Math 320-2: Midterm 2 Solutions
Northwestern University, Winter 2020

1. Give an example of each of the following. You do not have to justify your answer.
(a) A non-constant function $f(x)$ such that $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$ for all $n \in \mathbb{N}$.
(b) A sequence of non-constant functions which converges in $C[0,1]$ with the sup metric.
(c) A non-empty metric space for which every subset is both closed and open.
(d) A dense subset of $[e, \pi]$ with respect to the standard metric.

Solution. (a) By the orthogonality relations, a function like $f(x) = \cos mx$ for $m \neq 0$ works. (In fact, any non-constant even function works.)
(b) The sequence $f_n(x) = \frac{x}{n}$ works. This converges to 0 uniformly on $[0, 1]$.
(c) Any nonempty set with the discrete metric works.
(d) Something like (e, π) or $[e, \pi] \cap \mathbb{Q}$ works. (In fact technically, $[e, \pi]$ itself works.) □

2. Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is C^2 (i.e. continuously twice-differentiable). Show that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. You can take it for granted that for $n \geq 1$ the following relation between the Fourier coefficients of f and those of f' holds:

$$a_n(f') = nb_n(f) \quad \text{and} \quad b_n(f') = -na_n(f).$$

Hint: Relate the Fourier coefficients of f to those of f''. Here's another hint: M-test.

Proof. This is Problem 4 on Homework 4. Check the solutions to Homework 4. □

3. Let $\mathbb{R}^+$ denote the set of positive real numbers and define a metric on $\mathbb{R}^+$ by

$$d(x, y) = \left| \ln \frac{y}{x} \right|.$$

Take it for granted that this does define a metric.
(a) Determine explicitly the open ball $B_1(1)$ with respect to this metric.
(b) Show that this metric space is complete. (Take for granted the continuity of any single-variable function you might need to use. The fact that $\mathbb{R}$ is complete with respect to the standard metric is important.) Hint: There is an alternate way of expressing the logarithm of a fraction.

Solution. (a) By definition, $x \in B_1(1)$ precisely when $d(1, x) = |\ln \frac{x}{1}| < 1$. But $|\ln x| < 1$ is the same as

$$-1 < \ln x < 1,$$

and since the exponential function is continuous this gives

$$e^{-1} < e^{\ln x} < e^1.$$

Thus $x \in B_1(1)$ if and only if $\frac{1}{e} < x < e$, so $B_1(1) = (\frac{1}{e}, e)$.
(b) Suppose (x_n) is Cauchy in $\mathbb{R}^+$ with respect to d. Then for any $\epsilon > 0$ there exists N such that

$$d(x_n, x_m) < \epsilon \text{ for } m, n \geq N.$$

But using the definition of d, this becomes

$$\left| \ln \frac{x_m}{x_n} \right| = |\ln(x_m) - \ln(x_n)| < \epsilon$$

for $m, n \geq N$. This says precisely that the sequence $(\ln x_n)$ is Cauchy in $\mathbb{R}$ with respect to the standard metric, so since $\mathbb{R}$ complete with respect to the standard metric we get that $\ln x_n$ converges, say to $y \in R$. But then for any $\epsilon > 0$, there exists N such that

$$|\ln x_n - y| < \epsilon \text{ for } n \geq N,$$

which is the same as

$$d(x_n, e^y) < \epsilon \text{ for } n \geq N$$

since $d(x_n, e^y) = |\ln x_n - y|$. Thus x_n converges to $e^y \in \mathbb{R}^+$ with respect to d, showing that $\mathbb{R}^+$ is complete with respect to d. □

4. Suppose $B_r(p)$ and $B_s(q)$ are two open balls in a metric space X. Show that $B_r(p) \cap B_s(q)$ is open in X, by finding for each $x \in B_r(p) \cap B_s(q)$ a radius $t > 0$ such that

$$B_t(x) \subseteq B_r(p) \cap B_s(q).$$

(Don't forget to prove that your claimed radius actually works. A picture will give the right intuition, but is not itself enough justification.)

Proof. Let $x \in B_r(p) \cap B_s(q)$ and set

$$t := \min\{r - d(p, x), s - d(q, x)\}.$$

Note that since $d(p, x) < r$ and $d(q, x) < s$, t is positive. If $y \in B_t(x)$, we have:

$$\begin{aligned} d(y, p) &\leq d(y, x) + d(x, p) \\ &\leq t + d(x, p) \\ &\leq (r - d(p, x)) + d(x, p) \\ &= r \end{aligned}$$

and

$$\begin{aligned} d(y, q) &\leq d(y, x) + d(x, q) \\ &\leq t + d(x, q) \\ &\leq (s - d(q, x)) + d(x, q)) \\ &= s. \end{aligned}$$

Thus $y \in B_r(p)$ and $y \in B_s(q)$, so $y \in B_r(p) \cap B_s(q)$. Thus $B_t(x) \subseteq B_r(p) \cap B_s(q)$, so $B_r(p) \cap B_s(q)$ is open in X as claimed. □

5. Suppose X is a metric space and $A \subseteq X$. Suppose p is in the closure of A but not in A itself. Show that there exists a sequence of *distinct* points of A which converges to p. (The characterization of the closure of A as the set of points $q \in X$ such that every open ball around q contains an element of A may be useful.)

Proof. (This is similar to Problem 5 on the second 2015 Midterm, only in that case there was no convergence requirement.) First, since p is in the closure of A, there exists a point a_1 of A in the ball of radius 1 around p. Now, since p is not in A, $d(a_1, p) > 0$, so $\min\{d(a_1, p), 1/2\}$ is positive. Thus there exists a point a_2 of A in the ball of this radius $\min\{d(a_1, p), 1/2\}$ around p.

Again we have $d(a_2, p) > 0$ since $a_2 \neq p$, so $\min\{d(a_2, p), 1/3\}$ is positive and there is a point a_3 of A in the ball of this radius around p. Continuing in this manner, picking at the n-th stage a point a_n of A in the ball of radius $\min\{d(a_{n-1}, p), 1/n\} > 0$ around p, results in a sequence (a_n) of A such that

$$d(a_n, p) > d(a_{n+1}, p) \text{ and } d(a_n, p) < \frac{1}{n}$$

for all n. The first property guarantees that the a_n are all distinct, and the second that they converge to p, so this is the sequence we want. □