Math 320-2: Final Exam
Northwestern University, Winter 2020

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A convergent series $\sum a_n$ which is not guaranteed to converge after rearranging terms.

(b) A sequence of functions on $[0, 2]$ which converges pointwise but not uniformly.

(c) A function which is analytic on all of $\mathbb{R}$.

(d) A subset of $\mathbb{Q}$ which is closed and open with respect to the standard metric.

(e) A nonempty compact subset of $C[0, 1]$ with respect to the sup metric.

Problem	Score
1	
2	
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Total	

2. (10 points) Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions which converges uniformly to $f : \mathbb{R} \to \mathbb{R}$. Suppose also that $\lim_{x \to 0} f_n(x)$ exists and is the same for all n. Call this common limit L, and show that

$$\lim_{x \to 0} f(x) = L.$$

3. (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is analytic and $f'(\frac{1}{n}) = \cos(\frac{1}{n})$ for all $n \in \mathbb{N}$. If $f(0) = 1$, show that $f(x) = 1 + \sin x$ for all $x \in \mathbb{R}$. Hint: f' is also analytic.

4. (10 points) Suppose X is a metric space and $p \in X$. Show that the set

$$U := \{q \in X \mid d(q,p) > 3\}$$

is open in X by verifying the definition of "open" in terms of open balls directly. Hint: Draw a picture!

5. (10 points) Suppose A and B are compact subsets of a metric space X. Show that their intersection $A \cap B$ is compact as well.

6. (10 points) Let $K := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 8 \text{ and } x^2 + y^2 + z^2 = 100\}$. Show that there is a point in K which is closer to $(0, 0, 0)$ than any other point of K, and a point in K which farther from $(0, 0, 0)$ than any other point of K. Hint: The function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ which computes distance away from $(0, 0, 0)$ is continuous.

7. (10 points) Recall that $C[a,b]$ denotes the space of continuous functions $[a,b] \to \mathbb{R}$ equipped with the sup metric. Define $T : C[-10,10] \to C[-3,2]$ by

$$(Tf)(x) = f(x) + \int_0^{x^2+1} f(t)^2 \, dt.$$

(To be clear, T sends a function $f \in C[-10,10]$ to the function $Tf \in C[-3,2]$ whose value at x is the given expression.) Show that T is continuous.