Math 320-3: Final Exam
Northwestern University, Spring 2016

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(\mathbf{0})$ exists but $f_{xx}(\mathbf{0})$ does not.

(b) A function $f : \mathbb{R}^2 \to \mathbb{R}$ where one iterated integral exists but the other does not.

(c) A C^1 surface in $\mathbb{R}^3$ which is not smooth at $(1, 0, 0)$.

(d) A curve C over which the integral of $\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is nonzero.

(e) A compact subset $K \subseteq \mathbb{R}^3$ which does not satisfy the assumptions of the Mean Value Inequality for functions $K \to \mathbb{R}^3$.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in \mathbb{R}^n$. If $f(\mathbf{x}_0) \neq \mathbf{0}$, show that there exists a ball $B_\delta(\mathbf{x}_0)$ around $\mathbf{x}_0$ on which f is nonzero.

3. (10 points) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = \begin{cases} x + 2y + 3z + \frac{x^2yz}{x^2+y^2+z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0). \end{cases}$$

Show that f is differentiable at $(1, 1, 1)$ and at $(0, 0, 0)$. (Only one of these should involve an actual limit computation.)

4. (10 points) Suppose C is a smooth, C^1 curve in $\mathbb{R}^2$. Show that C has Jordan measure zero. Hint: Argue that locally near each point of C you can express C as the graph of a continuous function, and then argue that such graphs have Jordan measure zero.

5. (10 points) Suppose a smooth, C^1 curve C in $\mathbb{R}^n$ has parametrization $\mathbf{x}(t),\ a \le t \le b$, and that $\mathbf{y}(u),\ c \le u \le d$ is another parametrization of C such that $\mathbf{y} = \mathbf{x} \circ \tau$ for some bijective, C^1 function $\tau : [c, d] \to [a, b]$ with nonzero derivative everywhere. If the tangent vectors determined by $\mathbf{x}$ and by $\mathbf{y}$ point in the same direction at each point, show that

$$\int_c^d \mathbf{F}(\mathbf{y}(u)) \cdot \mathbf{y}'(u)\, du = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

for any C^1 vector field $\mathbf{F}$ on C. (In other words, show that vector line integrals are independent of parametrization as long as the parametrizations in question induce the same orientation.)

6. (10 points) Let $\mathbf{F} : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right).$$

If S_1 and S_2 are two smooth, closed C^1 surfaces in $\mathbb{R}^3$ which each enclose the origin and are each oriented with outward-pointing normal vectors, show that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(There was a problem on the final practice problems which would be applicable, except that here you cannot simply quote the result of that problem. The point is to justify this result in the case of this specific vector field.) Hint: Compute the divergence of $\mathbf{F}$.

7. (10 points) Suppose $E \subseteq \mathbb{R}^2$ is Jordan measurable and that $f : E \to \mathbb{R}$ is integrable. Recall that, by definition, this means that if R is a rectangle containing E, the extended function

$$f_E^{ext}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in E \\ 0 & \mathbf{x} \notin E \end{cases}$$

is integrable over R. If $A \subseteq E$ is also Jordan measurable, show that f is integrable on A, which means that the extended function

$$f_A^{ext}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in A \\ 0 & \mathbf{x} \notin A \end{cases}$$

is integrable over R. **Careful: f_A^{ext} can be zero at points where f_E^{ext} was nonzero.** You can use without justification the fact that upper and lower integrals can be approximated to whatever degree of accuracy we want using small rectangles contained *fully* within the region of integration, as opposed to ones which only intersect that region.