Math 320-1: Final Exam
Northwestern University, Fall 2019

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A sequence which diverges but has a convergent subsequence.
(b) A function $f(x)$ for which $\lim_{x\to 0} |f(x)|$ exists but $\lim_{x\to 0} f(x)$ does not.
(c) A function which is continuous only at 1 and is nowhere zero.
(d) A differentiable function which is not twice-differentiable.
(e) An integrable function on $[1, 3]$ which does not have an antiderivative.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Show that if (x_n) converges, then (x_n) is Cauchy. Hint: if x is the number to which (x_n) converges, find a way to relate $|x_m - x_n|$ to expressions of the form $|x_k - x|$.

3. (10 points) Suppose $f : (0, \infty) \to \mathbb{R}$ is continuous and always positive. Show, by verifying the ϵ-δ definition directly, that the function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{2}{3f(x)}$ is continuous at any $a \in (0, \infty)$.

4. (10 points) Show that for any $x \in \mathbb{R}$ there exists c between 0 and x such that

$$e^x = 1 + x + \frac{e^c}{2}x^2.$$

Hint: For a fixed $x \in \mathbb{R}$, define $h(y) = e^y - 1 - y - \left(\frac{e^x - 1 - x}{x^2}\right) y^2$ as a function of y. Verify that $h'(0) = 0$ and somehow use the Mean Value Theorem to get another point d between 0 and x at which $h'(d) = 0$, and then use the Mean Value Theorem again to get a point c at which $h''(c) = 0$. The derivatives of h here are taken with respect to y. (This is proving a special case of Taylor's Theorem for the function $f(x) = e^x$, so you cannot use Taylor's Theorem here.)

5. (10 points) Suppose $f : [a, b] \to \mathbb{R}$ is bounded and that P and Q are partitions of $[a, b]$. Show that

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

You cannot take it for granted that including more points in a partition can only make lower sums larger and upper sums smaller—rather, you should justify this as a first step. For this, first think about what happens if you throw an extra single point into your partition.

6. (10 points) Suppose $f : [0, 1] \to \mathbb{R}$ is a function which agrees with $g(x) = \sin x$ except at two points $c < d$ in $(0, 1)$. Show f is integrable on $[0, 1]$ by showing that, for each $\epsilon > 0$, there exists a partition P of $[0, 1]$ for which $U(f, P) - L(f, P) < \epsilon$.

7. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 5\cos(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{-1}^{x^3} f(t)\, dt.$$

Show that F is differentiable on all of $\mathbb{R}$, and determine if it is continuously differentiable.