1.4. The Substitution Rule

1.4.1. The Substitution Rule. The substitution rule is a trick for evaluating integrals. It is based on the following identity between differentials (where $u$ is a function of $x$):

$$du = u' \, dx.$$ 

Hence we can write:

$$\int f(u) \, u' \, dx = \int f(u) \, du$$

or using a slightly different notation:

$$\int f(g(x)) \, g'(x) \, dx = \int f(u) \, du$$

where $u = g(x)$.

Example: Find $\int \sqrt{1 + x^2} \, 2x \, dx$.

Answer: Using the substitution $u = 1 + x^2$ we get

$$\int \sqrt{1 + x^2} \, 2x \, dx = \int \sqrt{u} \, u' \, dx = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + x^2)^{3/2} + C.$$ 

Most of the time the only problem in using this method of integration is finding the right substitution.

Example: Find $\int \cos 2x \, dx$.

Answer: We want to write the integral as $\int \cos u \, du$, so $\cos u = \cos 2x \Rightarrow u = 2x$, $u' = 2$. Since we do not see any factor 2 inside the
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In general we need to look at the integrand as a function of some expression (which we will later identify with $u$) multiplied by the derivative of that expression.

**Example:** Find $\int e^{-x^2} x \, dx$.

**Answer:** We see that $x$ is “almost”, the derivative of $-x^2$, so we use the substitution $u = -x^2$, $u' = -2x$, hence in order to get $u'$ inside the integral we do the following:

$$\int e^{-x^2} x \, dx = -\frac{1}{2} \int e^{-x^2} (-2x) \, dx = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C.$$
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Answer: Here the idea is to write \( \tan x = \frac{\sin x}{\cos x} \) and use that \((\cos x)' = -\sin x\), so we make the substitution \( u = \cos x \), \( u' = -\sin x \):

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{u'}{u} \, dx = -\int \frac{1}{u} \, du
\]
\[
= -\ln |u| + C = -\ln |\cos x| + C.
\]

In general we need to identify inside the integral some expression of the form \( f(u) \, u' \), where \( f \) is some function with a known antiderivative.

Example: Find \( \int \frac{e^x}{e^{2x} + 1} \, dx \).

Answer: Let’s write

\[
\int \frac{e^x}{e^{2x} + 1} \, dx = k \int f(u) \, u' \, dx
\]

(where \( k \) is some constant to be determined later) and try to identify the function \( f \), the argument \( u \) and its derivative \( u' \). Since \((e^x)' = e^x\) it seems natural to chose \( u = e^x \), \( u' = e^x \), so \( e^{2x} = u^2 \) and

\[
\int \frac{e^x}{e^{2x} + 1} \, dx = \int \frac{u'}{u^2 + 1} \, dx = \int \frac{1}{u^2 + 1} \, du
\]
\[
= \tan^{-1} u + C = \tan^{-1} (e^x) + C.
\]

There is no much more that can be said in general, the way to learn more is just to practice.

1.4.2. Other Changes of Variable. Sometimes rather than making a substitution of the form \( u = \text{function of } x \), we may try a change of variable of the form \( x = \text{function of some other variable such as } t \), and write \( dx = x'(t) \, dt \), where \( x' = \text{derivative of } x \text{ respect to } t \).

Example: Find \( \int \sqrt{1 - x^2} \, dx \).

Answer: Here we write \( x = \sin t \), so \( dx = \cos t \, dt \), \( 1 - x^2 = 1 - \sin^2 t = \cos^2 t \), and

\[
\int \sqrt{1 - x^2} \, dx = \int \cos t \, \cos t \, dt = \int \cos^2 t \, dt.
\]
Since we do not know yet how to integrate $\cos^2 t$ we leave it like this and will be back to it later (after we study integrals of trigonometric functions).

1.4.3. The Substitution Rule for Definite Integrals. When computing a definite integral using the substitution rule there are two possibilities:

1. Compute the indefinite integral first, then use the evaluation theorem:

$$
\int f(u) u' \, dx = F(x) ;
$$

$$
\int_{a}^{b} f(u) u' \, dx = F(b) - F(a) .
$$

2. Use the substitution rule for definite integrals:

$$
\int_{a}^{b} f(u) u' \, dx = \int_{u(a)}^{u(b)} f(u) \, du .
$$

The advantage of the second method is that we do not need to undo the substitution.

**Example**: Find $\int_{0}^{4} \sqrt{2x + 1} \, dx$.

**Answer**: Using the first method first we compute the indefinite integral:

$$
\int \sqrt{2x + 1} \, dx = \frac{1}{2} \int \sqrt{2x + 1} \, 2 \, dx \quad (u = 2x + 1)
$$

$$
= \frac{1}{2} \int \sqrt{u} \, du
$$

$$
= \frac{1}{3} u^{3/2} + C
$$

$$
= \frac{1}{3} (2x + 1)^{3/2} + C .
$$

Then we use it for computing the definite integral:

$$
\int_{0}^{4} \sqrt{2x + 1} \, dx = \left[ \frac{1}{3} (2x + 1)^{3/2} \right]_{0}^{4} = \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} .
$$
In the second method we compute the definite integral directly adjusting the limits of integration after the substitution:

\[
\int_0^4 \sqrt{2x+1} \, dx = \frac{1}{2} \int_0^4 \sqrt{2x+1} \, 2 \, dx \quad (u = 2x + 1; \, u' = 2)
\]

\[
= \frac{1}{2} \int_1^9 \sqrt{u} \, du
\]

(note the change in the limits of integration to \( u(0) = 1 \) and \( u(4) = 9 \))

\[
= \left[ \frac{1}{3} u^{3/2} \right]_1^9
\]

\[
= \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2}
\]

\[
= \frac{27}{3} - \frac{1}{3} = \frac{26}{3}
\]