Math 320-1: Midterm 2
Northwestern University, Fall 2019

Name: ______________________________

1. (10 points) Give an example of each of the following. You do not have to justify your answer.
(a) A function on $\mathbb{R}$ which is continuous only at 2.
(b) An unbounded function on $\mathbb{R}$ which is uniformly continuous on any bounded interval.
(c) A function on $\mathbb{R}$ which does not have an anti-derivative.
(d) A differentiable function $\mathbb{R}$ which is not continuously differentiable.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Show, by verifying the ϵ-δ definition directly, that the function $f(x) = x^3 - 2x$ is continuous on the interval $(-10, 3)$. You will need the following: $x^3 - a^3 = (x^2 + ax + a^2)(x - a)$.

3. (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and let M denote the supremum of the values of f:

$$M = \sup\{f(x) \mid x \in \mathbb{R}\}$$

Show that for any $\epsilon > 0$, there exists a **rational** number $a \in \mathbb{R}$ such that $M - \epsilon < f(a)$. Hint: First take (why does this exist?) a real number $y \in \mathbb{R}$ such that $M - \frac{\epsilon}{2} < f(y)$, and consider a sequence of rationals converging to y.

4. (10 points) Determine, with justification, the largest k for which the following function $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable, and if its k-th derivative is continuous.

$$f(x) = \begin{cases} x^3 & x > 0 \\ x^2 & x \leq 0. \end{cases}$$

5. (10 points) Suppose $f : [0, 1] \to \mathbb{R}$ is differentiable and nonnegative, satisfies $f(0) = 0$, and that there exists $0 < M < 1$ such that

$$f'(x) \leq M f(x) \text{ for all } x \in [0, 1].$$

If f is not decreasing, show that f is the constant zero function. Hint: $f(x) = f(x) - f(0)$.