Math 320-1: Midterm 1 Solutions
Northwestern University, Fall 2019

1. Give an example of each of the following. You do not have to justify your answer.
(a) A subset of $\mathbb{R}$ with rational infimum and irrational supremum.
(b) A monotone sequence which does not converge.
(c) A Cauchy sequence whose terms are in the interval $(1, 5)$ but which does not converge to an element of this interval.

Solution. (a) The interval $(0, \pi)$ works.
(b) The sequence $1, 2, 3, 4, \ldots$ (defined by $x_n = n$) is increasing but does not converge.
(c) The sequence $x_n = 1 + \frac{1}{n}$ works. Note this converges to 1 in $\mathbb{R}$, but 1 is not in $(1, 5)$. □

2. Show that the supremum of the following set S is 3.

$$S = \left\{ \frac{3n+1}{n+\sqrt{n}} \;\middle|\; n \in \mathbb{N} \text{ and } n \geq 10 \right\}$$

Proof. Note: The $n \geq 10$ requirement in the definition of this set is leftover from a previous version of this problem, and is not actually important in this particular example.

First, for any $n \in \mathbb{N}$ we have:

$$\frac{3n+1}{n+\sqrt{n}} \leq \frac{3n+3\sqrt{n}}{n+\sqrt{n}} = 3,$$

so 3 is an upper bound of S. Now let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\epsilon^2}{9},$$

which is possible by the Archimedean Property of $\mathbb{R}$. Then

$$3 - \frac{3N+1}{N+\sqrt{N}} = \frac{3\sqrt{N}-1}{N+\sqrt{N}} < \frac{3\sqrt{N}}{N} = \frac{3}{\sqrt{N}} < \epsilon,$$

so

$$3 - \epsilon < \frac{3N+1}{N+\sqrt{N}}.$$

Thus $\frac{3N+1}{N+\sqrt{N}}$ is an element of S (simply make N larger if need be if you really want to satisfy $N \geq 10$) which is larger than $3 - \epsilon$, so $3 - \epsilon$ is not an upper bound of S, and hence $3 = \sup S$ is the least upper bound as claimed. □

3. Suppose (x_n) is a sequence which converges to 2. Show, using the precise definition of convergence, that the sequence $(\frac{1}{x_n^2})$ converges to $\frac{1}{4}$. Hint: Figure out how to bound $|\frac{1}{x_n^2} - \frac{1}{4}|$ by a constant times $|x_n - 2|$, for large enough n.

Proof. Let $\epsilon > 0$. Since $x_n \to 2$, there exists $N \in \mathbb{N}$ such that

$$|x_n - 2| < \min\left\{1, \frac{4}{5}\epsilon\right\} \text{ for } n \geq N.$$

In particular, $|x_n - 2| < 1$, so $x_n \in (2-1, 2+1) = (1,3)$ for $n \geq N$. Thus 1 is a lower bound on these x_n and 3 is an upper bound.

Thus for $n \geq N$, we have:

$$\left|\frac{1}{x_n^2} - \frac{1}{4}\right| = \frac{|x_n+2||x_n-2|}{4|x_n|^2} \leq \frac{(|x_n|+2)|x_n-2|}{4|x_n|^2} \leq \frac{5|x_n-2|}{4} < \epsilon.$$

To be clear, in the second inequality we used the fact that $1 \leq x_n$ in order to bound $\frac{1}{4|x_n|^2}$ by $\frac{1}{4}$. This shows that $\frac{1}{x_n^2} \to \frac{1}{4}$ as claimed. $\square$

4. Suppose (x_n) is a convergent sequence. Show that the sequence (y_n) defined by

$$y_n = 4x_n + \frac{4\sin(n^2) - 3 + n^2\cos(n+1)}{4n^2 - n}$$

has a convergent subsequence.

Proof. By the Bolzano-Weierstrass Theorem, it suffices to show that (y_n) is bounded. Since (x_n) converges, it is bounded, say by M. Then we have:

$$\begin{aligned}
|y_n| &\leq 4|x_n| + \frac{|4\sin(n^2) - 3 + n^2\cos(n+1)|}{|4n^2 - n|} \\
&\leq 4M + \frac{4|\sin n^2| + 3 + n^2|\cos(n+1)|}{4n^2 - n} \\
&\leq 4M + \frac{4+3+n^2}{4n^2 - n} \\
&\leq 4M + \frac{8n^2}{3n^2} \\
&= 4M + \frac{8}{3}.
\end{aligned}$$

To be clear, in the second line we used the triangle inequality in the numerator of the second term, an in the fourth line we bounded $7+n^2$ by $7n^2+n^2 = 8n^2$ and $4n^2-n$ from below by $4n^2-n^2 = 3n^2$. Since (y_n) is bounded, it has a convergent subsequence. $\square$

5. Suppose (x_n) is a sequence such that $x_n < 5$ for all $n \geq 100$. If (x_n) converges to x, show that $x \leq 5$. Hint: Show that $x > 5$ is not possible. (You cannot simply quote the "comparison theorem" from the book. The point is to give a proof of a special case of that theorem.)

Proof. By way of contradiction, suppose $x > 5$. Then $x - 5 > 0$, so since $x_n \to x$ there exists $N \in \mathbb{N}$ such that

$$|x - x_n| < x - 5 \text{ for } n \geq N.$$

This implies that $x - x_n < x - 5$, so $5 < x_n$ for $n \geq N$. In particular, picking $n \geq \max\{N, 100\}$ gives a term for which $x_n < 5$ is not true, which is a contradiction. Thus $x \leq 5$. $\square$