Math 320-2: Midterm 1 Solutions
Northwestern University, Winter 2016

1. Give an example of each of the following. You do not have to justify your answer.
(a) A sequence (a_n) which converges to 0 but for which $\sum a_n$ diverges.
(b) A sequence of continuous functions on $[2, 3]$ which converges pointwise but not uniformly.
(c) A uniformly convergent series $\sum f_n(x)$ on $(-\frac{1}{2}, \frac{1}{2})$ such that $\sum f_n'(x)$ converges to $\frac{1}{(1-x)^2}$.
(d) A power series centered at 5 with radius of convergence $\frac{1}{3}$.

Solution. Here are some possible example.
(a) The sequence $a_n = \frac{1}{n}$ works.
(b) The sequence $f_n(x) = (x - 2)^n$ works. This converges pointwise to the function which is zero on $[2, 3)$ and 1 at $x = 3$, but not uniformly since each f_n is continuous but the limit is not.
(c) The sequence $\sum x^n$ works. This converges to $\frac{1}{1-x}$ uniformly on the given interval and $\sum nx^{n-1}$ converges to the derivative of $\frac{1}{1-x}$.
(d) The series $\sum 3^n(x-5)^n$ works. Writing this as $\sum(3(x-5))^n$, this converges when $3|x-5| < 1$, so when $|x - 5| < \frac{1}{3}$. □

2. Suppose (a_n) is a decreasing sequence of numbers for which $\sum_{n=1}^{\infty} a_n$ converges. Show that the sequence (na_{2n}) converges to 0. Hint: Use the fact that (a_n) is decreasing to bound $na_{2n} = \underbrace{a_{2n} + \cdots + a_{2n}}_{n \text{ times}}$.

Proof. Note: This was a Warm-Up example from my old lectures notes. We didn't do it in class this quarter, but you did go through it in discussion.
Since (a_n) is decreasing, we have

$$na_{2n} = \underbrace{a_{2n} + \cdots + a_{2n}}_{n \text{ times}} \leq a_{n+1} + \cdots + a_{2n}$$

for any n since each a_{n+k} is larger than or equal to a_{2n} for $1 \leq k \leq n$. Since the sequence (a_n) converges to 0 (because the series $\sum a_n$ converges), each a_n must be nonnegative, so the above inequality still holds if we take absolute values:

$$|na_{2n}| = |\underbrace{a_{2n} + \cdots + a_{2n}}_{n \text{ times}}| \leq |a_{n+1} + \cdots + a_{2n}|.$$

Let $\epsilon > 0$. Since $\sum a_n$ converges, the Cauchy criterion says there exists $N \in \mathbb{N}$ such that

$$|a_{n+1} + \cdots + a_{n+k}| < \epsilon \text{ for all } n \geq N \text{ and } k \geq 0.$$

Hence in particular, for $n \geq N$ we have

$$|na_{2n}| \leq |a_{n+1} + \cdots + a_{2n}| < \epsilon,$$

so (na_{2n}) converges to 0 as desired. □

3. Determine the value of the following limit.

$$\lim_{n\to\infty} \int_0^4 \left(x^2 e^{x/n} - \frac{xn}{n+1}\right) dx$$

Solution. We first show that the sequence

$$f_n(x) = x^2 e^{x/n} - \frac{xn}{n+1}$$

converges uniformly on $[0, 4]$. The pointwise limit is:

$$f(x) = x^2 - x$$

since for a fixed $x \in [0, 4]$, $e^{x/n} \to e^0 = 1$ (because the exponential function is continuous) and $\frac{xn}{n+1} \to x$. Let $\epsilon > 0$. Since the exponential function is continuous and $\frac{4}{n} \to 0$, $e^{4/n} \to 1$ so there exists $N \in \mathbb{N}$ such that

$$|e^{4/n} - 1| < \frac{\epsilon}{32} \text{ for } n \geq N.$$

By making N larger if need be, we may also assume

$$\frac{1}{n+1} < \frac{\epsilon}{8} \text{ for } n \geq N.$$

Note that for any $x \in [0, 4]$ and $n \in \mathbb{N}$, $e^{x/n} \geq e^0 = 1$ so

$$\left|e^{x/n} - 1\right| = e^{x/n} - 1 \leq e^{4/n} - 1 = \left|e^{4/n} - 1\right|.$$

Thus if $n \geq N$ and $x \in [0, 4]$, we have:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|x^2 e^{x/n} - \frac{xn}{n+1} - x^2 + x\right| \\
&= \left|\left(x^2 e^{x/n} - x^2\right) - \left(\frac{xn}{n+1} - x\right)\right| \\
&\leq |x^2| \left|e^{x/n} - 1\right| + |x| \left|\frac{n}{n+1} - 1\right| \\
&\leq 16|e^{x/n} - 1| + 4\left(\frac{1}{n+1}\right) \\
&\leq 16|e^{4/n} - 1| + 4\left(\frac{1}{n+1}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $f_n \to f$ uniformly as claimed.

Thus we may exchange the limit with the integration in the expression we are asked to compute, so:

$$\lim_{n \to \infty} \int_0^4 \left(x^2 e^{x/n} - \frac{xn}{n+1}\right) dx = \int_0^4 (x^2 - x)\, dx = \frac{64}{3} - \frac{16}{2}$$

is the required value. □

4. Suppose for each $n \in \mathbb{N}$ the function $f_n : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies

$$|f_n(x)| \leq \frac{|x|}{n} \quad \text{and} \quad |f_n'(x)| \leq \frac{1 + \sin^2 x}{n} \text{ for all } x \in \mathbb{R}.$$

Show that $\sum_{n=1}^{\infty} f_n(x)^2$ converges pointwise to a differentiable function on $\mathbb{R}$.

Proof. Given any interval $[-b, b]$ with $b > 0$, for $x \in [-b, b]$ we have:

$$|f_n(x)^2| \leq \frac{|x|^2}{n^2} \leq \frac{b}{n^2}.$$

Since $\sum \frac{b}{n^2}$ converges, the M-test implies that $\sum f_n(x)^2$ converges uniformly on any $[-b, b]$. Any $x \in \mathbb{R}$ will be in some such interval, so uniform convergence on each $[-b, b]$ implies pointwise converges on $\mathbb{R}$.

Given any interval $[-b, b]$ with $b > 0$, for $\mathbf{x} \in [-b, b]$ we have:

$$|2f_n(x)f_n'(x)| \leq \frac{2|x||1 + \sin^2 x|}{n^2} \leq \frac{4b}{n^2}.$$

Since $\sum \frac{4b}{n^2}$ converges, the M-test implies that

$$\sum (f_n(x)^2)' = \sum 2f_n(x)f_n'(x)$$

converges uniformly on any $[-b, b]$, and hence the function

$$f(x) = \sum f_n(x)^2$$

is differentiable on any $[-b, b]$. Since any $x \in \mathbb{R}$ is contained in some such interval, this implies that f is differentiable on all of $\mathbb{R}$ as claimed. ☐

5. Determine the radius of convergence of the following series, and the explicit function to which it converges on its interval of convergence.

$$\sum_{k=1}^{\infty} k 4^{2k} x^{4k}$$

Proof. Writing this series as

$$\sum_{k=1}^{\infty} k 4^{2k} x^{4k} = \sum_{n=1}^{\infty} a_n x^n,$$

we have

$$a_n = \begin{cases} k4^{2k} = \frac{n}{4} 4^{n/2} & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$|a_n|^{1/n} = \begin{cases} \frac{n^{1/n}}{4^{1/n}} 4^{1/2} & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

The value of $\limsup |a_n|^{1/n}$ can only come from the positive terms above. Since $n^{1/n} \to 1$ and $4^{1/n} \to 1$, we get that

$$\limsup |a_n|^{1/n} = 4^{1/2} = 2.$$

Thus the radius of convergence of the given series is $\frac{1}{2}$.

Since

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k \text{ for } |y| < 1,$$

differentiating gives

$$\frac{1}{(1-y)^2} = \sum_{k=1}^{\infty} k y^{k-1} \text{ for } |y| < 1.$$

Setting $y = 2x^4$, we have

$$\frac{1}{(1-16x^4)^2} = \sum_{k=1}^{\infty} k(16x^4)^{k-1} = \sum_{k=1}^{\infty} k4^{2k-2}x^{4k-4} \text{ for } |16x^4| < 1.$$

Multiplying through by $16x^4$ gives:

$$\frac{16x^4}{(1-16x^4)^2} = \sum_{k=1}^{\infty} k4^{2k}x^{4k} \text{ for } |x| < \frac{1}{2},$$

which is the desired explicit function. $\square$