Math 320-2: Final Exam Solutions
Northwestern University, Winter 2016

1. Give an example of each of the following. You do not have to justify your answer.

(a) A convergent series $\sum a_n$ of numbers such that $\sum a_n^2$ diverges.

(b) A sequence of functions on $\mathbb{R}$ which converges uniformly on $[0, \frac{1}{2}]$ but not on $[0, 1]$.

(c) A metric on $\mathbb{R}$ relative to which $\mathbb{Q}$ is bounded.

(d) Two connected subsets A, B of $\mathbb{R}^2$ such that $A \cap B$ is disconnected.

(e) A nonempty compact subset of $\mathbb{Q}$ with respect to the Euclidean metric.

Solution. (a) The series $\sum (-1)^n \frac{1}{\sqrt{n}}$ works: this converges but $\sum \frac{1}{n}$ does not.

(b) The sequence $f_n(x) = x^n$ works: this converges uniformly to 0 on $[0, \frac{1}{2}]$ but not on $[0, 1]$ since the pointwise limit on $[0, 1]$ is not continuous.

(c) The discrete metric works: $\mathbb{Q}$ is contained in $B_2(0)$.

(d) This problem was omitted since the original version was nonsense: it had $\mathbb{R}$ instead of $\mathbb{R}^2$. In $\mathbb{R}^2$ you can draw pictures of possible A and B, say A the unit square $[0, 1] \times [0, 1]$ with the bottom edge removed and B the unit square with the top edge removed. These are both connected but their intersection is the union of the two vertical edges of the unit square, which is not connected.

(e) Any set consisting of a single element works. □

2. Suppose (f_n) is a sequence of continuous functions on $\mathbb{R}$ which converges uniformly to a function f. If (x_n) is a sequence in $\mathbb{R}$ which converges to x, show that the sequence $(f_n(x_n))$ in $\mathbb{R}$ converges to $f(x)$. (To be clear, $(f_n(x_n))$ is the sequence of numbers whose n-th term is what you get when you evaluate f_n at x_n.) Hint:

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)|$$

Proof. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ for } n \geq N \text{ and all } x \in \mathbb{R}.$$

In particular, taking such an x to be x_n itself, we get

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ for } n \geq N.$$

Since each f_n is continuous, the uniform limit f is continuous as well, so since $x_n \to x$ we must have $f(x_n) \to f(x)$. Thus there exists $M \in \mathbb{N}$ such that

$$|f(x_n) - f(x)| < \frac{\epsilon}{2} \text{ for } n \geq M.$$

Hence if $n \geq \max\{N, M\}$, we get:

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $f_n(x_n) \to f(x)$ as claimed. □

3. Show that the following series converges uniformly on any compact subset of $\mathbb{R}$.

$$\sum_{n=1}^{\infty} \frac{e^x}{n} \sin\left(\frac{x}{n}\right)$$

Proof. Let K be a compact subset of $\mathbb{R}$. Then K is bounded, say by $M > 0$. Then for $x \in K$ we have:

$$\left| \frac{e^x}{n} \sin\left(\frac{x}{n}\right) \right| = \left| \frac{e^x}{n} \right| \left| \sin \frac{x}{n} \right| \le \frac{e^M}{n} \left| \frac{x}{n} \right| \le \frac{Me^M}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{Me^M}{n^2}$ does as well since Me^M is a constant, so the Weierstrass M-test implies that the series in question converges uniformly on K as desired. □

4. Show that the following subset S of $\mathbb{R}^3$ is closed in $\mathbb{R}^3$ with respect to the Euclidean metric.

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + xy + \sin(xyz) = 1\}$$

Proof 1. Let (x_n, y_n, z_n) be a sequence in S and suppose $(x_n, y_n, z_n) \to (x, y, z) \in \mathbb{R}$. Then since converges in $\mathbb{R}^3$ with respect to the Euclidean metric is equivalent to converges of each individual component sequence, we have

$$x_n \to x, \ y_n \to y, \text{ and } z_n \to z.$$

Since the limit of a product of converges sequences in $\mathbb{R}$ is the product of the individual limits, we get

$$x_n^2 \to x^2, \ x_n y_n \to xy, \text{ and } x_n y_n z_n \to xyz.$$

Since the single variable sine function is continuous, we then get $\sin(x_n y_n z_n) \to \sin(xyz)$. Thus

$$x_n^2 + x_n y_n + \sin(x_n y_n z_n) \to x^2 + xy + \sin(xyz),$$

and since the left-hand side is the constant sequence 1 (since $(x_n, y_n, z_n) \in S$), we must have that the right-hand side is 1 as well. Thus $(x, y, z) \in S$, so S is closed in $\mathbb{R}^3$. □

Proof 2. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + xy + \sin(xyz)$$

is continuous since it is a sum of products and compositions of continuous functions. Thus the preimage of the closed set $\{1\} \subseteq \mathbb{R}$ is closed in $\mathbb{R}^3$, and this preimage is precisely S. □

5. Suppose (X, d) is a metric space and that K and L are compact subsets of X. Show that the union $K \cup L$ is compact as well.

Proof 1. Let $\{U_\alpha\}$ be an open cover of $K \cup L$. Since $K \subseteq K \cup L$ and $L \subseteq K \cup L$, this same cover is then also an open cover of both K and L. Considering it as a cover of K, the compactness of K implies that there is a finite subcover:

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n},$$

and considering it as a cover of L, the compactness L gives a finite subcover

$$L \subseteq U_{\beta_1} \cup \cdots \cup U_{\beta_m}.$$

Then

$$K \cup L \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup U_{\beta_1} \cup \cdots \cup U_{\beta_m},$$

so $U_{\alpha_1}, \ldots, U_{\alpha_n}, U_{\beta_1}, \cdots, U_{\beta_m}$ is a finite subcover of the original open cover of $K \cup L$. Thus any open cover of $K \cup L$ has a finite subcover, so $K \cup L$ is compact. □

Proof 2. Let (x_n) be a sequence in $K \cup L$. If only finitely many terms $x_{n_1}, \ldots, x_{n_k}$ were in K and only finitely many $x_{m_1}, \ldots, x_{m_\ell}$ were in L, then

$$x_{n_1}, \ldots, x_{n_k}, x_{m_1}, \ldots, x_{m_\ell}$$

would be the only terms in the sequence in $K \cup L$, which is nonsense since every term of (x_n) is supposed to be in $K \cup L$ and a sequence contains infinitely many terms. Thus at least one of K or L contains infinitely many terms from the given sequence.

Without loss of generality, suppose K contains infinitely many of these terms and call them (x_{n_k}). Since K is compact, this sequence (x_{n_k}) in K has a convergent subsequence, say $x_{n_{k_\ell}}$ converges to some $x \in K$. Then $x_{n_{k_\ell}}$ is a convergent subsequence of the original sequence (x_n) converging to $x \in K \cup L$, so $K \cup L$ is compact. □

6. Suppose (X, d_X) and (Y, d_Y) are metric spaces and that $f, g : X \to Y$ are both continuous functions. If A is a dense subset of X such that

$$f(a) = g(a) \text{ for all } a \in A,$$

show that $f(x) = g(x)$ for all $x \in X$. (This says that continuous functions which agree on a dense set must be the same.)

Proof. Let $x \in X$. Since A is dense in X, there exists a sequence $a_n \to x$ with all $a_n \in A$. Since f and g are each continuous,

$$f(a_n) \to f(x) \quad \text{and} \quad g(a_n) \to g(x).$$

But $f(a_n) = g(a_n)$ for all n, so the sequences above are the same sequence, and hence their limits must be the same. Thus $f(x) = g(x)$ as claimed. □

7. Recall that $C[a, b]$ denotes the space of continuous functions $[a, b] \to \mathbb{R}$ equipped with the sup metric. Define $T : C[0, 5] \to C[0, 2]$ by

$$(Tf)(x) = 3 + \int_0^{x^2+1} (f(t) + 2e^{\cos t})\, dt.$$

(To be clear, T sends a function $f \in C[0, 5]$ to the function $Tf \in C[0, 2]$ whose value at x is the given expression.) Show that T is continuous. Hint: Figure out how to relate $d(Tf, Tg)$ and $d(f, g)$.

Proof. Note that here I made a slight change to the problem, which does not actually affect the solution at all. In the original version the domain of T was $C[0, 2]$, but in the given integral definition we actually need f to be defined on $[0, 5]$ since the largest upper limit of integration will be 5 when $x = 2$. Thus the domain of T should actually consist of functions defined on $[0, 5]$.

Let $f, g \in C[0, 5]$. For any $x \in [0, 2]$ we have:

$$\begin{aligned}
|(Tf)(x) - (Tg)(x)| &= \left| 3 + \int_0^{x^2+1} (f(t) + 2e^{\cos t})\, dt - 3 - \int_0^{x^2+1} (g(t) + 2e^{\cos t})\, dt \right| \\
&= \left| \int_0^{x^2+1} (f(t) - g(t))\, dt \right| \\
&= \int_0^{x^2+1} |f(t) - g(t)|\, dt
\end{aligned}$$

$$\leq \int_0^{x^2+1} d(f,g)\, dt,$$

where in the last step we use the fact that $|f(t) - g(t)| \leq \sup_{y \in [0,5]} |f(y) - g(y)|$ for any $t \in [0,5]$. Thus

$$|(Tf)(x) - (Tg)(x)| \leq \int_0^{x^2+1} d(f,g)\, dt = (x^2+1)d(f,g) \leq 5d(f,g).$$

Thus the supremum of the terms on the left is also bounded by $5d(f,g)$, so we get

$$d(Tf, Tg) \leq 5d(f,g).$$

Hence for $\epsilon > 0$, setting $\delta = \frac{\epsilon}{5}$ gives that

$$d(f,g) < \delta \text{ implies } d(Tf,Tg) \leq 5d(f,g) < 5\delta = \epsilon,$$

which shows that T is (uniformly) continuous as claimed. $\square$