Math 320-1: Midterm 2 Solutions
Northwestern University, Fall 2019

1. Give an example of each of the following. You do not have to justify your answer.
(a) A function on $\mathbb{R}$ which is continuous only at 2.
(b) An unbounded function on $\mathbb{R}$ which is uniformly continuous on any bounded interval.
(c) A function on $\mathbb{R}$ which does not have an anti-derivative.
(d) A differentiable function $\mathbb{R}$ which is not continuously differentiable.

Solution. (a) The function defined by $f(x) = x - 2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$ works.

(b) The function $f(x) = x$ works. This is uniformly continuous on any bounded interval since it is continuous on any $[a, b]$.

(c) Any function with a jump discontinuity works, say the one defined by $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. This does not have an anti-derivative since it does not have the intermediate value property.

(d) The function defined by $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ works, as we saw in class. □

2. Show, by verifying the ϵ-δ definition directly, that the function $f(x) = x^3 - 2x$ is continuous on the interval $(-10, 3)$. You will need the following: $x^3 - a^3 = (x^2 + ax + a^2)(x - a)$.

Proof. First note that for $x \in (-10, 3)$, $|x| \leq 10$. Fix $a \in (-10, 3)$ and let $\epsilon > 0$. Set

$$\delta = \frac{\epsilon}{100 + 100|a| + |a|^2} > 0.$$

Then if $|x - a| < \delta$, we have:

$$|x^3 - a^3| = |x^2 + ax + a^2||x - a| \leq (|x|^2 + |a||x| + |a|^2)|x - a| \leq (100 + 100|a| + |a|^2)|x - a| < \epsilon.$$

Thus f is continuous at a, and since $a \in (-10, 3)$ was arbitrary, f is continuous on $(-10, 3)$. □

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and let M denote the supremum of the values of f:

$$M = \sup\{f(x) \mid x \in \mathbb{R}\}$$

Show that for any $\epsilon > 0$, there exists a **rational** number $a \in \mathbb{R}$ such that $M - \epsilon < f(a)$.

Proof. Let $\epsilon > 0$. Then $M - \frac{\epsilon}{2}$ is not an upper bound of the set of values of f, so there exists such a value $f(y)$ such that $M - \frac{\epsilon}{2} < f(y)$. Take a sequence of rationals r_n which converges to y. Then $f(r_n)$ converges to $f(y)$ since f is continuous, so there exists some N such that $f(r_N)$ is within $\frac{\epsilon}{2}$ of $f(y)$:

$$|f(r_N) - f(y)| < \frac{\epsilon}{2}.$$

For the rational $a = r_N$ we thus have:

$$M - \epsilon = M - \frac{\epsilon}{2} - \frac{\epsilon}{2} < f(y) - \frac{\epsilon}{2} < f(a)$$

as desired. (The point is that $f(a)$ is within $\frac{\epsilon}{2}$ of $f(y)$, which in turn is within $\frac{\epsilon}{2}$ of M, which implies that $f(a)$ is within $\frac{\epsilon}{2} + \frac{\epsilon}{2}$ of M.)

Alternatively, after we have $f(y)$ as above, we can get a as follows. Since f is continuous at y, there exists $\delta > 0$ such that

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Thus for a rational a in $(y-\delta, y+\delta)$, which exists by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we get $|f(a)-f(y)| < \frac{\epsilon}{2}$, and the same reasoning as above implies that $M - \epsilon < f(a)$. □

4. Determine, with justification, the largest k for which the following function $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable, and if its k-th derivative is continuous.

$$f(x) = \begin{cases} x^3 & x > 0 \\ x^2 & x \leq 0. \end{cases}$$

Solution. First, for f is differentiable at any $x \neq 0$, since near any such point f agrees completely with either x^3 or x^2, each of which are differentiable at nonzero x. Since:

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0$$

and

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^3}{x} = \lim_{x\to 0^+} x^2 = 0,$$

we have $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists and equals zero, so $f'(0) = 0$. Thus f is at least 1-time differentiable and

$$f'(x) = \begin{cases} 3x^2 & x > 0 \\ 0 & x = 0 \\ 2x & x < 0. \end{cases}$$

Next, we have:

$$\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^-} \frac{2x}{x} = 2$$

and

$$\lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^-} \frac{3x^2}{x} = 0.$$

Since these do not agree, $\lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0}$ does not exist, so f is not twice differentiable at 0. Thus $k = 1$ is the largest k for which f is k-times differentiable.

Since

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2x = 0$$

and

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 3x^2 = 0,$$

we have $\lim_{x\to 0} f'(x) = 0 = f'(0)$, so f' is continuous at 0; it is continuous at nonzero x since $3x^2$ and $2x$ are each continuous. □

5. Suppose $f : [0, 1] \to \mathbb{R}$ is differentiable and nonnegative, satisfies $f(0) = 0$, and that there exists $0 < M < 1$ such that

$$f'(x) \leq M f(x) \text{ for all } x \in [0, 1].$$

If f is not decreasing, show that f is the constant zero function. Hint: $f(x) = f(x) - f(0)$.

Proof. Fix $x \in (0, 1]$. By the Mean Value Theorem there exists $c \in (0, x)$ such that

$$f(x) - f(0) = f'(c)(x - 0), \text{ which becomes } f(x) = f'(c)x.$$

By the given assumption, $f'(c) \leq Mf(c)$, so

$$f(x) = f'(c)x \leq Mf(c)x.$$

Since $x < 1$ and $f(c) \leq f(x)$ because f is not decreasing (and $c < x$), this gives:

$$0 \leq f(x) \leq Mf(c)x \leq Mf(x).$$

Since $0 < M < 1$, this implies $f(x) = 0$, so f is the constant zero function, given that we already know $f(0) = 0$. $\square$