Math 320-3: Midterm 1
Northwestern University, Spring 2020

Name: ____

1. (10 points) Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{R}$ whose boundary is all of $\mathbb{R}$.

(b) A function $f(x,y)$ such that $f_x(0,0)$ does not exist but $f_y(0,0)$ does.

(c) A differentiable function $f(x,y)$ such that f_x is not continuous at $(0,0)$.

(d) A differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$D(f \circ g)(x,y) = \begin{bmatrix} 4xy + x^2 & 2xy + x^2 \end{bmatrix}$$

where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is the function $g(x,y) = (2x+y, x+y)$. (Hint: You can determine $Df(x,y)$ explicitly from the given information. Recall that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$.)

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Let A be the region in $\mathbb{R}^2$ which lies within the the square $[-5, 5] \times [-5, 5]$ and outside the square $[-1, 1] \times [-1, 1]$. Show that A is connected. (Recall $[a, b] \times [c, d]$ denotes the rectangle consisting of points (x, y) with $a \leq x \leq b$ and $c \leq y \leq d$. A proof which relies on pictures alone is not enough.)

3. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x,y) = \begin{cases} \left(\frac{2x^2y - 3x^4}{x^2+y^2}, 4x + y^2\right) & (x,y) \neq (0,0) \\ (0,0) & (x,y) = (0,0). \end{cases}$$

Show that f is continuous but not differentiable at $(0,0)$.

4. (10 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = xf(x, y)$. Show that g is differentiable at any $(x, y) \in \mathbb{R}^2$ using the definition of differentiability directly.

5. (10 points) Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}^2$ are differentiable and satisfy

$$F(x, g_1(x), g_2(x)) = \mathbf{0} \text{ for all } x \in \mathbb{R}$$

where $g(x) = (g_1(x), g_2(x))$. Write the Jacobian matrix of F at a point $(x, g_1(x), g_2(x))$ as

$$DF(x, g_1(x), g_2(x)) = \begin{bmatrix} \mathbf{b} & A \end{bmatrix}$$

where $\mathbf{b}$ is the 2×1 matrix making up the first column of $DF(x, g_1(x), g_2(x))$ and A the 2×2 matrix making up the final two columns. If A is invertible, show that

$$Dg(x) = -A^{-1}\mathbf{b}.$$

Hint: View $F(x, g_1(x), g_2(x))$ as the result of composing the function $h(x) = (x, g_1(x), g_2(x))$ with F. We did a similar problem as a Warm-Up when discussing the chain rule, only in that case g (or perhaps we called it f) was a function with only one component.