Math 320-3: Midterm 1

Northwestern University, Spring 2016

Name: ______________________________

1. (10 points) Give an example of each of the following. You do not have to justify your answer.

(a) A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(\mathbf{0})$ exists but $f_y(\mathbf{0})$ does not.

(b) An open $U \subseteq \mathbb{R}^2$ and non-constant differentiable $f : U \to \mathbb{R}$ such that $Df(\mathbf{x}) = 0$ for all $\mathbf{x}$.

(c) A differentiable $f : \mathbb{R} \to \mathbb{R}$ such that $u(x, y) = f(xy)$ has Jacobian $Du(x, y) = \begin{pmatrix} 2xy^2 & 2x^2y \end{pmatrix}$.

(d) A point (a, b) such that $f(x, y) = (x + y, x^2y^3)$ is invertible near (a, b).

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined below is continuous but not differentiable at the origin.

$$f(x,y) = \begin{cases} 1 - 3x^2 + 4y + \frac{x^3y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

3. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function and A is an $m \times n$ matrix such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| + \|A\| \, \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|^2 \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Show that f has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^m$. Hint: First show that $g(\mathbf{x}) = f(\mathbf{x}) - A\mathbf{x}$ satisfies $\|g(\mathbf{x}) - g(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. What property of g is equivalent to required claim about f? Why does g have this property?

4. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$. Show that for any $\mathbf{u} \in \mathbb{R}^m$, there exists $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$ such that

$$\mathbf{u} \cdot (f(\mathbf{x}) - f(\mathbf{a})) = \mathbf{u} \cdot [Df(\mathbf{c})(\mathbf{x} - \mathbf{a})],$$

where $\cdot$ denotes the usual dot product: $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$. Hint: Consider the single-variable function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = \mathbf{u} \cdot f(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))$.

5. (10 points) Let A be the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying

$$xyz + \sin(x + y + z) = 0.$$

(a) Show that there exists an open set $W \in \mathbb{R}^2$ containing $(0, 0)$ and a differentiable function $g : W \to \mathbb{R}$ such that $(x, y, g(x, y)) \in A$ for all $(x, y) \in W$.

(b) Let B denote the set of all points satisfying

$$x^2 + y^4 - y + z = 0.$$

Note that $(0, 0, 0)$ is in the intersection of A and B. Show that near $(0, 0, 0)$ this intersection is a curve given by parametric equations of the form

$$x = x(t),\ \ y = y(t),\ \ z = t.$$