Math 320-2: Final Exam
Northwestern University, Winter 2016

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A convergent series $\sum a_n$ of numbers such that $\sum a_n^2$ diverges.
(b) A sequence of functions on $\mathbb{R}$ which converges uniformly on $[0, \frac{1}{2}]$ but not on $[0, 1]$.
(c) A metric on $\mathbb{R}$ relative to which $\mathbb{Q}$ is bounded.
(d) Two connected subsets A, B of $\mathbb{R}^2$ such that $A \cap B$ is disconnected.
(e) A nonempty compact subset of $\mathbb{Q}$ with respect to the Euclidean metric.

Problem	Score
1	
2	
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Total	

2. (10 points) Suppose (f_n) is a sequence of continuous functions on $\mathbb{R}$ which converges uniformly to a function f. If (x_n) is a sequence in $\mathbb{R}$ which converges to x, show that the sequence $(f_n(x_n))$ in $\mathbb{R}$ converges to $f(x)$. (To be clear, $(f_n(x_n))$ is the sequence of numbers whose n-th term is what you get when you evaluate f_n at x_n.) Hint:

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)|$$

3. (10 points) Show that the following series converges uniformly on any compact subset of $\mathbb{R}$.

$$\sum_{n=1}^{\infty} \frac{e^x}{n} \sin\left(\frac{x}{n}\right)$$

4. (10 points) Show that the following subset S of $\mathbb{R}^3$ is closed in $\mathbb{R}^3$ with respect to the Euclidean metric.

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + xy + \sin(xyz) = 1\}$$

5. (10 points) Suppose (X, d) is a metric space and that K and L are compact subsets of X. Show that the union $K \cup L$ is compact as well.

6. (10 points) Suppose (X, d_X) and (Y, d_Y) are metric spaces and that $f, g : X \to Y$ are both continuous functions. If A is a dense subset of X such that

$$f(a) = g(a) \text{ for all } a \in A,$$

show that $f(x) = g(x)$ for all $x \in X$. (This says that continuous functions which agree on a dense set must be the same.)

7. (10 points) Recall that $C[a,b]$ denotes the space of continuous functions $[a,b] \to \mathbb{R}$ equipped with the sup metric. Define $T : C[0,5] \to C[0,2]$ by

$$(Tf)(x) = 3 + \int_0^{x^2+1} (f(t) + 2e^{\cos t})\, dt.$$

(To be clear, T sends a function $f \in C[0,5]$ to the function $Tf \in C[0,2]$ whose value at x is the given expression.) Show that T is continuous. Hint: Figure out how to relate $d(Tf, Tg)$ and $d(f, g)$.