Math 320-2: Midterm 2 Solutions
Northwestern University, Winter 2016

1. Give an example of each of the following. You do not have to justify your answer.

(a) A metric on $\mathbb{R}$ relative to which the sequence $\left(\frac{1}{n}\right)$ does not converge to 0.

(b) A subset of $\mathbb{Q}$ which is closed and open in $\mathbb{Q}$ with respect to the Euclidean metric.

(c) A non-closed subset of $\mathbb{R}^2$ which does not equal its interior relative to the Euclidean metric.

(d) A metric space (X, d) which is not complete.

Solution. (a) The discrete metric works, since the only convergent sequences with respect to a discrete metric are ones which are eventually constant.

(b) Any subset of the form $(a, b) \cap \mathbb{Q}$ with $a < b$ and both irrational works.

(c) Any subset which contains part but not all of its boundary works, such as the unit disk with only the top half of the unit circle included.

(d) $\mathbb{Q}$ with the Euclidean metric works. $\square$

2. Suppose that (X, d) is a metric space, $p \in X$, and r_1, r_2 are real numbers such that $r_2 > r_1 > 0$. Let U be the subset of X consisting of all points whose distance to p is strictly between r_1 and r_2:

$$U := \{x \in X \mid r_1 < d(x, p) < r_2\}$$

For $x \in U$, give an explicit radius r such that $B_r(x) \subseteq U$ and prove that your answer is correct. To be clear, an "explicit" radius can still depend on data given in the problem, such as p and the values of r_1 and r_2.

Proof. Set $r := \min\{r_2 - d(x, p), d(x, p) - r_1\}$, which is positive since both terms in the set we are taking the minimum of is positive. To show that this works, let $q \in B_r(x)$. Then $d(q, x) < r$. Hence

$$d(q, p) \leq d(q, x) + d(x, p) < r + d(x, p) \leq (r_2 - d(x, p)) - d(x, p) = r_2.$$

Also,

$$d(q, p) \geq d(p, x) - d(q, x) > d(p, x) - r \geq d(p, x) - (d(x, p) - r_1) = r_1.$$

Thus $r_1 < d(q, p) < r_2$, so $q \in U$ as required. $\square$

3. Consider the metric space $C[-2, 1]$ of continuous functions $f : [-2, 1] \to \mathbb{R}$ equipped with the sup metric:

$$d(f, g) = \sup_{x \in [-2,1]} |f(x) - g(x)|.$$

Show that the sequence (f_n) in $C[-2, 1]$ defined by

$$f_n(x) = x \sin\left(\frac{x}{n}\right).$$

is Cauchy with respect to the sup metric. Hint: $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$.

Proof. We show that (f_n) converges, which implies that it is Cauchy. In particular, we show $f_n \to 0$. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\frac{4}{N} < \epsilon$. If $n \geq N$, we have:

$$|f_n(x) - 0| = |x| \left|\sin \frac{x}{n}\right| \leq \frac{|x|^2}{n} \leq 4n \leq \frac{4}{N}$$

for every $x \in [-2,1]$. Thus

$$\sup_{x\in[-2,1]} |f_n(x) - 0| \leq \frac{4}{N} < \epsilon$$

for $n \geq N$, showing that $f_n \to 0$ with respect to the sup metric as desired. $\square$

4. Let (X, d) be a metric space. Show that a subset $A \subseteq X$ has empty boundary in X if and only if both A and its complement A^c are open in X.

Proof. Suppose A has empty boundary and let $p \in A$. Then p is not a boundary point of A, so there exists a ball $B_r(p)$ which either doesn't intersect A or doesn't intersect A^c. But since this ball definitely contains $p \in A$, it must be that it doesn't intersect A^c, so $B_r(p) \subseteq A$ and hence A is open. Similarly, If $q \in A^c$, there is a ball $B_s(q)$ which doesn't intersect one of A or A^c, so it must be that $B_s(q)$ doesn't intersect A since $q \in B_s(q)$. Hence $B_s(q) \subseteq A^c$.

Conversely suppose that A and A^c are both open in X. Let $p \in X$. If $p \in A$, then there exists a ball $B_r(p)$ contained in A, so this ball does not intersect A^c and hence p is not a boundary point of A. Similarly, if $p \in A^c$, there exists a ball $B_s(q)$ contained in A^c, so this ball does not intersect A and hence p is not a boundary point of A. Thus no point of X is a boundary point of A, so the boundary of A is empty. $\square$

5. Consider $\mathbb{R}^2$ with respect to the Euclidean metric. Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be three points in $\mathbb{R}^2$. Show that the subset A of $\mathbb{R}^2$ obtained by removing these points:

$$A := \{q \in \mathbb{R}^2 \mid q \neq p_1,\ q \neq p_2,\ \text{and } q \neq p_3\},$$

otherwise known as the complement of $\{p_1, p_2, p_3\}$ in $\mathbb{R}^2$, is dense in $\mathbb{R}^2$.

Proof. Let $B_r(x)$ be an open ball in $\mathbb{R}^2$. Since

$$s = \min\{r, d(x, p_1), d(x, p_2), d(x, p_3)\}$$

is strictly positive, the ball $B_s(x)$ contains none of p_1, p_2, p_3. Thus this ball contains some element of A (since any ball contains infinitely many points), and this then also gives an element of A is in the original ball $B_r(x)$ since $B_s(x) \subseteq B_r(x)$. Thus any open ball in $\mathbb{R}^2$ contains an element of A, so A is dense in $\mathbb{R}^2$. $\square$

<u>Correction</u> this doesn't work if x is one of p_1, p_2, p_3 since then $s = 0$.

<u>Fix</u> If $x = p_1, p_2,$ or p_3, $B_r(x)$ still contains infinitely many points which are not among p_1, p_2, p_3, and any of these is then a point of A in $B_r(x)$.