Math 320-1: Midterm 1 Solutions
Northwestern University, Fall 2015

1. Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{R}\backslash\mathbb{Q}$ with a rational infimum and irrational supremum.

(b) A sequence which has no convergent subsequence.

(c) A sequence (x_n) which does not converge but for which $(|x_n|)$ does converge.

(d) A Cauchy sequence (x_n) whose terms are in $\mathbb{Q}$ which does not have a limit in $\mathbb{Q}$.

Solutions. (a) The set $\{x \in \mathbb{R}\backslash\mathbb{Q} \mid 2 < x < \pi\}$ as infimum 2 and supremum π.

(b) The sequence $a_n = n$ has no convergent subsequence since every subsequence is unbounded.

(c) The sequence $a_n = (-1)^n$ does not converge but $|a_n| = 1$ does converge.

(d) The sequence defined recursively by $a_1 = 1,\ a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geq 1$ consists of rational numbers and is Cauchy in $\mathbb{Q}$ since it converges in $\mathbb{R}$ (as we saw on the homework), but its limit in $\mathbb{R}$ is $\frac{1+\sqrt{5}}{2}$, which is not in $\mathbb{Q}$. ☐

2. Determine the supremum of the following set and prove that your answer is correct.

$$\left\{ \frac{2n^3 - 4n^2}{n^3 - n^2 + 1} \;\middle|\; n \in \mathbb{N} \right\}$$

Solution. We claim the supremum is 2. Indeed,

$$2n^3 - 4n^2 \leq 2n^3 - 2n^2 + 2 \text{ for all } n \in \mathbb{N}$$

since $4n^2 \geq 2n^2 - 2$, so dividing both sides by $n^3 - n^2 + 1 > 0$ gives

$$\frac{2n^3 - 4n^2}{n^3 - n^2 + 1} \leq 2 \text{ for all } n \in \mathbb{N},$$

showing that 2 is an upper bound of the given set.

To see that 2 is the least upper bound, let $\epsilon > 0$ and pick $n \in \mathbb{N}$ such that $\frac{4}{n-1} < \epsilon$. Then

$$2 - \frac{2n^3 - 4n^2}{n^3 - n^2 + 1} = \frac{2n^2 + 2}{n^3 - n^2 + 1} \leq \frac{2n^2 + 2n^2}{n^3 - n^2} = \frac{4}{n-1} < \epsilon,$$

so

$$2 - \epsilon < \frac{2n^3 - 4n^2}{n^3 - n^2 + 1}.$$

Thus $2 - \epsilon$ is not an upper bound of the given set, and since $\epsilon > 0$ was arbitrary, nothing smaller than 2 is an upper bound so 2 is the least upper bound as claimed. ☐

3. Suppose $x_n \to x$ and $y_n \to y$. Using the fact that

$$x_n y_n - xy = x_n y_n - x_n y + x_n y - xy,$$

show that $x_n y_n \to xy$.

Proof. Let $\epsilon > 0$. Since (x_n) converges, it is bounded, say by $M > 0$. Pick $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2(|y|+1)} \text{ for } n \geq N_1$$

and pick $N_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\epsilon}{2M} \text{ for } n \geq N_2.$$

Then for $n \geq \max\{N_1, N_2\}$, we have:

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&\leq |x_n||y_n - y| + |x_n - x||y| \\
&\leq M|y_n - y| + |x_n - x|(|y|+1) \\
&< M\frac{\epsilon}{2M} + \frac{\epsilon}{2(|y|+1)}(|y|+1) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $x_n y_n \to xy$ as claimed. □

4. Show that the sequence (x_n) defined by

$$x_n = \frac{3^n}{4^n}$$

is monotone and bounded, and that it converges to 0. (When showing $x_n \to 0$ you cannot just quote the fact that $a^n \to 0$ when $|a| < 1$; you must prove that this is true in this particular case.) Hint: What is the relation between x_{n+1} and x_n?

Proof. For any $n \geq 1$, we have

$$x_{n+1} = \frac{3^{n+1}}{4^{n+1}} = \frac{3}{4}\frac{3^n}{4^n} = \frac{3}{4}x_n < x_n,$$

so (x_n) is decreasing. Furthermore,

$$|x_n| = \frac{3^n}{4^n} \leq \frac{4^n}{4^n} = 1,$$

so (x_n) is bounded. Hence (x_n) converges—call its limit L. Since

$$x_{n+1} = \frac{3}{4}x_n$$

and x_{n+1} also converges to L (because it is a subsequence of (x_n)), taking limits in this equation gives

$$L = \frac{3}{4}L,$$

which implies that $L = 0$. Hence $x_n \to 0$ as claimed. □

5. Suppose that (x_n) is a convergent sequence and that (y_n) is a sequence such that

$$|y_m - y_n| \leq \frac{4}{m+n}|x_m - x_n|^3 \text{ for all } m, n \in \mathbb{N}.$$

Show that (y_n) converges.

Proof. Let $\epsilon > 0$. Since (x_n) is convergent, it is Cauchy so there exists $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \sqrt[3]{\frac{\epsilon}{4}} \text{ for } m, n \geq N.$$

Thus for $m, n \geq N$ we have:

$$|y_m - y_n| \leq \frac{4}{m+n}|x_m - x_n|^3 \leq 4|x_m - x_n|^3 < 4\left(\sqrt[3]{\frac{\epsilon}{4}}\right)^3 = \epsilon,$$

so (y_n) is Cauchy and hence converges. □