Math 320-2: Midterm 2
Northwestern University, Winter 2016

Name: ____

1. (10 points) Give an example of each of the following. You do not have to justify your answer.

(a) A metric on $\mathbb{R}$ relative to which the sequence $\left(\frac{1}{n}\right)$ does not converge to 0.

(b) A subset of $\mathbb{Q}$ which is closed and open in $\mathbb{Q}$ with respect to the Euclidean metric.

(c) A non-closed subset of $\mathbb{R}^2$ which does not equal its interior relative to the Euclidean metric.

(d) A metric space (X, d) which is not complete.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose that (X, d) is a metric space, $p \in X$, and r_1, r_2 are real numbers such that $r_2 > r_1 > 0$. Let U be the subset of X consisting of all points whose distance to p is strictly between r_1 and r_2:

$$U := \{x \in X \mid r_1 < d(x, p) < r_2\}$$

For $x \in U$, give an explicit radius r such that $B_r(x) \subseteq U$ and prove that your answer is correct. To be clear, an "explicit" radius can still depend on data given in the problem, such as p and the values of r_1 and r_2.

3. (10 points) Consider the metric space $C[-2,1]$ of continuous functions $f : [-2,1] \to \mathbb{R}$ equipped with the sup metric:

$$d(f,g) = \sup_{x \in [-2,1]} |f(x) - g(x)|.$$

Show that the sequence (f_n) in $C[-2,1]$ defined by

$$f_n(x) = x \sin\left(\frac{x}{n}\right).$$

is Cauchy with respect to the sup metric. Hint: $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$.

4. (10 points) Let (X, d) be a metric space. Show that a subset $A \subseteq X$ has empty boundary in X if and only if both A and its complement A^c are open in X.

5. (10 points) Consider $\mathbb{R}^2$ with respect to the Euclidean metric. Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be three points in $\mathbb{R}^2$. Show that the subset A of $\mathbb{R}^2$ obtained by removing these points:

$$A := \{q \in \mathbb{R}^2 \mid q \neq p_1,\ q \neq p_2,\ \text{and } q \neq p_3\},$$

otherwise known as the complement of $\{p_1, p_2, p_3\}$ in $\mathbb{R}^2$, is dense in $\mathbb{R}^2$.