PUTNAM TRAINING
PIGEONHOLE PRINCIPLE

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REMARK. This is a list of exercises on the Pigeonhole Principle. —Miguel A. Lerma

EXERCISES

1. Prove that any $(n+1)$-element subset of $\{1, 2, \ldots, 2n\}$ contains two integers that are relatively prime.

2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \ldots, 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

4. Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A, two of them must coincide in at least 2 places.

5. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

6. (Putnam, 2006-B2.) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

7. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

8. Prove that among any seven real numbers $y_1, \ldots, y_7$, there are two such that

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

9. Prove that among five different integers there are always three with sum divisible by 3.

10. Prove that there exist an integer n such that the first four digits of 2^n are $2, 0, 0, 9$.

11. Prove that every convex polyhedron has at least two faces with the same number of edges.

Hints

1. Divide the set into n subsets each of which has only pairwise relatively prime numbers.

2. Divide the set into n subsets each of which contains only numbers which are multiple or divisor of the other ones.

3. Look at pairs of numbers in that sequence whose sum is precisely 104. Those pairs may not cover the whole progression, but that can be fixed...

4. Prove that for each $k = 1, 2, \dots, 8$, at least 2 of the elements given coincide at place k. Consider a pair of elements which coincide at place 1, another pair of elements which coincide at place 2, and so on. How many pairs of elements do we have?

5. Consider the sequences a_i = number of games played from the 1st through the jth day of the month, and $b_j = a_j + 14$. Put them together and use the pigeonhole principle to prove that two elements must be the equal.

6. Consider the fractional part of sums of the form $s_i = x_1 + \cdots + x_i$.

7. Consider the number of different subsets of a ten-element set, and the possible number of sums of at most ten two-digit numbers.

8. Write $y_i = \tan x_i$, with $-\frac{\pi}{2} \le x_i \le \frac{\pi}{2}$ $(i = 1, \dots, 7)$. Find appropriate "boxes" for the x_is in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

9. Classify the numbers by their reminder when divided by 3.

10. We must prove that there are positive integers n, k such that

$$2009 \cdot 10^k \le 2^n < 2010 \cdot 10^k \,.$$

11. Look at the face with the maximum number of edges and its neighbors.

Solutions

1. We divide the set into n-classes $\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}$. By the pigeonhole principle, given $n + 1$ elements, at least two of them will be in the same class, $\{2k - 1, 2k\}$ $(1 \leq k \leq n)$. But $2k - 1$ and $2k$ are relatively prime because their difference is 1.

2. For each odd number $\alpha = 2k - 1$, $k = 1, \dots, n$, let C_α be the set of elements x in S such that $x = 2^i\alpha$ for some i. The sets $C_1, C_3, \dots, C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one.

3. The given set can be divided into 18 subsets $\{1\}$, $\{4, 100\}$, $\{7, 97\}$, $\{10, 94\}$,..., $\{49, 55\}$, $\{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.

4. For $k = 1, 2, \dots, 8$, look at the digit used in place k for each of the 4 given elements. Since there are only 3 available digits, two of the elements will use the same digit in place k, so they coincide at that place. Hence at each place, there are at least two elements that coincide at that place. Pick any pair of such elements for each of the 8 places. Since there are 8 places we will have 8 pairs of elements, but there are only $\binom{4}{2} = 6$ two-element subsets in a 4-element set, so two of the pairs will be the same pair, and the elements of that pair will coincide in two different places.

5. Let a_j the number of games played from the 1st through the jth day of the month. Then $a_1, a_2, \dots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Likewise, $b_j = a_j + 14$, $j = 1, \dots, 30$ is also an increasing sequence of distinct positive integers with $15 \leq b_j \leq 59$. The 60 positive integers $a_1, \dots, a_{30}, b_1, \dots, b_{30}$ are all less than or equal to 59, so by the pigeonhole principle two of them must be equal. Since the a_j's are all distinct integers, and so are the b_j's, there must be indices i and j such that $a_i = b_j = a_j + 14$. Hence $a_i - a_j = 14$, i.e., exactly 14 games were played from day $j + 1$ through day i.

6. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \dots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \dots, \{s_n\}$ into ascending order, and call the result $t_0, \dots, t_n$. Since $0 = t_0 \leq \cdots \leq t_n < 1$, the differences

$$t_1 - t_0, \dots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n + 1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \leq i < j \leq n$. Put $S = \{x_{i+1}, \dots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$;

then

$$\begin{aligned}\left|m+\sum_{s\in S}s\right| &= |m+s_j-s_i| \\ &= |\{s_j\}-\{s_i\}| \\ &\le \frac{1}{n+1},\end{aligned}$$

as desired. In case $1-t_n \le 1/(n+1)$, we take $S=\{x_1,\dots,x_n\}$ and $m=-\lceil s_n\rceil$, and again obtain the desired conclusion.

7. A set of 10 elements has $2^{10}-1=1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10\cdot 99=990$. There are more subsets than possible sums, so two different subsets S_1 and S_2 must have the same sum. If $S_1\cap S_2=\emptyset$ then we are done. Otherwise remove the common elements and we get two non-intersecting subsets with the same sum.

8. Writing $y_i=\tan x_i$, with $-\frac{\pi}{2}\le x_i\le\frac{\pi}{2}$ $(i=1,\dots,7)$, we have that

$$\frac{y_i-y_j}{1+y_iy_j}=\tan\left(x_i-x_j\right),$$

so all we need is to do is prove that there are x_i,x_j such that $0\le x_i-x_j\le\frac{\pi}{6}$. To do so we divide the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ into 6 subintervals each of length $\frac{\pi}{6}$. By the box principle, two of the x_is will be in the same subinterval, and their difference will be not larger than $\frac{\pi}{6}$, as required.

9. Classify the numbers by their reminder when divided by 3. Either three of them will yield the same reminder, and their sum will be a multiple of 3, or there will be at least a number x_r for each possible reminder $r=0,1,2$, and their sum $x_0+x_1+x_2$ will be a multiple of 3 too.

10. We must prove that there are positive integers n,k such that

$$2009\cdot 10^k\le 2^n<2010\cdot 10^k\,.$$

That double inequality is equivalent to

$$\log_{10}(2009)+k\le n\log_{10}(2)<\log_{10}(2010)+k\,.$$

where $\log_{10}$ represents the decimal logarithm. Writing $\alpha=\log_{10}(2009)-3$, $\beta=\log_{10}(2010)-3$, we have $0<\alpha<\beta<1$, and the problem amounts to showing that for some integer n, the fractional part of $n\log_{10}(2)$ is in the interval $[\alpha,\beta)$. This is true because $\log_{10}(2)$ is irrational, and the integer multiples of an irrational number are dense modulo 1 (their fractional parts are dense in the interval $[0,1)$).

11. Let F be the face with the largest number m of edges. Then for the $m+1$ faces consisting of F and its m neighbors the possible number of edges are $3,4,\dots,m$. These are only $m-2$ possibilities, hence the number of edges must occur more than once.