Math 320-1: Final Exam
Northwestern University, Fall 2015

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A nonempty bounded set $S \in \mathbb{R}$ such that $(\sup S)^2 \neq \sup S^2$, where $S^2 = \{x^2 \mid x \in S\}$.

(b) A uniformly continuous differentiable function on $(0, \infty)$ with unbounded derivative.

(c) A non-integrable function f on $[2, 3]$ such that $f(2) = f(3) = 10$.

(d) A positive integrable function f on $[1, 2]$ such that $\frac{1}{f}$ is not integrable on $[1, 2]$

(e) A differentiable function $f : (1, 2) \to \mathbb{R}$ such that $f'(x) = \sin(x^2)$ for all $x \in (1, 2)$.

Problem	Score
1	
2	
3	
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7	
Total	

2. (10 points) Suppose that S is a nonempty bounded subset of $\mathbb{R}$. Show that there exists a sequence (x_n) with each $x_n \in S$ which converges to $\inf S$. Hint: For any $\epsilon > 0$, $\inf S + \epsilon$ is not a lower bound of S.

3. (10 points) Define the sequence (x_n) by

$$x_n = \frac{2}{1^3} + \frac{2}{2^3} + \frac{2}{3^3} + \cdots + \frac{2}{n^3}$$

Show that (x_n) converges. You can use the fact from a previous homework assignment that the sequence $y_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ converges.

4. (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Show that for any $x, y \in \mathbb{R}$ with $x \neq y$, there exists a **rational** c between x and y such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(c) \right| < \frac{1}{1000}.$$

Hint: Use the Mean Value Theorem to rewrite $\frac{f(x)-f(y)}{x-y}$.

5. (10 points) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - \frac{1}{n} & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$ and determine the value of $\int_0^1 f(x)\, dx$.

6. (10 points) Suppose $f : [0, 5] \to \mathbb{R}$ is continuous and define $g : [0, 5] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & x \neq 2, 5 \\ 10 & x = 2 \\ -4 & x = 5. \end{cases}$$

Show that g is integrable on $[0, 5]$. You **cannot** simply quote the practice problem which says that changing the value of an integrable function at a finite number of points still results in an integrable function—the point here is to prove this in the special case where we change the value at 2 points.

7. (10 points) Define $f : [-2, 2] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \cos\frac{1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

and $F : [-2, 2] \to \mathbb{R}$ by

$$F(x) = \int_{-2}^{x^4 e^x} t f(t)\, dt \text{ for all } x \in [-2, 2].$$

Show that $F'(0)$ exists. Careful: f is not continuous at 0