Math 320-2: Midterm 1 Solutions
Northwestern University, Winter 2020

1. Give an example of each of the following. You do not have to justify your answer.

(a) A sequence (a_n) for which $\sum a_n$ diverges but $\sum a_n^3$ converges.

(b) Continuous functions on $[-1, 0]$ which converge pointwise to a discontinuous function.

(c) A pointwise convergent series $\sum f_n(x)$ on $(-1, 1)$ such that $\sum f_n'(x)$ converges to $\frac{1}{1-x}$.

(d) A function which is not analytic on $(2, 3)$.

Solution. (a) For $a_n = \frac{1}{n}$, $\sum \frac{1}{n}$ diverges but $\frac{1}{n^3}$ converges.

(b) The functions $f_n(x) = (-x)^n$ are continuous and converge pointwise to the function which is 0 for $-1 < x < 0$ and 1 for $x = -1$.

(c) The series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ works, since the derivative series is the geometric series $\sum x^n$, which does converge to $\frac{1}{1-x}$.

(d) Any noncontinuous function works, since an analytic function must at the very least be analytic. (Similarly, any continuous but non-differentiable function works.) If you really want an example which is infinitely differentiable but not analytic, then the function which is $f(x) = e^{-1/(x-2.5)}$ for $x > 2.5$ and $f(x) = 0$ for $x \leq 2.5$ works. (This is a “shifted” version of the standard example we saw in class.) □

2. Suppose (b_n) is a decreasing sequence of positive numbers which converges to 0. Show that the series $\sum_{n=0}^{\infty}(-1)^n b_n$ converges. Hint: How does the value of

$$b_n - b_{n+1} + b_{n+2} - b_{n+3} + \cdots + (-1)^k b_{n+k}$$

compare to the value of b_n?

Proof. For any n, we have that

$$b_n - b_{n+1} + b_{n+2} - b_{n+3} + \cdots + (-1)^k b_{n+k} \leq b_n$$

since on the left we start subtracting away from b_n and never add an amount larger than what was subtracted before; i.e. we subtract away b_{n+1}, then add back on the smaller amount b_{n+2} so that $b_n - b_{n+1} + b_{n+2}$ is not where b_n was originally, and so on. (This is where we use the fact that the the b_n are *decresaing.*) Thus for $\epsilon > 0$, pick N such that

$$b_n < \epsilon \text{ for } n \geq N,$$

which we can do since (b_n) converges to 0. Then for $n \geq N$ and $k \geq 0$, we have:

$$|(-1)^n b_n + (-1)^{n+1} b_{n+1} + (-1)^{n+k} b_{n+k}| = b_n - b_{n+1} + \cdots + (-1)^k b_{n+k} \leq b_n < \epsilon,$$

where in the first step we factored out $|(-1)^n| = 1$ and were left with the positive quantity $b_n - b_{n+1} + b_{n+2} - b_{n+3} + \cdots + (-1)^k b_{n+k}$. This shows that $\sum (-1)^n b_n$ by the Cauchy criterion for series convergence. (Concretely, this shows that the sequence of partial sums of $\sum (-1)^n b_n$ is Cauchy.) □

3. Determine, with justification, the value of ONE of the following limits:

$$\lim_{n \to \infty} \int_{-3}^{1} x^2 e^{x^2/n}\, dx \qquad \text{or} \qquad \lim_{n \to \infty} \int_{-3}^{1} (x^2 + \sin^2(\tfrac{x}{n}))\, dx$$

You can use any inequality you've seen in class or on homework without justification.

Proof. For the first limit, for a fixed x the sequence $\frac{x^2}{n}$ converges to 0, so $e^{x^2/n} \to e^0 = 1$ by the continuity of the exponential function. Thus the function $x^2 \cdot 1 = x^2$ is the pointwise limit of the sequence $x^2 e^{x^2/n}$. Now, let $\epsilon > 0$ and pick N such that

$$|e^{9/n} - 1| < \frac{\epsilon}{9} \text{ for } n \geq N,$$

which exists since $e^{9/n} \to 1$. Then for $x \in [-3, 1]$ we have:

$$\begin{aligned} |x^2 e^{x^2/n} - x^2| &= |x^2||e^{x^2/n} - 1| \\ &\leq 9|e^{9/n} - 1| \\ &< \epsilon, \end{aligned}$$

where in the second step we use the fact that the exponential function is increasing in order to say that $e^{x^2/n} - 1 < e^{9/n} - 1$ for $x \in [-3, 1]$. Thus $x^2 e^{x^2/n} \to x^2$ uniformly on $[-3, 1]$, and since uniform convergence preserves integrals we get:

$$\lim_{n\to\infty} \int_{-3}^{1} x^2 e^{x^2/n}\, dx = \int_{-3}^{1} \lim_{n\to\infty} x^2 e^{x^2/n}\, dx = \int_{-3}^{1} x^2\, dx = \frac{1}{3}(1 + 27).$$

For the second limit, for a fixed x we have $\frac{x}{n} \to 0$, so $\sin \frac{x}{n} \to \sin 0 = 0$ by the continuity of sine. Thus $x^2 + 0 = x^2$ is the pointwise limit of $x^2 + \sin^2(\frac{x}{n})$. Let $\epsilon > 0$ and pick N such that

$$\frac{9}{n^2} < \epsilon \text{ for } n \geq N.$$

Then for $x \in [-3, 1]$, we have:

$$\begin{aligned} |(x^2 + \sin^2(\tfrac{x}{n})) - x^2| &= |\sin^2(\tfrac{x}{n})| \\ &\leq \left|\frac{x^2}{n^2}\right| \\ &\leq \frac{9}{n^2} \\ &< \epsilon, \end{aligned}$$

where in the second step we use the fact that $|\sin y| \leq |y|$ for all y. Thus $x^2 + \sin^2(\frac{x}{n})$ converges uniformly to x^2 on $[-3, 1]$, and since uniform converges preserves integrals we get:

$$\lim_{n\to\infty} \int_{-3}^{1} (x^2 + \sin^2(\tfrac{x}{n}))\, dx = \int_{-3}^{1} \lim_{n\to\infty} (x^2 + \sin^2(\tfrac{x}{n}))\, dx = \int_{-3}^{1} x^2\, dx = \frac{1}{3}(1 + 27).$$

□

4. Show that the following series converges uniformly on any interval $[-M, M]$ centered at 0 in $\mathbb{R}$ and defines a differentiable function on all of $\mathbb{R}$.

$$\sum_{n=1}^{\infty} \left(1 - e^{x/n}\right)^2$$

You can take it for granted that for any $x \in \mathbb{R}$, $1 - e^{x/n} = \frac{x}{n} e^c$ for some c between 0 and $\frac{x}{n}$.

Proof. For each $x \in \mathbb{R}$, we have $1 - e^{x/n} = e^c x/n$ for some c between 0 and $\frac{x}{n}$ by the Mean Value Theorem, so

$$|1 - e^{x/n}| = |e^c| \left|\frac{x}{n}\right| \le e^{|x|} \left|\frac{x}{n}\right|,$$

where $|e^c| \le e^{|x|}$ since c, between 0 and $\frac{x}{n}$, is thus also between 0 and x, and the exponential function is increasing. (Note this inequality applies even if x and hence c is negative, in which case e^c is already smaller than $1 \le e^{\text{non-negative}}$.) Thus, for $x \in [-M, M]$, we have:

$$\left|1 - e^{x/n}\right|^2 \le e^{2|x|} \left|\frac{x}{n}\right|^2 \le e^{2M} \frac{M^2}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, multiplying by the constant $M^2 e^{2M}$ still results in a convergent series, so the Weierstrass M-test guarantees that the given series converges uniformly on $[-M, M]$.

Now, the term-by-term derivative series is

$$\sum_{n=1}^{\infty} -\frac{2}{n}(1 - e^{x/n})e^{x/n}.$$

As above, for $x \in [-M, M]$ we have

$$\left|-\frac{2}{n}(1 - e^{x/n})e^{x/n}\right| \le \frac{2}{n} e^{|x|} \left|\frac{x}{n}\right| |e^{x/n}| \le \frac{2Me^{2M}}{n^2}.$$

Since $\sum 2Me^{2M}/n^2$ converges, the M-test implies that this term-by-term derivative converges uniformly on $[-M, M]$. Thus, since the original series and its term-by-term derivative converge uniformly on $[-M, M]$, the original series is differentiable on this interval, and since these intervals cover all of $\mathbb{R}$ as M ranges over all positive numbers, we find that the original series defined a differentiable function on all of $\mathbb{R}$. □

5. Suppose $\sum_{n=0}^{\infty} a_n x^n$ has finite radius of convergence $R > 0$. Determine the largest open interval $(-L, L)$ centered at 0 on which the following series defines a differentiable function:

$$\sum_{n=0}^{\infty} 2^n a_n^2 x^{5n}$$

Proof. The largest such interval is precisely the interval of convergence of this power series, so we must determine its radius of convergence. Writing this series as $\sum b_k x^k$, we must compute $\limsup |b_k|^{1/k}$, where

$$b_k = \begin{cases} 2^n a_n^2 & k = 5n \\ 0 & \text{otherwise.} \end{cases}$$

The supremums used in computing $\limsup |b_k|^{1/k}$ can only be affected by the positive coefficients b_{5n} since the other zero coefficients will have no effect, so

$$\limsup |b_k|^{1/k} = \limsup |b_{5n}|^{1/5n} = \limsup (2^n a_n^2)^{1/5n} = 2^{1/5} \limsup (a_n)^{2/5n}.$$

Since $\sum a_n x^n$ has radius of convergence R, we have $\limsup |a_n|^{1/n} = \frac{1}{R}$, so

$$\limsup |a_n|^{2/5n} = \left(\frac{1}{R}\right)^{2/5}.$$

Thus $\limsup |b_k|^{1/k} = 2^{1/5} R^{-2/5}$, so the given series has radius of convergence $(R^2/2)^{1/5}$, so the largest interval on which it defines a differentiable function is $(-\sqrt[5]{R^2/2}, \sqrt[5]{R^2/2})$. □