Math 320-3: Final Exam Solutions
Northwestern University, Spring 2016

1. Give an example of each of the following. You do not have to justify your answer.

(a) A function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(\mathbf{0})$ exists but $f_{xx}(\mathbf{0})$ does not.

(b) A function $f : \mathbb{R}^2 \to \mathbb{R}$ where one iterated integral exists but the other does not.

(c) A C^1 surface in $\mathbb{R}^3$ which is not smooth at $(1, 0, 0)$.

(d) A curve C over which the integral of $\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is nonzero.

(e) A compact subset $K \subseteq \mathbb{R}^3$ which does not satisfy the assumptions of the Mean Value Inequality for functions $K \to \mathbb{R}^3$.

Solution. (a) The function defined by $f(x, y) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0, y) = 0$ works. We saw back in the fall that this function is differentiable with respect to x, but not twice differentiable.

(b) The function defined by $f(x, y) = \frac{y}{x^3}$ if $x > 0$ and $-x < y < x$ and $f(x, y) = 0$ otherwise on the rectangle $[0, 1] \times [-1, 1]$ works. For fixed $x > 0$, we have

$$\int_{-1}^{1} f(x, y)\, dy = \int_{-x}^{x} \frac{y}{x^3}\, dy = 0$$

and $\int_{-1}^{1} f(0, y)\, dy = \int_{-1}^{1} 0\, dy = 0$ as well. Thus the inner integral in $\int_0^1 \int_{-1}^1 f(x, y)\, dy\, dx$ is always zero, so this iterated integral exists and equals 0. For fixed $y \in [0, 1]$ we get:

$$\int_0^1 f(x, y)\, dx = \int_y^1 \frac{y}{x^3}\, dx = -\frac{1}{2}(y - \tfrac{1}{y})$$

and similarly for $y \in [-1, 0]$ we get

$$\int_0^1 f(x, y)\, dx = \int_{-y}^1 \frac{y}{x^3}\, dx = -\frac{1}{2}(y - \tfrac{1}{y}).$$

But these resulting functions are unbounded on $[-1, 1]$ and hence not integrable, so the inner integral in $\int_{-1}^1 \int_0^1 f(x, y)\, dx\, dy$ never exists, and hence neither does this iterated integral.

(c) ***TO BE FINISHED***

(d) ***TO BE FINISHED***

(e) Any compact but non-convex subset will work, such as the region between two spheres defined by $1 \leq \|(x, y, z)\| \leq 2$. □

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in \mathbb{R}^n$. If $f(\mathbf{x}_0) \neq \mathbf{0}$, show that there exists a ball $B_\delta(\mathbf{x}_0)$ around $\mathbf{x}_0$ on which f is nonzero.

Proof. Since $f(\mathbf{x}_0) \neq \mathbf{0}$, $\|f(\mathbf{x}_0)\| > 0$. Since f is continuous at $\mathbf{x}_0$, there exists $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \quad \text{implies} \quad \|f(\mathbf{x}_0) - f(\mathbf{x})\| < \|f(\mathbf{x}_0)\|.$$

By the reverse triangle inequality, this gives

$$\|f(\mathbf{x}_0)\| - \|f(\mathbf{x})\| < \|f(\mathbf{x}_0)\|, \text{ so } 0 = \|f(\mathbf{x}_0)\| - \|f(\mathbf{x}_0)\| < \|f(\mathbf{x})\|$$

for $\mathbf{x} \in B_\delta(\mathbf{x}_0)$. Hence f is nonzero on $B_\delta(\mathbf{x}_0)$. □

3. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = \begin{cases} x + 2y + 3z + \frac{x^2yz}{x^2+y^2+z^2} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0). \end{cases}$$

Show that f is differentiable at $(1,1,1)$ and at $(0,0,0)$. (Only one of these should involve an actual limit computation.)

Proof. On some ball around $(1,1,1)$ the function f has the same values as the function $g(x,y,z) = x+2y+3z+\frac{x^2yz}{x^2+y^2+z^2}$. Thus since g is differentiable at $(1,1,1)$—in particular because the fraction used has differentiable numerator and denominator with nonzero denominator near $(1,1,1)$—f is differentiable at $(1,1,1)$ too.

Now, we have:

$$f(x,0,0) = x \qquad f(0,y,0) = 2y \qquad f(0,0,z) = 3z,$$

so $f_x(0,0,0) = 1$, $f_y(0,0,0) = 2$, and $f_z(0,0,0) = 3$. The Jacobian matrix of f at $(0,0,0)$ is thus $Df(0,0,0) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, so we must verify that

$$\lim_{(h,k,\ell)\to(0,0,0)} \frac{f(h,k,\ell) - f(0,0,0) - \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \\ \ell \end{bmatrix}}{\|(h,k,\ell)\|} = 0.$$

The numerator of this quotient becomes:

$$h + 2k + 3\ell + \frac{h^2k\ell}{\|(h,k,\ell)\|^2} - h - 2k - 3\ell = \frac{h^2k\ell}{\|(h,k,\ell)\|^2},$$

so we are considering the limit of

$$\frac{h^2k\ell}{\|(h,k,\ell)\|^3}.$$

Since $|h|, |k|, |\ell|$ are all at most $\|(h,k,\ell)\|$, we have:

$$\frac{|h^2k\ell|}{\|(h,k,\ell)\|^3} \leq \frac{\|(h,k,\ell)\|^4}{\|(h,k,\ell)\|^3} = \|(h,k,\ell)\|,$$

so the squeeze theorem implies that

$$\lim_{(h,k,\ell)\to(0,0,0)} \frac{h^2k\ell}{\|(h,k,\ell)\|^3} = 0.$$

Hence f is differentiable at $(0,0,0)$ as claimed. $\square$

4. Suppose C is a smooth, C^1 curve in $\mathbb{R}^2$. Show that C has Jordan measure zero. Hint: Argue that locally near each point of C you can express C as the graph of a continuous function, and then argue that such graphs have Jordan measure zero.

Proof. Let $\phi : [a,b] \to \mathbb{R}^2$ be a smooth C^1 parametrization of C. Then at any $t \in [a,b]$ we have $\phi(t) = (\phi_1(t), \phi_2(t)) \neq (0,0)$, so at least one of $\phi_1(t)$ or $\phi_2(t)$ is nonzero. If it is the former, the Inverse Function Theorem implies that ϕ_1 is invertible with C^1 inverse near t, while if the latter we get that ϕ_2 is invertible with C^1 inverse near t. Thus either

$$y = \phi_2(t) = (\phi_2 \circ \phi_1^{-1})(x) \text{ or } x = (\phi_1 \circ \phi_2^{-1})(y)$$

near $\phi(t) \in C$. The upshot is that near any point on C we can express C as the graph of a C^1 function.

To be clear, for any $p \in C$ there exists an "interval" I_p on C (if $p = \phi(t')$, I_p is the image under ϕ an some interval around t' in $[a,b]$) such that I_p is the graph of a continuous function. Since C is compact (it is closed and bounded in $\mathbb{R}^2$), we can cover C using finitely many such intervals $I_{p_1}, \ldots, I_{p_n}$. We showed in class and on the homework that the graph of a continuous function has Jordan measure zero (the difference $U(f,P) - L(f,P)$ upper and lower sums can be made arbitrarily small since a continuous function is integrable, and this difference is precisely an outer sum $V(I_p, G)$ for some grid), so each I_{p_i} has Jordan measure zero. Hence C, being the union of a finite number of sets of Jordan measure zero, has Jordan measure zero itself. $\square$

5. Suppose a smooth, C^1 curve C in $\mathbb{R}^n$ has parametrization $\mathbf{x}(t)$, $a \le t \le b$, and that $\mathbf{y}(u)$, $c \le u \le d$ is another parametrization of C such that $\mathbf{y} = \mathbf{x} \circ \tau$ for some bijective, C^1 function $\tau : [c,d] \to [a,b]$ with nonzero derivative everywhere. If the tangent vectors determined by $\mathbf{x}$ and by $\mathbf{y}$ point in the same direction at each point, show that

$$\int_c^d \mathbf{F}(\mathbf{y}(u)) \cdot \mathbf{y}'(u)\, du = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

for any C^1 vector field $\mathbf{F}$ on C. (In other words, show that vector line integrals are independent of parametrization as long as the parametrizations in question induce the same orientation.)

Proof. ***TO BE FINISHED*** $\square$

6. Let $\mathbf{F} : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right).$$

If S_1 and S_2 are two smooth, closed C^1 surfaces in $\mathbb{R}^3$ which each enclose the origin and are each oriented with outward-pointing normal vectors, show that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(There was a problem on the final practice problems which would be applicable, except that here you cannot simply quote the result of that problem. The point is to justify this result in the case of this specific vector field.) Hint: Compute the divergence of $\mathbf{F}$.

Proof. ***TO BE FINISHED*** $\square$

7. Suppose $E \subseteq \mathbb{R}^2$ is Jordan measurable and that $f : E \to \mathbb{R}$ is integrable. Recall that, by definition, this means that if R is a rectangle containing E, the extended function

$$f_E^{ext}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in E \\ 0 & \mathbf{x} \notin E \end{cases}$$

is integrable over R. If $A \subseteq E$ is also Jordan measurable, show that f is integrable on A, which means that the extended function

$$f_A^{ext}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in A \\ 0 & \mathbf{x} \notin A \end{cases}$$

is integrable over R. **Careful: f_A^{ext} can be zero at points where f_E^{ext} was nonzero.** You can use without justification the fact that upper and lower integrals can be approximated to whatever degree of accuracy we want using small rectangles contained *fully* within the region of integration, as opposed to ones which only intersect that region.

Proof. The key point in all of this is to use only rectangles contained within A or E when approximating the required integrals, so that we can avoid the issue of how f^{ext} is defined over A vs how it is defined over E. Still, it is tricky to get all the details right!

Let R be a rectangular box containing E, which is thus also a rectangular box containing A. Let $\epsilon > 0$ and pick a grid G on R such that

$$\left| \sum_{R_i \subseteq E} M_i |R_i| - \int_E f\, dA \right| < \epsilon \quad \text{and} \quad \left| \sum_{R_i \subseteq E} m_i |R_i| - \int_E f\, dA \right| < \epsilon$$

where M_i, m_i respectively denote the supremum and infimum of f over R_i. These inequalities imply that

$$\sum_{R_i \subseteq E} (M_i - m_i)|R_i| < \left(\int_E f\, dA + \epsilon \right) - \left(\int_E f\, dA - \epsilon \right) = 2\epsilon.$$

Now, the rectangles $R_i \subseteq E$ can be separated into those contained in A and those not, so:

$$\sum_{R_i \subseteq E} (M_i - m_i)|R_i| = \sum_{R_i \subseteq A} (M_i - m_i)|R_i| + \sum_{R_i \not\subseteq A} (M_i - m_i)|R_i| \geq \sum_{R_i \subseteq A} (M_i - m_i)|R_i|$$

where we use the fact that $M_i - m_i$ is never negative. The left side is smaller than 2ϵ, and hence so is the right.

By picking a possibly finer grid we get that

$$\left| \sum_{R_i \subseteq A} M_i |R_i| - (U) \int_A f\, dA \right| < \epsilon \quad \text{and} \quad \left| \sum_{R_i \subseteq A} m_i |R_i| - (L) \int_A f\, dA \right| < \epsilon.$$

These inequalities imply that

$$(U) \int_A f\, dA - (L) \int_A f\, dA < \left(\sum_{R_i \subseteq A} M_i |R_i| + \epsilon \right) - \left(\sum_{R_i \subseteq A} m_i |R_i| - \epsilon \right) = \sum_{R_i \subseteq A} (M_i - m_i)|R_i| + 2\epsilon.$$

Using the inequality derived above we thus get

$$(U) \int_A f\, dA - (L) \int_A f\, dA < 4\epsilon.$$

The left side is non-negative, so since $\epsilon > 0$ here is arbitrary we get that

$$(U) \int_A f\, dA = (L) \int_A f\, dA,$$

showing that f is integrable over A. $\square$