ALGEBRA PRELIMINARY EXAM

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such. Solve **three** problems from each part below. Full credit requires proving that your answer is correct.

1. Part 1: Groups, rings and modules

(1) Let A_6 be the group of even permutations on 6 elements. Construct a p-Sylow subgroup of A_6 for every prime p for which such a subgroup exists. For every p, determine how many such subgroups are there.

(2) Let G be a group and H its normal subgroup. Denote by H^{ab} the abelianization (i.e. the maximal Abelian quotient) of H. Show that G/H acts on H^{ab} by conjugation, and in this manner get a homomorphism $G/H \to \mathrm{Aut}(H^{\mathrm{ab}})$.

(3) (a) Show that the group of invertible elements of the ring $\mathbb{Z}[\sqrt{3}]$ is infinite.
(b) Show that the ring $\mathbb{Z}[\sqrt{-3}]$ is not a principal ideal domain.

(4) Let R be a principal ideal domain. Prove that a submodule of a finitely generated free module is finitely generated and free.

(5) Let R be a ring. Let F be the functor from the category of R-modules to itself given by $M \mapsto M \oplus M$. Describe the group of invertible natural transformations from F to itself as best as you can.

2. Part 2: Linear algebra and Galois theory

(1) Find a complete set of representatives for the similarity classes of matrices A which has characteristic polynomial $(x-1)^2(x^2+1)$ over
(a) $\mathbb{C}$.
(b) $\mathbb{Q}$.

(2) Let $A_1, \cdots A_k$ be a set of pairwise commuting $n \times n$ matrices over $\mathbb{C}$.
(a) Show that $A_1, \cdots A_k$ have a common eigenvector.
(b) Show that $A_1, \cdots A_k$ can be simultaneously triangularized: there is an invertible matrix X such that $X^{-1}A_1X, \cdots X^{-1}A_kX$ are all upper triangular.

(3) Consider the automorphisms of the field of rational functions $\mathbb{C}(t)$ given by the formulas:

$$\sigma : f(t) \mapsto f(t^{-1})$$

$$\tau : f(t) \mapsto f(e^{2\pi i/3}t)$$

and let G be the group of automorphisms generated by σ, τ.
(a) Show that G has order 6.
(b) Compute the degree $[\mathbb{C}(t^3) : \mathbb{C}(t^3 + t^{-3})]$.
(c) Show that $\mathbb{C}(t^3 + t^{-3})$ is the subfield $\mathbb{C}(t)^G$ fixed by G.

(4) Let p be a prime and let $F/\mathbb{F}_p$ be an extension of degree n.

   (a) Show that $F = \mathbb{F}_p(\zeta)$ for some root of unity ζ. What is the order of the root of unity ζ?
   (b) Suppose $n = \ell$ is a prime number. Find the number of primitive elements of $F/\mathbb{F}_p$, that is the number of $\alpha \in F$ such that $F = \mathbb{F}_p(\alpha)$.

(5) Determine the Galois group of the polynomial $x^4 + 3x^2 + 3$ over
   (a) $\mathbb{Q}$.
   (b) $\mathbb{F}_5$.

3. Part 3. Commutative algebra

(1) Let A be a commutative ring and let M be a finitely generated A-module. Suppose that $I \subseteq A$ is an ideal and that $f\colon M \to M$ is an A-module morphism such that $f(M) \subseteq IM$. Show that f satisfies an equation of the form $f^n + a_1 f^{n-1} + \cdots a_n = 0$, where $a_1, \ldots, a_n \in I$.
(2) Suppose that $A \subseteq B$ is an integral extension of integral domains and that A is integrally closed in its fraction field. Suppose that $\mathfrak{q} \subseteq B$ is a prime and that $\mathfrak{p} = \mathfrak{q} \cap A$. Show that $\mathrm{Spec} B_{\mathfrak{q}} \to \mathrm{Spec} A_{\mathfrak{p}}$ is surjective.
(3) Let A be a noetherian commutative ring. Show that the polynomial ring $A[x]$ is noetherian.
(4) Let A be a commutative ring and let $f_1, \ldots, f_r$ be elements of A such that the collection of open sets $\{X_{f_i} \subseteq \mathrm{SpecA}\}_{1 \leq i \leq r}$ forms an open cover of A. Show that $A \to \prod_{i=1}^{r} A[1/f_i]$ is injective.
(5) Let $A \to B$ be a map of commutative rings. Suppose that I is an injective A-module. Show that the B-module $\mathrm{Hom}_A(B, I)$ is injective.