Math 320-1: Final Exam Solutions
Northwestern University, Fall 2015

1. Give an example of each of the following. You do not have to justify your answer.

(a) A nonempty bounded set $S \in \mathbb{R}$ such that $(\sup S)^2 \neq \sup S^2$, where $S^2 = \{x^2 \mid x \in S\}$.

(b) A uniformly continuous differentiable function on $(0, \infty)$ with unbounded derivative.

(c) A non-integrable function f on $[2, 3]$ such that $f(2) = f(3) = 10$.

(d) A positive integrable function f on $[1, 2]$ such that $\frac{1}{f}$ is not integrable on $[1, 2]$

(e) A differentiable function $f : (1, 2) \to \mathbb{R}$ such that $f'(x) = \sin(x^2)$ for all $x \in (1, 2)$.

Solution. (a) The interval $S = (-10, 3)$ works. We have $S^2 = [0, 100)$, which has supremum 100 and not $(\sup S)^2 = 9$.

(b) The function $f(x) = \sqrt{x}$ works. This is uniformly continuous since for any $\epsilon > 0$, $\delta = \epsilon^2$ satisfies the required definition if we use the fact that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$, and its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, which is unbounded near 0.

(c) The function which is 10 at each rational and 0 at each irrational works. This is not integrable since all lower sums equal 0 and all upper sums equal $10(3 - 2) = 10$.

(d) The function defined by $f(x) = x - 1$ for $x \neq 1$ and $f(1) = 2$ works. This is integrable since it is continuous except at a single point, but its reciprocal—$\frac{1}{x-1}$ for $x \neq 1$ and $\frac{1}{2}$ at 1—is unbounded on $[1, 2]$ and so is not integrable.

(e) The function $F(x) = \int_1^x \sin(t^2)\, dt$ works by the Fundamental Theorem of Calculus. □

2. Suppose that S is a nonempty bounded subset of $\mathbb{R}$. Show that there exists a sequence (x_n) with each $x_n \in S$ which converges to $\inf S$. Hint: For any $\epsilon > 0$, $\inf S + \epsilon$ is not a lower bound of S.

Proof. For each $n \in \mathbb{N}$, $\inf S + \frac{1}{n}$ is not a lower bound of S, so there exists $x_n \in S$ such that

$$x_n < \inf S + \frac{1}{n}.$$

Since $\inf S \leq x_n$ ($\inf S$ is a lower bound of S), this gives

$$|x_n - \inf S| < \frac{1}{n}.$$

Thus for any $\epsilon > 0$, we can pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$, and get:

$$|x_n - \inf S| < \frac{1}{n} \leq \frac{1}{N} < \epsilon \text{ for any } n \geq N.$$

Hence the sequence (x_n) of elements of S thus constructed converges to $\inf S$. □

3. Define the sequence (x_n) by

$$x_n = \frac{2}{1^3} + \frac{2}{2^3} + \frac{2}{3^3} + \cdots + \frac{2}{n^3}$$

Show that (x_n) converges. You can use the fact from a previous homework assignment that the sequence $y_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ converges.

Proof. We will show that this sequence is Cauchy. Let $\epsilon > 0$. Since (y_n) converges, it is Cauchy so there exists $N \in \mathbb{N}$ such that

$$|y_{n+k} - y_n| < \frac{\epsilon}{2} \text{ for any } k \geq 0 \text{ and } n \geq N.$$

The difference $y_{n+k} - y_n$ equals:

$$y_{n+k} - y_n = \frac{1}{(n+k)^2} + \cdots + \frac{1}{(n+2)^2} + \frac{1}{(n+1)^2}$$

and the difference $x_{n+k} - x_n$ equals:

$$x_{n+k} - x_n = \frac{2}{(n+k)^3} + \cdots + \frac{2}{(n+2)^3} + \frac{2}{(n+1)^3}.$$

Since $\frac{1}{m^3} \leq \frac{1}{m^2}$ for any $m \geq \mathbb{N}$, we thus get that for any $n \geq N$ and $k \geq 0$, we have:

$$\begin{aligned}
|x_{n+k} - x_n| &= \frac{2}{(n+k)^3} + \cdots + \frac{2}{(n+2)^3} + \frac{2}{(n+1)^3} \\
&\leq \frac{2}{(n+k)^2} + \cdots + \frac{2}{(n+2)^2} + \frac{2}{(n+1)^2} \\
&= 2|y_{n+k} - y_n| \\
&< 2\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus (x_n) is Cauchy, so it converges. □

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Show that for any $x, y \in \mathbb{R}$ with $x \neq y$, there exists a **rational** c between x and y such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(c) \right| < \frac{1}{1000}.$$

Hint: Use the Mean Value Theorem to rewrite $\frac{f(x)-f(y)}{x-y}$.

Proof. By the Mean Value Theorem, for any $x, \neq y$ there exists d between x and y such that:

$$\frac{f(x) - f(y)}{x - y} = f'(d).$$

Since f' is continuous at d, there exists $\delta > 0$ such that

$$|f'(d) - f'(x)| < \frac{1}{1000} \text{ whenever } |d - x| < \delta.$$

Thus for a rational number c in $(d - \delta, d + \delta)$—which exists by the denseness of $\mathbb{Q}$ in $\mathbb{R}$—we have:

$$|f'(d) - f'(c)| < \frac{1}{1000}, \text{ which is equivalent to } \left| \frac{f(x) - f(y)}{x - y} - f'(c) \right| < \frac{1}{1000}$$

as desired. □

5. Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - \frac{1}{n} & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

is integrable on $[0,1]$ and determine the value of $\int_0^1 f(x)\,dx$.

Proof. Let $\epsilon > 0$. There are only finitely many numbers of the form $\frac{1}{n}$ where $n \in \mathbb{N}$ which are larger than $\frac{\epsilon}{2}$—call the n's which give these finite number $n_1, \ldots, n_k$, so that there are k in total. Take an interval I_j around each $\frac{1}{n_j}$ whose length is smaller than:

$$\text{length}(I_j) < \frac{\epsilon}{2k}$$

and furthermore if necessary shrink each I_j so that they do not intersect and lie completely within $[0,1]$. Take P to be the partition of $[0,1]$ defined by $0,1$, and the endpoints of all the I_j.

We break up the computation of $U(f,P) - L(f,P)$ into three types of subintervals: those taken over the subintervals I_j; those taken over $[0, \frac{\epsilon}{2}]$; and those taken over the remaining subintervals. Over the third type, $\sup f$ and $\inf f$ are both 1 since f is constant on these, so these contribute nothing to the difference $U(f,P) - L(f,P)$. Over the second type $[0, \frac{\epsilon}{2}]$, we have:

$$(\sup f - \inf f)(\text{length}) \leq 1(\text{length}) = \frac{\epsilon}{2}.$$

And finally over the first type, we have:

$$\sum_{I_j} (\sup f - \inf f)(\text{length}) \leq \sum_{I_j} 1(\text{length}) < \sum_{j=1}^{k} \frac{\epsilon}{2k} = \frac{\epsilon}{2}.$$

Thus after adding up all three contributions, we get:

$$U(f,P) - L(f,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + 0 = \epsilon,$$

which shows that f is integrable on $[0,1]$.

The value of all upper sums is $1(1-0) = 1$ since the supremum of f over any subinterval is 1, so the infimum of all upper sums, and hence the value of $\int_0^1 f(x)\,dx$, is 1. ☐

6. Suppose $f : [0,5] \to \mathbb{R}$ is continuous and define $g : [0,5] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & x \neq 2, 5 \\ 10 & x = 2 \\ -4 & x = 5. \end{cases}$$

Show that g is integrable on $[0,5]$. You **cannot** simply quote the practice problem which says that changing the value of an integrable function at a finite number of points still results in an integrable function—the point here is to prove this in the special case where we change the value at 2 points.

Proof. Since f is continuous, it is bounded, and since g differs from f at possibly only two points, it too is bounded. Let M be a bound on g, so that $|g(x)| \leq M$ for all $x \in [0,5]$, which then implies that $|g(x)-g(y)| \leq 2M$ for all $x, y \in [0,5]$. This in turn implies that $\sup g - \inf g \leq 2M$ on any subinterval within $[0,5]$.

Pick an interval $I = [2-\delta_1, 2+\delta_1]$ around 2 of length smaller than

$$\text{length}(I) < \frac{\epsilon}{8M}$$

and an interval $J = [5-\delta_2, 5]$ containing 5 of length smaller than

$$\text{length}(J) < \frac{\epsilon}{8M}.$$

Furthermore, if necessary shrink I and J so that they lie within $[0,5]$ and do not intersect. Since g is continuous on $[0, 2-\delta_1]$ and $[2+\delta_1, 5-\delta]$—because it equals f on each of these—g is integrable on these so there exist partitions P_1, P_2 of these two intervals respectively such that

$$U(g,P_1) - L(g,P_1) < \frac{\epsilon}{4} \quad \text{and} \quad U(g,P_2) - L(g,P_2) < \frac{\epsilon}{4}.$$

Let P be the partition of $[0,5]$ consisting of $0, 5$, all the points making up P_1, and all the points making up P_2. Then the subintervals determined by P come in four types: those determined by P_1, $[2-\delta_1, 2+\delta_1]$, those determined by P_2, and $[5-\delta_1, 5]$. The value of $U(g,P) - L(g,P)$ then consists of four contributions. The first type contributes $U(g,P_1) - L(g,P_1) < \frac{\epsilon}{4}$; the second contributes:

$$(\sup g - \inf g)(\text{length}) \leq 2M\frac{\epsilon}{8M} = \frac{\epsilon}{4};$$

the third contributes $U(g,P_2) - L(g,P_2) < \frac{\epsilon}{4}$; and the fourth contributes:

$$(\sup g - \inf g)(\text{length}) \leq 2M\frac{\epsilon}{8M} = \frac{\epsilon}{4}.$$

Thus altogether we get:

$$U(g,P) - L(g,P) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,$$

so g is integrable over $[0,5]$. □

7. Define $f : [-2,2] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \cos\frac{1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

and $F : [-2,2] \to \mathbb{R}$ by

$$F(x) = \int_{-2}^{x^4 e^x} t f(t)\, dt \text{ for all } x \in [-2,2].$$

Show that $F'(0)$ exists. Careful: f is not continuous at 0

Proof. We have:

$$\frac{F(x) - F(0)}{x - 0} = \frac{1}{x}\left(\int_{-2}^{x^4 e^x} t f(t)\, dt - \int_{-2}^{0} t f(t)\, dt\right) = \frac{1}{x}\int_{0}^{x^4 e^x} t f(t)\, dt.$$

In absolute value, we can found this by:

$$\frac{1}{|x|}\left|\int_{0}^{x^4 e^x} t f(t)\, dt\right| \leq \frac{1}{|x|}\int_{0}^{x^4 e^x} |t f(t)|\, dt \leq \frac{1}{|x|}\int_{0}^{x^4 e^x} 2\, dt = 2|x^3|e^x,$$

where we use the fact that $|tf(t)| \leq 2(1) = 2$ for $t \in [-2, 2]$. Since this final expression goes to 0 as $x \to 0$, the squeeze theorem implies that the initial expression on the left does too, and so

$$\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = 0$$

as well. Hence $F'(0)$ exists and equals 0. □