Math 320-3: Midterm 2
Northwestern University, Spring 2020

Name: ______________________________

1. (10 points) Give an example of each of the following. You do not have to justify your answer.
(a) A point at which $f(x, y) = (e^{xy}, x \sin y)$ is locally invertible.
(b) A point at which $f(x, y) = xy$ achieves its maximum value subject to $\|(x, y)\| = 1$.
(c) Sets $A, B \subseteq \mathbb{R}^2$ which are not Jordan regions but for which $A \cup B$ is a Jordan region.
(d) A bounded function $f(x, y, z)$ on the unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^3$ which is not integrable.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 and that $E \subseteq \mathbb{R}^n$ is compact and convex. Show that f is uniformly continuous on E. (You cannot simply quote the fact that any continuous function on a compact domain is automatically uniformly continuous.) Hint: Mean Value Theorem.

3. (a) (5 points) Consider the following system of nonlinear equations:

$$\begin{aligned} x^2y + yz - z &= 1 \\ x^3y^3z^2 - 3xy &= -2. \end{aligned}$$

The point $(1,1,1)$ is one solution. Show that there are infinitely many others.

(b) (5 points) Now consider the system:

$$\begin{aligned} x^2y + yz - z &= 1 \\ x^3y^3z^2 - 3xy &= -2 \\ xy - xz &= 0. \end{aligned}$$

Determine, with justification, the number of solutions this system has near $(1,1,1)$.

4. (10 points) Suppose $E \subseteq \mathbb{R}^n$ is a Jordan region. Define $3E$ to be the set obtained by scaling all points in E by 3:

$$3E := \{3\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in E\}.$$

Show, using only an argument based on grids and outer sums, that $3E$ is a Jordan region and that $\mathrm{Vol}(3E) = 3^n \,\mathrm{Vol}(E)$.

5. (10 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable on two closed Jordan regions $A, B \subseteq \mathbb{R}^2$ which intersect at only one point. Take it for granted that f is then integrable on $A \cup B$, and show that

$$\int_{A\cup B} f(\mathbf{x})\, d\mathbf{x} = \int_A f(\mathbf{x})\, d\mathbf{x} + \int_B f(\mathbf{x})\, d\mathbf{x}.$$

You cannot simply quote the result in the book (Theorem 12.23) which says that this is true; the point here is to give a proof of precisely this fact in this special case, which is much simpler than the full version in the book. You can use the book's version as a guide to figure out what you need to do, but do not simply reproduce the book's full version.