K)$. Show that Y is finite-dimensional. (Hint: let $Z = K^{-1}(Y) \subset X$ and consider $K : Z \to Y$, which is a surjective map.)

(4) Let X be a Banach space and S a closed proper subspace; take $x_0 \notin S$.
 (a) Prove that $\mathrm{dist}(x_0, S) = \inf\{\|x_0 - y\| : y \in S\}$ is strictly positive.
 (b) Show that there exists $u \in X^*$ with $u = 0$ on S and $u(x_0) = 1$, and with $\|u\| = \mathrm{dist}(x_0, S)^{-1}$.

(5) Let $\alpha \in (0,1)$ and let $f \in \mathcal{C}^\alpha(S^1)$, which means f is a periodic function on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leq C_0 |x - y|^\alpha$$

for some C_0.
 (a) By considering the Fourier coefficients of $f(x + a) - f(x)$, show that for some constant C_1,

$$\left| e^{ina} \widehat{f}(n) - \widehat{f}(n) \right| \leq C_1 |a|^\alpha.$$

 Use this to show that for some C

$$\left| \widehat{f}(n) \right| \leq C|n|^{-\alpha},\ n \neq 0.$$

 (Hint: choose a a small multiple of $1/n$.)
 (b) Show that for all $k \in \mathbb{N}$ there exists $N(k)$ such that the $N(k)$-fold convolution

$$\underbrace{f * \cdots * f}_{N(k)}$$

 lies in $\mathcal{C}^k(S^1)$; provide a concrete value of $N(k)$ (it doesn't have to be optimal).

Part III. Complex Analysis

(1) Suppose that $f, g : \Omega \to \mathbb{C}$ are holomorphic on an open set Ω such that $\overline{\mathbb{D}} \subset \Omega$ for the unit disk $\mathbb{D}$. Suppose that $|f(e^{i\theta})| \leq |g(e^{i\theta})|$ for all $\theta \in [0, 2\pi]$.
   (a) Assume in addition that $\mathrm{ord}_z f \geq \mathrm{ord}_z g$ for all $z \in \mathbb{D}$. Show that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{D}$.
   (b) Show that without the additional assumption in part (a) the conclusion that $|f(z)| \leq |g(z)|$ for $z \in \mathbb{D}$ may fail.

(2) Let $a \in \mathbb{C}$. Show that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |e^{it} + a| \, dt = \begin{cases} \log |a|, & \text{if } |a| > 1, \\ 0, & \text{if } |a| < 1. \end{cases}$$

(3) Compute the integral

$$\int_0^\infty \frac{\cos x}{x^2 + 1} \, dx.$$

Justify any limits that you take.

(4) Let $\Omega_1 \subset \mathbb{C}$ denote the unit disk with the interval $[0, 1]$ removed, and let Ω_2 be the plane with the interval $[0, \infty)$ removed. I.e.

$$\Omega_1 = \{z : |z| < 1, z \neq 0, \arg(z) \neq 0\},$$

and

$$\Omega_2 = \{z : z \neq 0, \arg(z) \neq 0\}.$$

Find a biholomorphism $\Omega_1 \to \Omega_2$.

(5) (a) Let $f : \Omega \to \mathbb{C}$ be holomorphic, where Ω contains the closed unit disk $\overline{\mathbb{D}}$. Suppose that $|f(z)| < 1$ for all $z \in \partial\mathbb{D}$. Show that there is exactly one fixed point of f in $\mathbb{D}$, i.e. one number $z \in \mathbb{D}$ such that $f(z) = z$.
   (b) Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic, nonconstant, such that $f'(0) = 0$. Show that for all $\delta > 0$ there exist distinct $z_1, z_2 \in D_\delta(0)$, such that $f(z_1) = f(z_2)$.