Math 320-3: Midterm 2 Solutions
Northwestern University, Spring 2016

1. Give an example of each of the following. You do not have to justify your answer.

(a) A bounded subset of $\mathbb{R}$ which is not a Jordan region.

(b) A bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ which is integrable on $B_1(0,0)$ but not on $B_2(0,0)$.

(c) An integrable function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$ does not exist.

(d) A function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\int_D 2\, dV = \int_{f(D)} 4\, dV$ for any Jordan region $D \subseteq \mathbb{R}^n$.

Solution. (a) The set $\mathbb{Q} \cap [0,1]$ works.

(b) The function which is the constant 0 on $B_1(0,0)$ but which outside this ball has value 1 at $\mathbf{x} \in \mathbb{Q}$ and -1 at $\mathbf{x} \notin \mathbb{Q}$ works.

(c) Define f via

$$f(x,y) = \begin{cases} 1 & x \in \mathbb{Q} \text{ and } y = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Since this function agrees with the constant zero function everywhere except for on a set of measure zero, f is integrable on the square since 0 is integrable. However, at $y = 1/2$ the single variable function $x \mapsto f(x,y)$ is not integrable since it is 1 at rationals and 0 at irrationals. Thus the single-variable integral $\int_0 !f(x,y)\, dx$ does not exist at this value of y, so the iterated integral $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$ does not exist. (Saying that an iterated integral exists requires that the inner integral exists for all values of the outer variable.)

(d) The function $f(x_1, x_2, \ldots, x_n) = (\frac{1}{2}x_1, x_2, \ldots, x_n)$ works. This is C^1, bijective, and has invertible Jacobian everywhere since

$$Df(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The change of variables formula gives

$$\int_D 4|\det Df(\mathbf{x})|\, dx = \int_{f(D)} 4\, dV,$$

and since $\det Df(\mathbf{x}) = \frac{1}{2}$ we get the desired equality. □

2. Suppose $A, B \subseteq \mathbb{R}^2$ are Jordan regions, which implies that $A \cup B$ is also Jordan measurable. If $A \cap B$ has Jordan measure zero, show that

$$\operatorname{Vol}(A \cup B) = \operatorname{Vol} A + \operatorname{Vol} B.$$

(This is an exercise in the book, but of course the point here is to prove this fact without simply quoting the result of that exercise.)

Proof. First, note that since A, B, and $A \cup B$ are all Jordan measurable, the volume of each is equal to the volume of its closure, so we may as well assume that each of these sets equals its closure, or in other words we assume that each of these is closed. (This is just so that in the definition of an outer sum we can consider rectangles which intersect each as opposed to rectangles which intersect their closures; a minor point.)

Pick a rectangle containing $A \cup B$ and a grid G on this rectangle. Then

$$V(A \cup B; G) = \sum_{R_i \cap (A \cup B) \neq \emptyset} |R_i|.$$

The rectangles being summed over here can be broken up into three categories: those (call these type I) which intersect A but not B, those which intersect B but not A (call these type II), and those which intersect $A \cap B$. This gives

$$V(A \cup B; G) = \sum_{R_i \text{ of type } I} |R_i| + \sum_{R_i \text{ of type } II} |R_i| + \sum_{R_i \cap (A \cap B) \neq \emptyset} |R_i|.$$

The first and final pieces together give all R_i intersecting A, so we can rewrite this expression as

$$V(A \cup B; G) = V(A; G) + \sum_{R_i \text{ of type } II} |R_i|.$$

Adding $V(A \cup B; G) = \sum_{R_i \cap (A \cap B) \neq \emptyset} |R_i|$ to both sides gives

$$V(A \cup B; G) + V(A \cap B; G) = V(A; G) + V(B; G).$$

Finally, using the fact that $\text{Vol}(A \cap B) = 0$, taking the infimum of both sides gives

$$V(A \cup B; G) = V(A; G) + V(B; G)$$

as claimed, where we use the fact that $\inf(S + T) = \inf S + \inf T$ for subsets S, T of $\mathbb{R}$. □

3. Suppose $B \subseteq \mathbb{R}^2$ is a rectangle and that $f : B \to \mathbb{R}$ is uniformly continuous, which, to recall, means that for any $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\text{if } \mathbf{x}, \mathbf{y} \in B \text{ and } \|\mathbf{x} - \mathbf{y}\| < \delta, \text{ then } |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon.$$

Show that f is integrable over B. (This is also in the book, but again the point is to prove this fact without simply quoting the result in the book.) Hint: Since f is continuous, it achieves a maximum (i.e. supremum) and a minimum value (i.e. infimum) value on any rectangle.

Proof. Let $\epsilon > 0$ and pick δ such that

$$\text{if } \mathbf{x}, \mathbf{y} \in B \text{ and } \|\mathbf{x} - \mathbf{y}\| < \delta, \text{ then } |f(\mathbf{x}) - f(\mathbf{y})| < \frac{\epsilon}{|B|},$$

where $|B|$ denotes the area of B and which we can do by uniform continuity. Let G be any grid fine enough so that $\mathbf{x}, \mathbf{y}$ belonging to the same smaller rectangle are at a distance less than δ from each other. On any small rectangle R_i, f has a maximum and a minimum, so there exist $\mathbf{x}', \mathbf{y}'$ such that

$$f(\mathbf{x}') = \sup f \text{ on } R_i, \text{ and } f(\mathbf{y}') = \inf f \text{ on } R_i.$$

This gives that on any small rectangle,

$$\sup f - \inf f = f(\mathbf{x}') - f(\mathbf{y}') < \frac{\epsilon}{|B|}.$$

Thus

$$U(f, G) - L(f, G) = \sum_{R_i} (\sup f - \inf f)|R_i|$$

$$
\begin{aligned}
&< \sum_{R_i} \frac{\epsilon}{|B|} |R_i| \\
&= \frac{\epsilon}{|B|} \sum_{R_i} |R_i| \\
&= \epsilon
\end{aligned}
$$

where we use the fact that $\sum_{R_i} |R_i| = |B|$. We conclude that f is integrable over B as claimed. □

4. Define $f : [0,1] \times [0,2] \to \mathbb{R}$ by

$$
f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \text{ and } y \leq 1 \\ 2 & \text{if } x \notin \mathbb{Q} \text{ and } y > 1. \end{cases}
$$

Of the two possible iterated integrals:

$$
\int_0^2 \int_0^1 f(x,y)\, dx\, dy \quad \text{and} \quad \int_0^1 \int_0^2 f(x,y)\, dy\, dx,
$$

one exists and the other does not. Determine which exists and find its value.

Solution. At a fixed $y \in [0,1]$, the single-variable function $x \mapsto f(x,y)$ is not integrable over $[0,1]$ since this has value 0 for $x \notin \mathbb{Q}$ an 1 for $x \in \mathbb{Q}$. Thus the inner integral in the first iterated integral does not exist for $0 \leq y \leq 1$ (in fact, it fails to exist for all y), so the first iterated integral does not exist.

Now, for a fixed $x \in \mathbb{Q}$, we have

$$
\int_0^2 f(x,y)\, dy = \int_0^2 1\, dy = 2.
$$

For a fixed $x \notin Q$, we have

$$
\int_0^2 f(x,y)\, dy = \int_0^1 f(x,y)\, dy + \int_1^2 f(x,y)\, dy = \int_0^1 0\, dy + \int_1^2 2\, dy = 2.
$$

Hence $\int_0^2 f(x,y)\, dy = 2$ for all x, so the second iterated integral exists and

$$
\int_0^1 \int_0^2 f(x,y)\, dy\, dx = \int_0^1 2\, dx = 2
$$

is its value. □

5. Show that

$$
\lim_{b \to \infty} \iint_{[-b,b]\times[-b,b]} e^{-(x^2+y^2)}\, dA = \pi.
$$

Hint: First argue that

$$
\iint_{B_b(0,0)} e^{-(x^2+y^2)}\, dA \leq \iint_{[-b,b]\times[-b,b]} e^{-(x^2+y^2)}\, dA \leq \iint_{B_{b\sqrt{2}}(0,0)} e^{-(x^2+y^2)}\, dA.
$$

(Side note: this fact is the key to showing that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, which is an important equality in probability and statistics.)

Proof. Note that

$$B_b(0,0) \subseteq [-b,b] \times [-b,b] \subseteq B_{b\sqrt{2}}(0,0).$$

Thus, since the integrand in question is always positive, we have that

$$\iint_{B_b(0,0)} e^{-(x^2+y^2)}\,dA \leq \iint_{[-b,b]\times[-b,b]} e^{-(x^2+y^2)}\,dA \leq \iint_{B_{b\sqrt{2}}(0,0)} e^{-(x^2+y^2)}\,dA$$

since at each step we are integrating a positive function over a larger region, so the extra contribution to the resulting integral will be positive. Now, we have:

$$\iint_{B_b(0,0)} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^b re^{-r^2}\,dr\,d\theta = -\pi e^{-r^2}\Big|_0^b = \pi(1-e^{-b^2})$$

and

$$\iint_{B_{b\sqrt{2}}(0,0)} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{b\sqrt{2}} re^{-r^2}\,dr\,d\theta = -\pi e^{-r^2}\Big|_0^{b\sqrt{2}} = \pi(1-e^{-2b^2}).$$

Since both of these expressions have limit π as $b \to \infty$, the squeeze theorem gives the desired equality. $\square$