Math 320-3: Midterm 2
Northwestern University, Spring 2016

Name: ______________________________

1. (10 points) Give an example of each of the following. You do not have to justify your answer.

(a) A bounded subset of $\mathbb{R}$ which is not a Jordan region.

(b) A bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ which is integrable on $B_1(0,0)$ but not on $B_2(0,0)$.

(c) An integrable function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ does not exist.

(d) A function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\int_D 2\,dV = \int_{f(D)} 4\,dV$ for any Jordan region $D \subseteq \mathbb{R}^n$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $A, B \subseteq \mathbb{R}^2$ are Jordan regions, which implies that $A \cup B$ is also Jordan measurable. If $A \cap B$ has Jordan measure zero, show that

$$\operatorname{Vol}(A \cup B) = \operatorname{Vol} A + \operatorname{Vol} B.$$

(This is an exercise in the book, but of course the point here is to prove this fact without simply quoting the result of that exercise.)

3. (10 points) Suppose $B \subseteq \mathbb{R}^2$ is a rectangle and that $f : B \to \mathbb{R}$ is uniformly continuous, which, to recall, means that for any $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\text{if } \mathbf{x}, \mathbf{y} \in B \text{ and } \|\mathbf{x} - \mathbf{y}\| < \delta, \text{ then } |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon.$$

Show that f is integrable over B. (This is also in the book, but again the point is to prove this fact without simply quoting the result in the book.) Hint: Since f is continuous, it achieves a maximum (i.e. supremum) and a minimum value (i.e. infimum) value on any rectangle.

4. (10 points) Define $f : [0,1] \times [0,2] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \text{ and } y \leq 1 \\ 2 & \text{if } x \notin \mathbb{Q} \text{ and } y > 1. \end{cases}$$

Of the two possible iterated integrals:

$$\int_0^2 \int_0^1 f(x,y)\, dx\, dy \quad \text{and} \quad \int_0^1 \int_0^2 f(x,y)\, dy\, dx,$$

one exists and the other does not. Determine which exists and find its value.

5. (10 points) Show that

$$\lim_{b\to\infty} \iint_{[-b,b]\times[-b,b]} e^{-(x^2+y^2)}\, dA = \pi.$$

Hint: First argue that

$$\iint_{B_b(0,0)} e^{-(x^2+y^2)}\, dA \leq \iint_{[-b,b]\times[-b,b]} e^{-(x^2+y^2)}\, dA \leq \iint_{B_{b\sqrt{2}}(0,0)} e^{-(x^2+y^2)}\, dA.$$

(Side note: this fact is the key to showing that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, which is an important equality in probability and statistics.)