Math 320-1: Midterm 1
Northwestern University, Fall 2015

Name: _____

1. (10 points) Give an example of each of the following. You do not have to justify your answer.
(a) A subset of $\mathbb{R}\backslash\mathbb{Q}$ with a rational infimum and irrational supremum.
(b) A sequence which has no convergent subsequence.
(c) A sequence (x_n) which does not converge but for which $(|x_n|)$ does converge.
(d) A Cauchy sequence (x_n) whose terms are in $\mathbb{Q}$ which does not have a limit in $\mathbb{Q}$.

2. (10 points) Determine the supremum of the following set and prove that your answer is correct.

$$\left\{ \frac{2n^3 - 4n^2}{n^3 - n^2 + 1} \;\middle|\; n \in \mathbb{N} \right\}$$

3. (10 points) Suppose $x_n \to x$ and $y_n \to y$. Using the fact that

$$x_n y_n - xy = x_n y_n - x_n y + x_n y - xy,$$

show that $x_n y_n \to xy$.

4. (10 points) Show that the sequence (x_n) defined by

$$x_n = \frac{3^n}{4^n}$$

is monotone and bounded, and that it converges to 0. (When showing $x_n \to 0$ you cannot just quote the fact that $a^n \to 0$ when $|a| < 1$; you must prove that this is true in this particular case.) Hint: What is the relation between x_{n+1} and x_n?

5. (10 points) Suppose that (x_n) is a convergent sequence and that (y_n) is a sequence such that

$$|y_m - y_n| \leq \frac{4}{m+n}|x_m - x_n|^3 \text{ for all } m, n \in \mathbb{N}.$$

Show that (y_n) converges.