Math 320-3: Midterm 1 Solutions
Northwestern University, Spring 2016

1. Give an example of each of the following. You do not have to justify your answer.

(a) A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(\mathbf{0})$ exists but $f_y(\mathbf{0})$ does not.

(b) An open $U \subseteq \mathbb{R}^2$ and non-constant differentiable $f : U \to \mathbb{R}$ such that $Df(\mathbf{x}) = 0$ for all $\mathbf{x}$.

(c) A differentiable $f : \mathbb{R} \to \mathbb{R}$ such that $u(x, y) = f(xy)$ has Jacobian $Du(x, y) = \begin{pmatrix} 2xy^2 & 2x^2y \end{pmatrix}$.

(d) A point (a, b) such that $f(x, y) = (x + y, x^2y^3)$ is invertible near (a, b).

Solution. (a) The function $f(x, y) = x + |y|$ works.

(b) Take U to be the union of $B_1(0, 0)$ and $B_1(5, 5)$, and define f to be 1 on $B_1(0, 0)$ and 2 on $B_1(5, 5)$. This is not constant but does have derivative zero everywhere since it is *locally* constant. Any valid example will have to involve a disconnected U.

(c) The chain rule gives

$$Du(x, y) = \begin{pmatrix} f'(xy)y & f'(xy)x \end{pmatrix}.$$

Thus we need $f'(xy) = 2xy$ in order to satisify the requirement, so $f(t) = t^2$ works.

(d) Since

$$Df(x, y) = \begin{pmatrix} 1 & 1 \\ 2xy^3 & 3x^2y^2 \end{pmatrix},$$

any point where $Df(x, y)$ is invertible, say $(1, 1)$ for instance, works by the Inverse Function Theorem. □

2. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined below is continuous but not differentiable at the origin.

$$f(x, y) = \begin{cases} 1 - 3x^2 + 4y + \frac{x^3y^2}{(x^2+y^2)^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$$

Proof. Since $|x|, |y| \leq \sqrt{x^2 + y^2}$, we have

$$\begin{aligned} |f(x, y) - 1| &= \left| -3x^2 + 4y + \frac{x^3y^2}{(x^2 + y^2)^2} \right| \\ &\leq 3|x|^2 + 4|y| + \frac{|x|^3|y|^2}{(x^2 + y^2)^2} \\ &\leq 3\sqrt{x^2 + y^2}^2 + 4\sqrt{x^2 + y^2} + \frac{\sqrt{x^2 + y^2}^5}{(x^2 + y^2)^2} \\ &= 8\sqrt{x^2 + y^2}. \end{aligned}$$

Thus for $\epsilon > 0$, $\delta = \frac{\epsilon}{8}$ satisfies

$$0 < \|\mathbf{x} - \mathbf{0}\| < \delta \text{ implies } |f(\mathbf{x}) - f(\mathbf{0})| < \epsilon,$$

so f is continuous at $\mathbf{0}$.

We have

$$f(x, 0) = -3x^2 \text{ for all } x \text{ and } f(0, y) = 4y \text{ for all } y.$$

Thus $\frac{\partial}{\partial x}(f(x,0)) = -6x$ exists, so $f_x(0,0) = 0$, and $\frac{\partial}{\partial y}(f(0,y)) = 4$ exists, so $f_y(0,0) = 4$. In order for f to be differential at $\mathbf{0}$, we would need

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{pmatrix}0 & 4\end{pmatrix}\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}.$$

With $\mathbf{h} = (h,k)$, we get

$$f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{pmatrix}0 & 4\end{pmatrix}\mathbf{h} = -3h^2 + \frac{h^3k^2}{(h^2+k^2)^2}.$$

After converting to polar coordinates, we get

$$\frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{pmatrix}0 & 4\end{pmatrix}\mathbf{h}}{\|\mathbf{h}\|} = -3r\cos^2\theta + \cos^3\theta\sin^2\theta.$$

The first term here as limit 0 as $r \to 0$ by the squeeze theorem, but the second term does not have a limit as $r \to 0$ since the limit depends on which value of θ we approach the origin along. Thus

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{pmatrix}0 & 4\end{pmatrix}\mathbf{h}}{\|\mathbf{h}\|}$$

does not exist, so f is not differentiable at $\mathbf{0}$. □

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function and A is an $m \times n$ matrix such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| + \|A\|\,\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|^2 \text{ for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^n.$$

Show that f has the form $f(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^m$. Hint: First show that $g(\mathbf{x}) = f(\mathbf{x}) - A\mathbf{x}$ satisfies $\|g(\mathbf{x}) - g(\mathbf{y})\| \leq \|\mathbf{x}-\mathbf{y}\|^2$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. What property of g is equivalent to required claim about f? Why does g have this property?

Proof. We have:

$$\begin{aligned}
\|g(\mathbf{x}) - g(\mathbf{y})\| &= \|f(\mathbf{x}) - A\mathbf{x} - (f(\mathbf{y}) - A\mathbf{y})\| \\
&= \|f(\mathbf{x}) - f(\mathbf{y}) - (A\mathbf{x} - A\mathbf{y})\| \\
&\leq \|f(\mathbf{x}) - f(\mathbf{y})\| + \|A(\mathbf{x}-\mathbf{y})\| \\
&\leq \|f(\mathbf{x}) - f(\mathbf{y})\| + \|A\|\,\|\mathbf{x}-\mathbf{y}\| \\
&= \|\mathbf{x}-\mathbf{y}\|^2.
\end{aligned}$$

Thus

$$\frac{\|g(\mathbf{x}) - g(\mathbf{y})\|}{\|\mathbf{x}-\mathbf{y}\|} \leq \|\mathbf{x}-\mathbf{y}\|,$$

so the squeeze theorem implies that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|g(\mathbf{x}) - g(\mathbf{y})\|}{\|\mathbf{x}-\mathbf{y}\|} = 0.$$

Hence

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{g(\mathbf{x}) - g(\mathbf{y}) - 0(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} = \mathbf{0},$$

so g is differentiable everywhere with Jacobian matrix 0 everywhere. Since $\mathbb{R}^n$ is connected, this implies that g is constant; say $g(\mathbf{x}) = \mathbf{b}$ for some $\mathbf{b}$ and all $\mathbf{x}$. Then

$$f(\mathbf{x}) - A\mathbf{x} = \mathbf{b}, \text{ so } f(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \text{ for all } \mathbf{x},$$

and thus f has the required form. □

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$. Show that for any $\mathbf{u} \in \mathbb{R}^m$, there exists $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$ such that

$$\mathbf{u} \cdot (f(\mathbf{x}) - f(\mathbf{a})) = \mathbf{u} \cdot [Df(\mathbf{c})(\mathbf{x} - \mathbf{a})],$$

where $\cdot$ denotes the usual dot product: $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$. Hint: Consider the single-variable function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = \mathbf{u} \cdot f(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))$.

Proof. The function $t \mapsto \mathbf{a} + t(\mathbf{x} - \mathbf{a})$ is differentiable for all t, so $h(t) = \mathbf{u} \cdot f(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))$ is as well. By the single-variable Mean Value Theorem, there exists $c \in (0, 1)$ such that

$$h(1) - h(0) = h'(c)(1 - 0),$$

which becomes

$$\mathbf{u} \cdot f(\mathbf{x}) - \mathbf{u} \cdot f(\mathbf{a}) = h'(c).$$

The chain rule applied to the composition of $t \mapsto a + t(\mathbf{x} - \mathbf{a})$ and $f(\mathbf{x})$ gives that the derivative of $f(\mathbf{a} + t(\mathbf{x} - \mathbf{a})$ with respect to t is

$$Df(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a}).$$

Setting $\mathbf{c} = \mathbf{a} + c(\mathbf{x} - \mathbf{a})$, we have that $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$ since $0 < c < 1$, and hence

$$\mathbf{u} \cdot f(\mathbf{x}) - \mathbf{u} \cdot f(\mathbf{a}) = h'(c) = \mathbf{u} \cdot [Df(\mathbf{c})(\mathbf{x} - \mathbf{a})]$$

as required. □

5. Let A be the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying

$$xyz + \sin(x + y + z) = 0.$$

(a) Show that there exists an open set $W \subseteq \mathbb{R}^2$ containing $(0, 0)$ and a differentiable function $g : W \to \mathbb{R}$ such that $(x, y, g(x, y)) \in A$ for all $(x, y) \in W$.

(b) Let B denote the set of all points satisfying

$$x^2 + y^4 - y + z = 0.$$

Note that $(0, 0, 0)$ is in the intersection of A and B. Show that near $(0, 0, 0)$ this intersection is a curve given by parametric equations of the form

$$x = x(t), \;\; y = y(t), \;\; z = t.$$

Proof. (a) Since

$$\frac{\partial g}{\partial z} = xy + \cos(x + y + z)$$

is nonzero at $z = 0$, the implicit function theorem implies that near $(0, 0)$ we can solve for z in terms of (x, y), or more precisely that there exists an open set W containing $(0, 0)$ and a C^1 function $g : W \to \mathbb{R}$ such that $(x, y, g(x, y)) \in A$ as claimed.

(b) Let $F(x, y, z) = (xyz + \sin(x + y + z), x^2 + y^4 - y + z)$. Note that $F(0, 0, 0) = (0, 0)$. We have

$$DF_{(x,y)} = \begin{pmatrix} yz + \cos(x + y + z) & xz + \cos(x + y + z) \\ 2x & 4y^3 - 1 \end{pmatrix},$$

so

$$DF_{(x,y)}(0,0,0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Since $DF_{(x,y)}(0,0,0)$ is invertible, the Implicit Function Theorem implies that near $(0,0,0)$ there exists C^1 function $x(z)$ and $y(z)$ such that

$$F(x(z), y(z), z) = (0,0).$$

This says that the parametric equations

$$x = x(t),\ \ y = y(t),\ \ z = t$$

describe the curve where the two surfaces intersect. $\square$