Math 320-2: Midterm 2
Northwestern University, Winter 2020

Name: ____________________

1. (10 points) Give an example of each of the following. You do not have to justify your answer.
(a) A non-constant function $f(x)$ such that $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$ for all $n \in \mathbb{N}$.
(b) A sequence of non-constant functions which converges in $C[0,1]$ with the sup metric.
(c) A non-empty metric space for which every subset is both closed and open.
(d) A dense subset of $[e, \pi]$ with respect to the standard metric.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is C^2 (i.e. continuously twice-differentiable). Show that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. You can take it for granted that for $n \geq 1$ the following relation between the Fourier coefficients of f and those of f' holds:

$$a_n(f') = nb_n(f) \quad \text{and} \quad b_n(f') = -na_n(f).$$

Hint: Relate the Fourier coefficients of f to those of f''. Here's another hint: M-test.

3. (10 points) Let $\mathbb{R}^+$ denote the set of positive real numbers and define a metric on $\mathbb{R}^+$ by

$$d(x, y) = \left| \ln \frac{y}{x} \right| .$$

Take it for granted that this does define a metric.

(a) Determine explicitly the open ball $B_1(1)$ with respect to this metric.

(b) Show that this metric space is complete. (Take for granted the continuity of any single-variable function you might need to use. The fact that $\mathbb{R}$ is complete with respect to the standard metric is important.) Hint: There is an alternate way of expressing the logarithm of a fraction.

4. (10 points) Suppose $B_r(p)$ and $B_s(q)$ are two open balls in a metric space X. Show that $B_r(p) \cap B_s(q)$ is open in X, by finding for each $x \in B_r(p) \cap B_s(q)$ a radius $t > 0$ such that

$$B_t(x) \subseteq B_r(p) \cap B_s(q).$$

(Don't forget to prove that your claimed radius actually works. A picture will give the right intuition, but is not itself enough justification.)

5. (10 points) Suppose X is a metric space and $A \subseteq X$. Suppose p is in the closure of A but not in A itself. Show that there exists a sequence of *distinct* points of A which converges to p. (The characterization of the closure of A as the set of points $q \in X$ such that every open ball around q contains an element of A may be useful.)